Proc. Estonian Acad. Sci. Phys. Math., 2006, **55**, 3, 150–158

Quaternary binary trilinear forms of norm unity

Bo Bernhardsson[a], Fernando Cobos[b], Thomas Kühn[c], Daniel Mondoc[d], and Jaak Peetre[e]

[a] Department of Automatic Control, Lund University, Box 118, SE-221 00 Lund, Sweden; bo.bernhardsson@ericsson.com

[b] Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad Complutense de Madrid, E-28040 Madrid, Spain; cobos@mat.ucm.es

[c] Fakultät für Mathematik und Informatik, Mathematisches Institut, Universität Leipzig, Augustusplatz 10/11, D-04109 Leipzig, Germany; kuehn@math.uni-leipzig.de

[d] Department of Mathematics, Royal Institute of Technology (KTH), SE-100 44 Stockholm, Sweden; mondoc@math.kth.se

[e] Centre for Mathematical Sciences, Department of Mathematics, Lund University, Box 118, SE-221 00 Lund, Sweden; jaak@maths.lth.se

Received 10 April 2006, in revised form 24 May 2006

Abstract. So far trilinear forms have mostly been considered in low dimensions, in particular the dimension two (*binary*) case, and when the ring of scalars $\mathbb{K}$ is either the real numbers $\mathbb{R}$ or the complex ones $\mathbb{C}$. The main aim in both situations has been to decide when a normalized form has norm unity. Here we consider the case of *quaternions*, $\mathbb{K} = \mathbb{H}$. This note is rather preliminary, and somewhat *experimental*, where the computer program Mathematica plays a certain role. A preliminary result obtained is that the form has norm unity if and only if the discriminant of a certain 5-dimensional quadratic form has all its principal minors non-negative. We found also a rather unexpected similarity between the noncommutative case of $\mathbb{H}$ and the commutative one of $\mathbb{R}$ and $\mathbb{C}$.

Key words: trilinear form, quaternion, discriminant, principal minor.

1. INTRODUCTION

Most studies on trilinear forms have been devoted to forms in low dimensions, in particular the dimension two or *binary* case, the ring of scalars $\mathbb{K}$ being the real numbers $\mathbb{R}$ or the complex ones $\mathbb{C}$. The main goal in both situations has been to find a criterion for a normalized (see 2.1) form to have norm unity.

Here we consider the case of *quaternions*, $\mathbb{K} = \mathbb{H}$. A motivation for this study was that the cases $\mathbb{R}$ and $\mathbb{C}$ may behave quite differently, for instance, from the point of view of the norm unity criterion. Therefore we began to look for a more general approach. Recall that the ring of quaternions $\mathbb{H}$ is obtained from $\mathbb{C}$ by adjoining, besides i, another imaginary unit j ($j^2 = -1$). As was discovered by Hamilton himself, $\mathbb{H}$ is *noncommutative*. Continuation of this procedure – known as the *Cayley–Dickson process* – gives next rise to the ring of *octonions* $\mathbb{O}$, which is an 8-dimensional ring over the reals. (In passing to octonions, one loses also the associativity.) So far we have not been able to extend our results fully to the latter case.

In this rather preliminary, and somewhat *experimental* note we present a result for quaternion forms, which exhibits a rather unexpected similarity between the noncommutative case of $\mathbb{H}$ and the commutative one of $\mathbb{R}$ and $\mathbb{C}$ (Section 5).

2. DEFINITIONS

In the *commutative* case ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) an N-dimensional trilinear form T can be defined by the formula

$$T(x, y, z) = a_{jk\ell} x^j y^k z^\ell. \tag{1}$$

Here, x, y, z denote vectors in three copies of $\mathbb{K}^N$; thus, $x = (x^0, \dots, x^{N-1})$, and similarly for y, z. The coefficients $a_{jk\ell}$ are here arbitrary elements of $\mathbb{K}$; they may be viewed as components of a 3-*tensor* or, as Gel'fand [1] says, elements of a 3-*dimensional matrix*. Furthermore, we use the *Einstein summation convention*.

In this case it is irrelevant in which order we take the coefficients and the variables. This is not so for the *noncommutative* case of $\mathbb{K} = \mathbb{H}$. So we modify (1) as follows:

$$T(x, y, z) = x^j y^k a_{jk\ell} z^\ell. \tag{2}$$

For a motivation of this choice see Section 2.2, where we associate to T a vector bilinear form $\tilde{T}$ (cf. [2]). From here on, we take $\mathbb{K} = \mathbb{H}$.

The norm of a vector is the square root of the sum of the squares of its components,

$$\|x\|^2 = \sum_{j=0}^{N-1} |x^j|^2,$$

similarly with y, z. The (maximum) norm of the form T is

$$\|T\| = \sup_{x,y,z \neq 0} \frac{|T(x, y, z)|}{\|x\| \|y\| \|z\|}. \tag{3}$$

2.1. Normalized forms

Our main concern in this paper is to describe the structure of forms of norm unity. Let us say that a form T is *normalized* (see [3]) if

$$a_{000} = 1; \quad a_{0k\ell} = a_{j0\ell} = a_{jk0} = 0. \tag{4}$$

Next, let T be any form of norm unity, $\|T\| = 1$. As we are in the finite-dimensional case, it follows that there exist unit vectors e, f, g in our three copies of the space $\mathbb{K}^N$ such that $T(e, f, g) = 1$. A variational argument (cf. [4]) shows that

$$T(x, f, g) = T(e, y, g) = T(e, f, z) = 0 \quad \forall x \perp e, \ \ \forall y \perp g, \ \ \forall z \perp f.$$

Making suitable unitary transformations, we can arrange that e, f, g are the standard first basis vectors in our spaces, $e = (1, 0, \dots)$, etc. Thus, we have reduced ourselves to normalized forms.

2.2. The associated vector bilinear form

As in [2], we define a *vector bilinear form* $\tilde{T}$, which may be viewed as a mapping from $\mathbb{H}^N \times \mathbb{H}^N$ to the *left dual* of $\mathbb{H}^N$. It is given by the formula

$$\langle x, \tilde{T}(y, z) \rangle = T(x, y, z) \ \ \forall x \in \mathbb{H}^N, \ \ y \in \mathbb{H}^N, \ \ z \in \mathbb{H}^N.$$

Its components $\tilde{T}_j(y, z)$ are given by

$$\tilde{T}_j(y, z) = y^k a_{jk\ell} z^\ell \ \ \forall j = 0, 1, \dots N-1.$$

From this we obtain the following expression for the norm of T:

$$\|T\| = \|\tilde{T}\| = \sup_{y,z} \frac{(\sum_{j=0}^{N-1} |y^k a_{jk\ell} z^\ell|^2)^{1/2}}{\|y\| \|z\|}.$$

In this way we have eliminated one variable, x.

We observe now that if $\|T\| \le 1$, then we have the inequality

$$\sum_{j=0}^{N-1} |y^k a_{jk\ell} z^\ell|^2 \le \|y\|^2 \|z\|^2. \tag{5}$$

2.3. Inhomogeneous coordinates

First, let us rewrite (5) in a more explicit form:

$$\sum_{j=0}^{N-1} \left| \sum_{k,\ell=0}^{N-1} y^k a_{jk\ell} z^\ell \right|^2 \leq \left(\sum_{k=0}^{N-1} |y^k|^2 \right) \cdot \left(\sum_{\ell=0}^{N-1} |z^\ell|^2 \right). \tag{6}$$

Next, we define *inhomogeneous coordinates* by putting $y^k = y^0 \eta^k$, $z^\ell = \zeta^\ell z^0$. Note that, formally, $\eta^0 = \zeta^0 = 1$. Inserting this into (6), we obtain

$$\sum_{j=0}^{N-1} \left| \sum_{k,\ell=0}^{N-1} \eta^k a_{jk\ell} \zeta^\ell \right|^2 \leq \left(1 + \sum_{k=1}^{N-1} |\eta^k|^2 \right) \cdot \left(1 + \sum_{\ell=1}^{N-1} |\zeta^\ell|^2 \right). \tag{7}$$

Our concern will be thus to decide if, given the quanties $a_{jk\ell}$, the inequality (7) is true or not. Here we can assume that the form is normalized, so that (4) is fulfilled. However, we will treat this question only in the *binary case*.

3. THE BINARY CASE. LITERAL NOTATION

It will be convenient to replace the tensor components $a_{jk\ell}$ by the numbers $d, a_1, a_2, a_3, b_1, b_2, b_3, c$. This transition is given by the following table (cf. [5], p. 512):

a_{000}	d
a_{001}	a_3
a_{010}	a_2
a_{011}	b_1
a_{100}	a_1
a_{101}	b_2
a_{110}	b_3
a_{111}	c

For the normalized case, $d = 1$, $a_1 = a_2 = a_3 = 0$, we obtain from (2)

$$\begin{aligned} T &= x^0 y^0 z^0 + x^0 y^1 b_1 z^1 + x^1 y^0 b_2 z^1 + x^1 y^1 b_3 z^0 + x^1 y^1 c z^1 \\ &= x^0 (y^0 z^0 + y^1 b_1 z^1) + x^1 (y^0 b_2 z^1 + y^1 b_3 z^0 + y^1 c z^1). \end{aligned} \tag{8}$$

Now we can turn to inhomogeneous coordinates. However, we make first a further notational change: instead of the Greek letters η, ζ we use Roman y, z. Thus, we end up with the following inequality:

$$|1 + y b_1 z|^2 + |b_2 z + y(b_3 + cz)|^2 \leq (1 + |y|^2)(1 + |z|^2). \tag{9}$$

This is the inequality we are going to solve. Having eliminated x, we want to eliminate also y, z.

4. ELIMINATIONS

The eliminations can be carried out tracing the steps of [4] or, even better, of [6] (see also the web page http://www.maths.lth.se/matematiklu/personal/jaak/engJP.html). This leads to the following inequality:

$$\mathbf{M} = \mathbf{M}(z,t) =: At^2 - 2\mathrm{Re}B(z)t + C|z|^2 - 2\mathrm{Re}D(z) \geq 0, \tag{10}$$

where t is a real variable, z is a quaternion, with A, B, C, D given by

$$\begin{aligned} A &= (1 - |\,b_1|^2)(1 - |\,b_2|^2) - |c|^2, \\ B(z) &= b_1 z b_2 \bar{c} + cz\bar{b}_3, \\ C &= 1 - |\,b_1|^2 - |\,b_2|^2 - |\,b_3|^2, \\ D(z) &= b_1 z b_2 z \bar{b}_3\,. \end{aligned} \tag{11}$$

Here the function $\mathbf{M}$ is regarded as a *quadratic form* (homogeneous!) in the space $\mathbb{H} \times \mathbb{R}$, which we can identify with $\mathbb{R}^5$. Thus we have to decide when this form is positive semidefinite. This is in principle easy, by just applying the classical "Determinant Criterion" [7]. Recall that it gives as a necessary and sufficient condition for a real n-dimensional quadratic form $\mathbf{A}$ with matrix $(a_{ik})_{0\leq i,k\leq n-1}$ to be positive semidefinite that all its *principal minors* are non-negative. The latter are labelled by sequences of integers $0 \leq i_1 < i_2 < \ldots i_p \leq n-1$, $p = 1, \ldots, n$. The corresponding principal minor will be written $A(i_1 i_2 \ldots i_p)$. (Following our above convention, we count the indices i, k from 0 to $n-1$, not from 1 to n as Gantmacher [7] did.)

In the case of the form $\mathbf{A} = \mathbf{M}$ we have $n = 5$ and denote the matrix $(m_{ik})_{0\leq i,k\leq 4}$. The principal minors – 31 ($= 2^5 - 1$) in number – are written as

$$\begin{aligned} p = 1: &\quad M(0), M(1), M(2), M(3), M(4) = A; \\[1em] p = 2: &\quad M(01), M(02), M(03), M(04), M(12), \\ &\quad M(13), M(14), M(23), M(24), M(34); \\[1em] p = 3: &\quad M(012), M(013), M(014), M(023), M(024), \\ &\quad M(034), M(123), M(124), M(134), M(223); \\[1em] p = 4: &\quad M(0123), M(0124), M(0134), M(0234), M(1234); \\[1em] p = 5: &\quad M(01234). \end{aligned}$$

Here the last one, $M(01234)$, the *discriminant* of the matrix $\mathbf{M}$, in other words, the *determinant* $\Delta = \det(m_{ik})_{i,k=0}^4$, is the most interesting.

We can now state our result as follows.

Main Theorem 1. *Let T be a normalized binary quaternion trilinear form as in* (8). *Then its norm is less than or equal to unity,* $\|T\| \leq 1$, *if and only if the quadratic form* $\mathbf{M}$ *in* (10) *has all its principal minors M non-negative.*

Unfortunately, it is not easy to compute these principal minors in terms of the quaternions b_1, b_2, b_3, c. Instead we use their real components b_1^h, b_2^h, b_3^h, c^h ($h = 0, 1, 2, 3$). We have found by hand the matrix components m_{ik}. With the help of the program Mathematica it is easy (*sic*!) to compute all 31 determinants needed. Writing them in TEX and/or printing the result on paper is less simple and quite pointless.

There arises also the question of expressing the result in terms of the quaternions themselves, rather than their components. This is easy, in principle, as there are simple formulae for expressing the components of a quaternion in terms of the quaternion itself, generalizing the formulae $x = (z + \bar{z})/2; y = (z - \bar{z})/2$ for the real part of a complex number $z = x + iy$ in the case of $\mathbb{C}$. But only in principle, because the noncommutativity makes it impossible to use the expression Expand in Mathematica.

Instead we apply now a somewhat different approach, which actually works as well as it does in the commutative case.

5. A REAL VARIABLE APPROACH

Our main idea in this section is to keep b_1, b_2, b_3 fixed and let c vary.

5.1. The case $\mathbb{R}$

We recall ([4,8]) that in this case a normalized binary quaternion trilinear form has norm unity if and only if

$$b_1^2 + b_2^2 + b_3^2 + 2b_1b_2b_3 + c^2 \leq 1. \tag{12}$$

Thus the variable c ranges over a certain *segment* on the line $\mathbb{R}$, symmetric with respect to the origin, entirely determined by b_1, b_2, b_3. In addition, three auxiliary inequalities must be fulfilled:

$$|b_1|, |b_2|, |b_3| \leq 1. \tag{13}$$

5.2. The case $\mathbb{C}$

This time we have a condition quite analogous to (12) (see [4], and – in the language of vector bilinear forms – [2]). (Note that, curiously enough, there is no immediate analogue of (13).) We do not state this condition here, but present its (real) geometric interpretation [5]: c ranges over a certain "elliptic disc"[1] in the Gaussian plane $\mathbb{C}$ or $\mathbb{R}^2$. Its axes are entirely determined by the two quantities $S = |b_1|^2 + |b_2|^2 + |b_3|^2$ and $P = |b_1b_2b_3|$.

---

[1] The interior of an ellipse.

5.3. The case $\mathbb{H}$. Some conjectures

The above suggests that in the case of quaternions the point c must range over the interior of an *ellipsoid* in the Hamiltonian plane $\mathbb{H}$ or $\mathbb{R}^4$, as before symmetric with respect to the origin and entirely determined by b_1, b_2, b_3. This would be a form of the sought norm-one criterion, and constitutes our main object in this research.

An obvious guess is that it is of the form $\Delta \geq 0$, with the discriminant Δ as above (see Section 4). (In addition, we must have – corresponding to the conditions (13) in the real case Subsection 5.1 – auxiliary inequalities given by the remaining 30 principal minors.)

Experiments using Mathematica point to that, indeed, Δ is an (inhomogeneous) quadratic form in the components of c. When expanded in Mathematica, it gives some 200 000 terms.

However, it is not hard to give a formal proof. From the formulae for the matrix elements m_{ik} referred to at the end of Section 4 one sees that

Lemma 1. *The quantities* c^h $(h = 0, 1, 2, 3)$ *appear to the first degree in* m_{04}, m_{14}, m_{24}, m_{34} *and, by reflexion in the diagonal, also in* $m_{40}, m_{41}, m_{42}, m_{43}$. *Moreover, by* (10) *and* (11) $m_{44} = A$, *so there we have them to degree* 2. *All other elements* m_{ik} *contain only* b_1, b_2, b_3.

Lemma 2. *The discriminant* Δ *is an inhomogeneous quadratic form in* c.

Proof. We use the rules about expanding determinants taught in any decent course in linear algebra.

1. Indeed, let us take a look at the element m_{04}. The corresponding *comatrix* is 3×3 and its lowest row contains only matrix elements which are of the first degree in c. All the other elements contain only components of b_1, b_2, b_3. As a result we get quadratic expressions.
2. The remaing elements on the same column (with $k = 4$), except for the last one m_{44}, are treated in the same way.
3. On the other hand, $m_{44} = A$ (see (10) and (11)) and so is of the second degree. But the comatrix in question is now made up only by components of b_1, b_2, b_3. So, as a result we have again a quadratic expression. □

Remark 1. It follows from Lemma 2 that Δ can be written in the form

$$\Delta = g + \sum_{h=0,1,2,3} e_h c^h + \sum_{h,k=0,1,2,3} e_{hk} c^h c^k,$$

where g is the constant term, and e_{hk} is the coefficient of $c^h c^k$ in Δ with a similar expression for e_h.

Using Mathematica it was found that g consists of 10 682 terms. Incidentally, this shows that Δ is *not* symmetric, contradicting to what we said above.

We examined more closely the coefficients e_h and e_{hk}. Below the results are given in table form. The second column stands for the number of terms, the third for the number of factors in the factorization of the coefficient under view:

e_0	56 568	4
e_1	56 568	4
e_2	56 568	4
e_3	56 568	4

e_{00}	25 300	3
e_{11}	25 300	2
e_{22}	25 300	2
e_{33}	25 300	2
e_{01}	18 856	4
e_{02}	18 856	4
e_{03}	18 856	4
e_{12}	18 856	4
e_{13}	18 856	4
e_{23}	18 856	4

Here the "periodic" recurrence of some numbers is quite curious, and ought to be given a rational explanation.

We also verified that, for instance, e_{23} admits four factors; besides a trivial factor -4, also three more of length 18, 22, and 96. (The presence of -4 is an unwanted effect of Mathematica!)

5.4. Main result

We may summarize the result obtained in Lemmas 1 and 2 as follows.

Main Theorem 2. *Let T be a binary quaternion trilinear form as in* (8) *of norm unity. Let b_1, b_2, b_3 be fixed. Then c moves on one side of the quadric $\mathcal{Q}$ in $\mathbb{H}$ defined by the equation $\Delta = 0$.*

5.5. A remaining question

The above leads us to the following conjecture: the only thing missing is that $\mathcal{Q}$ is indeed the interior of an ellipsoid (possibly a degenerate one). For this we must consider the homogeneous part of Δ and prove that the matrix $(e_{hk})_{h,k=0,1,2,3}$ is positive semidefinite. To this end we must again apply the Determinant Criterion [[7]], now in the 4-dimensional case, a difficult problem which we have not yet attacked.

The full details of the present study will appear in a forthcoming paper based on [[3]], hopefully with the above conjecture also settled.

ACKNOWLEDGEMENTS

The authors would like to thank the referee for useful suggestions, which let us come somewhat closer to the final solution of our problem. F. Cobos, T. Kühn, and J. Peetre were supported in part by the Ministry of Education and Science of Spain (project MTM2004-01888).

REFERENCES

1. Gel'fand, I. M., Kapranov, M. M. and Zelevinsky, A. V. *Discriminants, Resultants, and Multilinear Determinants*. Birkhäuser–Boston, MA, 1994.
2. Arazy, J. and Friedman, Y. Contractive projections in C_1 and C_∞. *Mem. Amer. Math. Soc.*, 1978, **13**, No. 200.
3. Peetre, J. *Quaternary binary trilinear forms*. 2005 (http://www.maths.lth.se/matematiklu/personal/jaak/engJP.html).
4. Cobos, F., Kühn, T. and Peetre, J. Extreme points of the complex binary trilinear ball. *Studia Math.*, 2000, **138**, 81–92.
5. Bernhardsson, B. and Peetre, J. Singular values of trilinear forms. *Experiment. Math.*, 2001, **10**, 509–517.
6. Cobos, F., Kühn, T. and Peetre, J. Supplement to "Generalized non-commutative integration – trilinear forms and Jordan quintuples". Technical report, Lund, 1995.
7. Gantmacher, F. R. *Theory of Matrices*. 2nd edition, Nauka, Moscow, 1966 (in Russian). Second German translation: *Matrizentheorie*. Springer-Verlag, Berlin, 1986.
8. Grząślewicz, R. and John, K. Extreme elements of the unit ball of bilinear operators on ℓ_2^2. *Arch. Math. (Basel)*, 1988, **50**, 264–269.

Ühiknormiga kvaternaarsetest binaarsetest kolmlineaarsetest vormidest

Bo Bernhardsson, Fernando Cobos, Thomas Kühn, Daniel Mondoc ja Jaak Peetre

Siiani on enamik trilineaarsete vormidega töödest olnud suunatud madala dimensiooniga vormidele, eriti kahedimensionaalsele ehk binaarsele juhule. Skalaaride ring $\mathbb{K}$ on siin kas reaalsed numbrid $\mathbb{R}$ või kompleksnumbrid $\mathbb{C}$. Põhieesmärgiks on mõlemas situatsioonis otsustada, millal normaliseeritud vormi norm on ühik. On vaadeldud kvaternioonide juhust $\mathbb{K} = \mathbb{H}$. Artiklis esitatu on üsna ajutine ja osalt eksperimentaalne. Puudub täiuslik lõpptulemus, kuid ajutiseks tulemuseks on see, et vorm on ühiknormiga parajasti siis, kui teatud 5-dimensionaalse ruutvormi diskriminandi kõik peamiinorid osutuvad mittenegatiivseteks. On avastatud ootamatu seos mittekommutatiivse juhu $\mathbb{H}$ ja kommutatiivse juhu vahel. Viimasel juhul on skalaaride ring kas $\mathbb{R}$ või $\mathbb{C}$.