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Calderón–Zygmund type decompositions and applications

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Abstract. In the paper a variant of the abstract approach to the Calderón–Zygmund type decompositions is considered. Applications to K-closedness of spaces of analytic functions are given.

Key words: Calderón–Zygmund decomposition, real interpolation.

1. INTRODUCTION AND GENERAL THEORY

The classical Calderón–Zygmund decomposition appeared in 1952 in the fundamental paper by A. P. Calderón and A. Zygmund "On the existence of certain singular integrals" (see [1]) and was used to prove a weak-type estimate for singular integral operators.

Later on this decomposition played a very important and sometimes even crucial role in proofs of many fundamental results such as the John–Nirenberg inequality, the Fefferman–Stein vector-valued maximal theorem, the theory of weighted norm inequalities, etc.

In the classical Calderón–Zygmund decomposition a given function $f \in L_1$ with a given number $t > 0$ is decomposed into a "good" part $f_t \in L_\infty$ and a "bad" part $f - f_t \in L_1 : f = f_t + (f - f_t)$. So we can see that it is deeply connected with a concrete couple of spaces, namely (L_1, L_∞), therefore there naturally appears the following problem.

Problem 1. How to define an analogue of the Calderón–Zygmund decomposition for couples different from (L_1, L_∞)? Are these "abstract" decompositions useful?

Below we will propose a variant of the abstract definition and will show its usefulness. Let (X_0, X_1) be a Banach couple and $\mathrm{dist}_{X_0}(x, B_{X_1}(t))$ be the distance in the metric of X_0 from the element $x \in X_0 + X_1$ to the ball of radius t in the space X_1 with the centre at the point zero.

Definition 2. *We will say that a Banach couple (X_0, X_1) possesses the Calderón–Zygmund (CZ) property if for any element $x \in X_0$ and any $t > 0$ there exists a linear operator $P_{x,t} : X_0 \to X_0$ such that*
(a) *elements $x_t = P_{x,t}(x)$ form a family of almost optimal approximations of x, i.e.,*

$$\|x_t\|_{X_1} \leq ct, \quad \|x - x_t\|_{X_0} \leq c \cdot \mathrm{dist}_{X_0}\Big(x, B_{X_1}\Big(\frac{1}{c}\, t\Big)\Big)$$

(b) *the norms of operators $P_{x,t}$, $I - P_{x,t}$ can be estimated by*

$$\|P_{x,t}\|_{X_0 \to X_0} \leq c, \quad \|I - P_{x,t}\|_{X_1 \to X_0} \leq c\frac{1}{t}\, \mathrm{dist}_{X_0}\Big(x, B_{X_1}\Big(\frac{1}{c}\, t\Big)\Big)\ .$$

Here the constant $c > 0$ is independent of x and t.

The classical Calderón–Zygmund decomposition is in some sense "well adapted" to the class of singular integral operators. The following definition is modelled on this property.

Definition 3. *Let (X_0, X_1) be a Banach couple with the CZ property and let $T : D_T :\to X_0 + X_1$ be a linear operator defined on some linear subspace $D_T \subset X_0 + X_1$. We will say that the CZ property is well adapted to the operator T if from $x \in X_0$ and $Tx \in X_0$ it follows that*

$$\max(\|(I - P_{x,t})TP_{x,t}\, x\|_{X_0}, \|P_{x,t}T(I - P_{x,t})\, x\|_{X_0}) \leq c{\cdot}\mathrm{dist}_{X_0}\Big(f, B_{X_1}\Big(\frac{1}{c}\, t\Big)\Big).$$

Here the constant $c > 0$ is independent of x and t. Moreover, the inequality implies that from $x \in X_0$ and $Tx \in X_0$ it follows that the left-hand side has sense.

Remark 4. From Definition 2 it follows that a couple with the CZ property is well adapted to all operators T bounded in X_0 and X_1.

And now some theory. The next theorem shows that the CZ property is stable under interpolation.

Theorem 5. *If (X_0, X_1) is a Banach couple with the CZ property which is well adapted to an operator T, then the couple $(X_0, X_{\theta,q})$ possesses the CZ property which is well adapted to the operator T.*

The next theorem shows that not only $x_t = P_{t,x}x$ approximates x in an almost optimal way (see Definition 2), but also Tx_t approximates Tx quite well.

Theorem 6. *Let* (X_0, X_1) *be a Banach couple with the CZ property which is well adapted to a given operator* T. *Suppose also that the element* $x \in X_0$ *is such that* $Tx \in X_0$. *Then* Tx_t *approximates* Tx:

$$\|Tx - Tx_t\|_{X_0} \leq c\left(\mathrm{dist}_{X_0}\Big(x, B_{X_1}\Big(\frac{t}{c}\Big)\Big) + \mathrm{dist}_{X_0}\Big(Tx, B_{X_1}\Big(\frac{t}{c}\Big)\Big)\right).$$

Note that in the case where $Tx = x$ we only have one term in the right-hand side. This gives us the following result.

Corollary 7. *Suppose*, *in addition*, *that* $T^2 = T$ *and* T *is bounded in* X_1. *Then* $(T(X_0), T(X_1))$ *is a* K*-closed subcouple of the couple* (X_0, X_1).

Applications of the above theory are based on the following result (see [2,3]).

Theorem 8. *Let* X *be any of the spaces* L_p, BMO, B_p^σ, W_p^k *with* $1 < p \leq \infty$, $\sigma > 0$. *Then the couple* (L_1, X) *possesses the CZ property which is well adapted to all singular integral operators bounded on* X. *For example*, *the couple* (L_1, Lip_α) *possesses the CZ property which is well adapted to singular integral operators bounded on* Lip_α.

Remark 9. In fact, in Theorem 8 instead of singular integral operators, we can consider the class of operators, which contains singular integral operators and satisfies several general properties (see [3] for details).

Remark 10. Previous results in this direction can be found in papers [4–6].

2. APPLICATION TO K-CLOSEDNESS OF SPACES OF ANALYTIC FUNCTIONS

The notion of K-closed subcouples was introduced by J. Peetre in 1971. However, the importance of this notion for some difficult and important analytic problems was discovered much later by G. Pisier (1992–1993). In fact, Pisier in several papers showed that the notion of K-closedness simplifies the proofs and gives a general point of view for two remarkable and deep results due to Jean Bourgain (an analogue of Grothendieck's theorem for the disk algebra of analytic functions) and Peter Jones (the existence of a solution of the $\bar{\partial}$-equation, which satisfies simultaneously good L_1 and L_∞ estimates).

To formulate the basic result due to Pisier (see [7]), let us consider the space L_1 on the unit circle and its closed subspace A of analytic functions, i.e., the functions $f \in L_1$ such that $f \sim \sum_{k \geq 0} c_k e^{ikx}$. Then Pisier proved that the couple $(A \cap L_1, A \cap L_p)$ is K-closed in the couple (L_1, L_p) for all $p > 1$. From the results of the previous section follows (compare with [2] and [3])

Theorem 11. *Pisier's result can be extended to Besov and Sobolev spaces. More exactly,*

(a) *the couple* $(A \cap L_1, A \cap B_p^\sigma)$ *is* K*-closed in the couple* (L_1, B_p^σ) *for all* $1 < p \leq \infty, \sigma > 0$;

(b) *the couple* $(A \cap L_1, A \cap W_p^k)$ *is* K*-closed in the couple* (L_1, W_p^k) *for all* $1 < p < \infty$.

Proof. Let us consider the Riesz projection on the space of analytic functions

$$Rf \sim \sum_{k \geq 0} c_k e^{ikx}, \quad f \sim \sum_{k \in \mathbb{Z}} c_k e^{ikx}.$$

It is well known that the Riesz projector is a singular integral operator, which is bounded in B_p^σ for $1 < p \leq \infty$, $\sigma > 0$ and is bounded in W_p^k for $1 < p < \infty$. Moreover, it is clear that $R^2 = R$. Therefore the theorem follows from Theorem 8 and Corollary 7. $\square$

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Calderóni–Zygmundi-tüüpi dekompositsioonidest ja nende rakendamisest

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On vaadeldud varianti abstraktsest lähenemisest Calderóni–Zygmundi-tüüpi dekompositsioonidele. On antud rakendused K-kinnisusele analüütilistest funktsioonidest moodustatud ruumides.