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Optimality of theoretical error estimates for spline collocation methods for linear weakly singular Volterra integro-differential equations

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Abstract. Two spline collocation methods for solving linear weakly singular Volterra integro-differential equations are considered. A result on the superconvergence at the collocation points is proved and optimality of several theoretical error estimates is demonstrated by extensive numerical experiments. Based on numerical results, a conjecture about the theoretical error estimates at the collocation points is stated for the cases not covered by known theorems.

Key words: weakly singular, Volterra, integro-differential equations, spline collocation, superconvergence.

1. INTRODUCTION

Consider the linear Volterra integro-differential equation (VIDE)

$$y'(t) = p(t)y(t) + q(t) + \int_0^t K(t,s)y(s)\mathrm{d}s, \quad t \in [0,T],\ T > 0, \tag{1}$$

with a given initial condition

$$y(0) = y_0,\ y_0 \in \mathbb{R} = (-\infty, \infty). \tag{2}$$

It will be assumed that

$$p, q \in C^{m,\nu}(0,T],\ K \in \mathcal{W}^{m,\nu}(\Delta_T),\ m \in \mathbb{N} = \{1,2,\ldots\},\ \nu \in \mathbb{R},\ \nu < 1. \tag{3}$$

Here $C^{m,\nu}(0,T]$, $m \in \mathbb{N}$, $\nu < 1$, is the set of all m times continuously differentiable functions $x: (0,T] \to \mathbb{R}$ such that the estimates

$$\left|x^{(j)}(t)\right| \le c \begin{cases} 1 & \text{if} \quad j < 1-\nu, \\ 1 + |\log t| & \text{if} \quad j = 1-\nu, \\ t^{1-\nu-j} & \text{if} \quad j > 1-\nu \end{cases}$$

hold with a constant $c = c(x)$ for all $t \in (0,T]$ and $j = 0, 1, \ldots, m$. The set $\mathcal{W}^{m,\nu}(\Delta_T)$, with $m \in \mathbb{N}$, $\nu < 1$, $\Delta_T = \{(t,s) \in \mathbb{R}^2 : 0 \le t \le T,\ 0 \le s < t\}$ consists of m times continuously differentiable functions $K : \Delta_T \to \mathbb{R}$ satisfying

$$\left| \left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j K(t,s) \right| \le c \begin{cases} 1 & \text{if} \quad \nu + i < 0, \\ 1 + |\log(t-s)| & \text{if} \quad \nu + i = 0, \\ (t-s)^{-\nu-i} & \text{if} \quad \nu + i > 0, \end{cases} \tag{4}$$

with a constant $c = c(K)$ for all $(t,s) \in \Delta_T$ and all integers $i, j \ge 0$ such that $i + j \le m$.

It is well known (see [1]) that under the assumptions (3) Eq. (1) has a unique solution $y \in C^{m+1,\nu-1}(0,T]$.

Spline collocation methods for weakly singular VIDEs have been examined by many authors (see, for example, [1–7]). There are two different approaches to solving weakly singular VIDEs with piecewise polynomial spline collocation methods:

1. Consider Eq. (1) as an integral equation with respect to y'. The spline collocation method is applied to Eq. (1) for finding an approximate solution for y' first. Then, the approximation for y can be constructed by integration.
2. The second reformulation is based on integrating both sides of Eq. (1) over $(0,t)$, which gives us a linear Volterra integral equation with respect to y. The spline collocation method is then applied to that equation for finding an approximate solution for y.

The most general theoretical results about the attainable orders of convergence of the proposed methods (Method 1 and Method 2) are proved in papers [1,3]. A result about superconvergence in the maximum norm for Method 1 is proved in [4].

The purpose of this paper is to perform numerical experiments in order to verify the optimality of available theoretical results, to state a conjecture for the cases not covered by existing theorems, and to provide data for comparison with other methods.

For Method 2, we prove the superconvergence at the collocation points for sufficiently large values of r, which improves an analogous theorem of [1]. For other values of r, we present a conjecture about the theoretical error estimate, based on our numerical results.

2. TEST PROBLEMS

For numerical verification of theoretical results we consider the following integro-differential equation

$$y'(t) = -y(t) + q_\nu(t) + \int_0^t K_\nu(t,s)y(s)\mathrm{d}s, \quad 0 \le t \le 1, \tag{5}$$

where

$$K_\nu(t,s) = \begin{cases} -(t-s)^{-\nu} & \text{if} \quad \nu \ne 0 \\ -\log(t-s) & \text{if} \quad \nu = 0 \end{cases}$$

and

$$q_\nu(t) = \begin{cases} (2-\nu)t^{1-\nu} + t^{2-\nu} + t^{3-2\nu}\gamma_1 & \text{if} \quad \nu \ne 0, \\ \frac{1}{3}t^3(\log t)^2 - (\frac{13}{18}t^3 - t^2 - 2t)\log t + t + t^3\gamma_2 & \text{if} \quad \nu = 0, \end{cases}$$

$$\gamma_1 = \int_0^1 (1-x)^{-\nu}x^{2-\nu}\mathrm{d}x, \ \gamma_2 = \int_0^1 x^2 \log x \log(1-x)\mathrm{d}x,$$

with the initial condition $y(0) = 0$. The choice of the function q_ν corresponds to the exact solution $y(t) = t^{2-\nu}$ in the case $\nu \ne 0$ and $y(t) = t^2 \log t$ in the case $\nu = 0$. Equation (5) is an equation of type (1) with $p(t) = -1$, and for any $\nu \in (-\infty, 1)$ the assumptions (3) hold with arbitrary m.

3. GRID, SPLINE SPACE

For solving problem $\{(1),(2)\}$ we use piecewise polynomial collocation methods on graded grids. Fix $r \in \mathbb{R}$, $r \ge 1$. For $N \in \mathbb{N}$ define a graded grid Π_N^r on the interval $[0,T]$ by

$$\Pi_N^r = \left\{ t_0, t_1, \ldots, t_N : t_j = T\left(\frac{j}{N}\right)^r, \ j = 0, \ldots, N \right\}.$$

Here r is a parameter describing the nonuniformity of the grid Π_N^r. If $r = 1$, we get the uniform grid and if r increases, the density of the gridpoints (near 0) also increases.

For given integers $m \ge 0$ and $-1 \le d \le m-1$, let $S_m^{(d)}(\Pi_N^r)$ be the spline space of piecewise polynomial functions on the grid Π_N^r:

$$S_m^{(d)}(\Pi_N^r) = \big\{ u : u\big|_{[t_{j-1},t_j]} =: u_j \in \pi_m, \quad j = 1, \ldots, N;$$
$$u_j^{(k)}(t_j) = u_{j+1}^{(k)}(t_j), \qquad 0 \le k \le d, \ j = 1, \ldots, N-1 \big\}.$$

Here π_m denotes the set of polynomials of degree not exceeding m and $u\big|_{[t_{j-1},t_j]}$ is the restriction of u to the subinterval $[t_{j-1}, t_j]$. Note that the elements of $S_m^{(-1)}(\Pi_N^r) = \{u : u\big|_{[t_{j-1},t_j]} \in \pi_m, j = 1, \ldots, N\}$ may have jump discontinuities at the interior grid points $t_1, \ldots, t_{N-1}$.

In the following we consider spline collocation methods for two equivalent reformulations of problem {(1),(2)}. In both cases the collocation points

$$t_{jk} = t_{j-1} + \eta_k(t_j - t_{j-1}), \quad k = 1, \ldots, m,\ j = 1, \ldots, N\,, \tag{6}$$

where $\eta_1, \ldots, \eta_m$ do not depend on j and N and satisfy $0 \le \eta_1 < \ldots < \eta_m \le 1$, are used.

4. METHOD 1

The first reformulation is based on introducing a new unknown function $z = y'$. Using $y' = z$ and (2), Eq. (1) may be rewritten as a linear Volterra integral equation of the second kind with respect to z:

$$z(t) = f_1(t) + p(t)\int_0^t z(s)\mathrm{d}s + \int_0^t K(t,s)\left(\int_0^s z(\tau)\mathrm{d}\tau\right)\mathrm{d}s\,, \tag{7}$$

where

$$f_1(t) = q(t) + y_0 p(t) + y_0\int_0^t K(t,s)\mathrm{d}s, \quad t \in [0,T]\,. \tag{8}$$

We look for an approximation v to the solution z of Eq. (7) in $S_{m-1}^{(-1)}(\Pi_N^r)$, $m, N \in \mathbb{N}$. We determine $v = v^{(N)} \in S_{m-1}^{(-1)}(\Pi_N^r)$ $(m \ge 1)$ by the collocation method from the following conditions:

$$v_j(t_{jk}) = f_1(t_{jk}) + p(t_{jk})\int_0^{t_{jk}} v(s)\mathrm{d}s + \int_0^{t_{jk}} K(t_{jk},s)\left(\int_0^s v(\tau)\mathrm{d}\tau\right)\mathrm{d}s, \tag{9}$$
$$k = 1, \ldots, m;\ j = 1, \ldots, N\,.$$

Here $v_j = v\big|_{[t_{j-1}-t_j]}$ is the restriction of v to $[t_{j-1}, t_j]$, $j = 1, \ldots, N$, and the function f_1 and the set of collocation points $\{t_{jk}\}$ are given by (8) and (6), respectively. Having determined the approximation v for $z = y'$, we can determine also the approximation u for y, the solution of the Cauchy problem {(1),(2)}, setting

$$u(t) = y_0 + \int_0^t v(s)\mathrm{d}s, \qquad t \in [0,T]\,. \tag{10}$$

Brunner et al. [3] have proved the following convergence result for method {(9),(10)}.

Theorem 1. *Assume* (3) *and let the collocation points* (6) *be used. Then for all sufficiently large* $N \in \mathbb{N}$ *and for every choice of parameters* $0 \leq \eta_1 < \ldots < \eta_m \leq 1$ *with* $\eta_1 > 0$ *or* $\eta_m < 1$*, Eqs.* (9) *and* (10) *determine unique approximations* $u \in S_m^{(0)}(\Pi_N^r)$ *and* $v \in S_{m-1}^{(-1)}(\Pi_N^r)$ *to the solution* y *of the Cauchy problem* $\{(1),(2)\}$ *and its derivative* y'*, respectively.*

The following error estimates hold for $k = 0$ *and* $k = 1$:

1) *if* $m < 2 - \nu - k$*, then*

$$||u^{(k)} - y^{(k)}||_\infty \leq cN^{-m} \qquad for \quad r \geq 1;$$

2) *if* $m = 2 - \nu - k$*, then*

$$||u^{(k)} - y^{(k)}||_\infty \leq c \begin{cases} N^{-m}(1 + |\log N|) & for \quad r = 1, \\ N^{-m} & for \quad r > 1; \end{cases}$$

3) *if* $m > 2 - \nu - k$*, then*

$$||u^{(k)} - y^{(k)}||_\infty \leq c \begin{cases} N^{-r(2-\nu-k)} & for \quad 1 \leq r < \frac{m}{2-\nu-k}, \\ N^{-m}(1 + |\log N|)^{1-k} & for \quad r = \frac{m}{2-\nu-k}, \\ N^{-m} & for \quad r > \frac{m}{2-\nu-k}. \end{cases} \tag{11}$$

Here c *is a positive constant which is independent of* N*.*

To illustrate the theoretical results, tables with numerical experiments in the case $m = 2$ for $\nu = -\frac{1}{4},\ 0,\ \frac{1}{2},\ \frac{9}{10}$ are presented. In order to estimate the errors $\|u - y\|_\infty$ and $\|u' - y'\|_\infty$, the points

$$\tau_{jk} = t_{j-1} + k\frac{t_j - t_{j-1}}{10},\ k = 1, \ldots, 9,\ j = 1, \ldots, N$$

are used. The corresponding error estimates are denoted by

$$\varepsilon_N = \{\max |u(\tau_{jk}) - y(\tau_{jk})| : k = 1, \ldots, 9;\ j = 1, \ldots, N\}$$

and

$$\varepsilon'_N = \{\max |u'(\tau_{jk}) - y'(\tau_{jk})| : k = 1, \ldots, 9;\ j = 1, \ldots, N\}.$$

The ratios of the actual errors $\varrho_N = \frac{\varepsilon_{N/2}}{\varepsilon_N}$ and $\varrho'_N = \frac{\varepsilon'_{N/2}}{\varepsilon'_N}$ are presented in the columns with headings in the form $\varrho(x)$ and $\varrho'(x)$, where x is a real number corresponding to the ratios of the error estimates. In order to save space, we have presented numerical results only for $N = 4, 32, 256, 1024$ although the computations were performed for all values $N = 2^j,\ j = 1, \ldots, 10$.

As we can see in Table 1, the observed errors of u' behave exactly according to the right-hand side of the estimate (11) starting from $N = 32$. We may conclude that in the case of the test equations, the estimate (11) corresponds to the leading term of the error $\|u' - y'\|_\infty$, which is dominating even for small values of N.

Table 1. Method 1; $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{3}{4}$

$\nu = -\frac{1}{4}$	$r = 1$		$r = 1.2$		$r = 1.4$		$r = 1.6$	
N	ε_N	ϱ(4.0)	ε_N	ϱ(4.0)	ε_N	ϱ(4.0)	ε_N	ϱ(4.0)
4	7.9E-4	4.1	6.8E-4	4.6	7.5E-4	4.4	8.6E-4	4.3
32	1.3E-5	3.9	8.9E-6	4.1	9.4E-6	4.2	1.1E-5	4.2
256	2.2E-7	3.9	1.4E-7	4.0	1.4E-7	4.0	1.6E-7	4.0
1024	1.4E-8	3.9	8.5E-9	4.0	8.8E-9	4.0	1.0E-8	4.0
N	ε'_N	ϱ'(2.4)	ε'_N	ϱ'(2.8)	ε'_N	ϱ'(3.4)	ε'_N	ϱ'(4.0)
4	1.3E-2	2.2	9.6E-3	2.6	6.9E-3	3.2	5.5E-3	3.5
32	1.0E-3	2.4	4.4E-4	2.8	1.9E-4	3.4	9.2E-5	4.0
256	7.9E-5	2.4	2.0E-5	2.8	4.9E-6	3.4	1.4E-6	4.0
1024	1.4E-5	2.4	2.5E-6	2.8	4.3E-7	3.4	9.0E-8	4.0
$\nu = 0$	$r = 1$		$r = 1.333$		$r = 1.667$		$r = 2.0$	
N	ε_N	ϱ(4.0)	ε_N	ϱ(4.0)	ε_N	ϱ(4.0)	ε_N	ϱ(4.0)
4	4.7E-3	3.5	3.7E-3	4.0	3.9E-3	4.1	4.6E-3	4.0
32	1.1E-4	3.6	5.9E-5	4.0	5.9E-5	4.0	6.9E-5	4.0
256	2.2E-6	3.7	9.2E-7	4.0	9.2E-7	4.0	1.1E-6	4.0
1024	1.6E-7	3.7	5.8E-8	4.0	5.7E-8	4.0	6.6E-8	4.0
N	ε'_N	ϱ'(2.0)	ε'_N	ϱ'(2.5)	ε'_N	ϱ'(3.2)	ε'_N	ϱ'(4.0)
4	7.3E-2	1.9	4.7E-2	2.4	3.0E-2	3.0	2.6E-2	3.6
32	9.6E-3	2.0	3.1E-3	2.5	9.6E-4	3.2	4.2E-4	4.0
256	1.2E-3	2.0	1.9E-4	2.5	3.0E-5	3.2	6.6E-6	4.0
1024	3.0E-4	2.0	3.0E-5	2.5	3.0E-6	3.2	4.1E-7	4.0
$\nu = \frac{1}{2}$	$r = 1$		$r = 1.2$		$r = 1.4$		$r = 4$	
N	ε_N	ϱ (2.8)	ε_N	ϱ(3.5)	ε_N	ϱ(4.0)	ε_N	ϱ(4.0)
4	3.2E-3	2.6	2.2E-3	3.2	1.4E-3	3.9	3.0E-3	3.8
32	1.5E-4	2.8	6.0E-5	3.2	3.0E-5	3.6	2.9E-5	4.6
256	7.0E-6	2.7	1.7E-6	3.3	5.9E-7	3.7	3.8E-7	4.1
1024	9.1E-7	2.8	1.6E-7	3.3	4.2E-8	3.8	2.3E-8	4.0
N	ε'_N	ϱ'(1.4)	ε'_N	ϱ'(1.5)	ε'_N	ϱ'(1.6)	ε'_N	ϱ'(4.0)
4	5.0E-2	1.3	4.5E-2	1.4	4.0E-2	1.5	1.9E-2	2.6
32	1.9E-2	1.4	1.4E-2	1.5	9.8E-3	1.6	3.3E-4	4.0
256	6.9E-3	1.4	4.0E-3	1.5	2.3E-3	1.6	5.2E-6	4.0
1024	3.5E-3	1.4	1.7E-3	1.5	8.7E-4	1.6	3.2E-7	4.0
$\nu = \frac{9}{10}$	$r = 1$		$r = 1.455$		$r = 1.909$		$r = 20$	
N	ε_N	ϱ(2.1)	ε_N	ϱ(3.0)	ε_N	ϱ(4.0)	ε_N	ϱ(4.0)
4	2.1E-3	1.7	1.3E-3	2.2	7.1E-4	3.1	5.7E-3	1.1
32	3.1E-4	2.0	5.7E-5	2.9	1.6E-5	3.6	1.4E-4	4.7
256	3.3E-5	2.1	2.4E-6	2.9	3.4E-7	3.6	9.3E-7	5.3
1024	7.1E-6	2.1	2.8E-7	2.9	2.5E-8	3.7	4.3E-8	4.4
N	ε'_N	ϱ'(1.1)	ε'_N	ϱ'(1.1)	ε'_N	ϱ'(1.1)	ε'_N	ϱ'(4.0)
4	2.3E-2	1.5	2.9E-2	0.9	3.3E-2	0.7	5.7E-2	1.0
32	3.3E-2	1.0	3.0E-2	1.1	2.6E-2	1.1	3.0E-3	4.1
256	2.9E-2	1.1	2.3E-2	1.1	1.8E-2	1.1	3.5E-5	4.0
1024	2.5E-2	1.1	1.8E-2	1.1	1.3E-2	1.1	2.2E-6	4.0

The observed errors of u are also in good agreement with the theoretical estimates of Theorem 1 except in the cases where r is close to the value $\frac{m}{2-\nu}$, after which the maximal theoretical convergence rate is achieved. If r is close to the critical value, then the observed convergence rate is smaller than the one predicted by the error estimate (11) but converges slowly to the theoretical value. To get a better picture of what happens near this value of r $(r = 1.333)$ in the case $\nu = \frac{1}{2}$, Table 2 is presented. This table shows the dependence of the convergence rate on the nonuniformity parameter r, when r is increasing by steps of 0.1.

In the proof of Theorem 1 (see [3]) it is actually shown that

$$\|u - y\|_\infty \le c'' N^{-r(2-\nu)} \sum_{l=1}^{N} l^{r(2-\nu)-m-1}, \tag{12}$$

which is asymptotically equivalent to (11). In Table 3 we see that the ratios of the right-hand side of (12) behave similarly to the observed convergence rate, which explains the slow convergence of the observed rate to the theoretical one.

By a careful choice of the collocation parameters η_j it is possible (assuming a little more regularity of functions p, q, and K) to improve the convergence rate.

Table 2. Method 1; $\eta_1 = \frac{1}{4}, \eta_2 = \frac{3}{4}$, and $\nu = \frac{1}{2}$

	$r = 1$		$r = 1.1$		$r = 1.2$		$r = 1.3$	
N	ε_N	ϱ(2.8)	ε_N	ϱ(3.1)	ε_N	ϱ(3.5)	ε_N	ϱ(3.9)
4	3.2E-3	2.6	2.6E-3	2.9	2.2E-3	3.2	1.8E-3	3.5
32	1.5E-4	2.8	9.1E-5	3.1	6.0E-5	3.2	4.1E-5	3.4
256	7.0E-6	2.7	3.4E-6	3.0	1.7E-6	3.3	9.7E-7	3.5
1024	9.1E-7	2.8	3.7E-7	3.0	1.6E-7	3.3	7.7E-8	3.6
	$r = 1.4$		$r = 1.5$		$r = 1.6$		$r = 1.7$	
N	ε_N	ϱ(4.0)	ε_N	ϱ(4.0)	ε_N	ϱ(4.0)	ε_N	ϱ(4.0)
4	1.4E-3	3.9	1.3E-3	3.9	1.2E-3	3.9	1.1E-3	3.9
32	3.0E-5	3.6	2.3E-5	3.8	1.8E-5	3.9	1.6E-5	4.0
256	5.9E-7	3.7	4.0E-7	3.9	3.0E-7	3.9	2.5E-7	4.0
1024	4.2E-8	3.8	2.7E-8	3.9	1.9E-8	4.0	1.6E-8	4.0

Table 3. The ratios of the right-hand side of (12) for $\nu = \frac{1}{2}$

N	$r = 1.1$	$r = 1.2$	$r = 1.3$	$r = 1.4$	$r = 1.5$	$r = 1.6$
4	2.5	2.6	2.8	3.0	3.2	3.4
8	2.6	2.8	3.0	3.2	3.3	3.5
16	2.8	3.0	3.2	3.3	3.5	3.6
32	2.9	3.1	3.3	3.4	3.6	3.7
64	2.9	3.2	3.4	3.5	3.7	3.8
128	3.0	3.2	3.4	3.6	3.7	3.8
256	3.0	3.3	3.5	3.7	3.8	3.9
512	3.1	3.3	3.5	3.7	3.8	3.9
1024	3.1	3.3	3.6	3.7	3.9	3.9

Table 4. Method 1; $\eta_1 = \frac{1}{4}, \eta_2 = \frac{5}{6}$

$\nu = -\frac{1}{4}$	$r = 1$		$r = 1.189$		$r = 1.378$		$r = 1.6$	
N	ε_N	ϱ(4.8)	ε_N	ϱ(6.4)	ε_N	ϱ(8.0)	ε_N	ϱ(8.0)
4	7.8E-4	4.5	4.4E-4	6.0	3.4E-4	7.1	4.2E-4	7.5
32	7.7E-6	4.7	1.8E-6	6.3	7.3E-7	7.8	8.4E-7	8.0
256	7.2E-8	4.8	6.8E-9	6.4	1.5E-9	7.8	1.6E-9	8.0
1024	3.2E-9	4.8	1.7E-10	6.4	2.5E-11	7.9	2.6E-11	8.0
N	ε'_N	ϱ'(2.4)	ε'_N	ϱ'(2.8)	ε'_N	ϱ'(3.3)	ε'_N	ϱ'(4.0)
4	1.4E-2	2.2	1.1E-2	2.6	7.7E-3	3.1	6.1E-3	3.5
32	1.1E-3	2.4	5.0E-4	2.8	2.2E-4	3.3	1.0E-4	4.0
256	8.5E-5	2.4	2.3E-5	2.8	6.2E-6	3.3	1.6E-6	4.0
1024	1.5E-5	2.4	2.9E-6	2.8	5.7E-7	3.3	1.0E-7	4.0
$\nu = 0$	$r = 1$		$r = 1.275$		$r = 1.550$		$r = 2$	
N	ε_N	ϱ(4.0)	ε_N	ϱ(5.9)	ε_N	ϱ(8.0)	ε_N	ϱ(8.0)
4	4.4E-3	3.8	2.1E-3	5.6	1.7E-3	7.3	2.7E-3	6.6
32	7.2E-5	4.0	1.1E-5	5.8	3.7E-6	7.7	5.8E-6	7.9
256	1.1E-6	4.0	5.4E-8	5.9	8.0E-9	7.8	1.1E-8	8.0
1024	7.1E-8	4.0	1.6E-9	5.9	1.3E-10	7.8	1.8E-10	8.0
N	ε'_N	ϱ'(2.0)	ε'_N	ϱ'(2.4)	ε'_N	ϱ'(2.9)	ε'_N	ϱ'(4.0)
4	7.8E-2	1.9	5.4E-2	2.3	3.8E-2	2.8	2.8E-2	3.7
32	1.0E-2	2.0	4.0E-3	2.4	1.5E-3	2.9	4.6E-4	4.0
256	1.3E-3	2.0	2.8E-4	2.4	6.2E-5	2.9	7.2E-6	4.0
1024	3.2E-4	2.0	4.8E-5	2.4	7.2E-6	2.9	4.5E-7	4.0
$\nu = \frac{1}{2}$	$r = 1$		$r = 1.533$		$r = 2.067$		$r = 4$	
N	ε_N	ϱ(2.8)	ε_N	ϱ(4.9)	ε_N	ϱ(8.0)	ε_N	ϱ(8.0)
4	3.3E-3	2.6	1.1E-3	4.5	7.7E-4	6.0	2.6E-3	4.5
32	1.6E-4	2.8	9.9E-6	4.9	2.0E-6	7.5	6.7E-6	7.8
256	7.0E-6	2.8	8.4E-8	4.9	4.6E-9	7.6	1.3E-8	8.0
1024	8.8E-7	2.8	3.4E-9	4.9	7.9E-11	7.7	2.1E-10	8.1
N	ε'_N	ϱ'(1.4)	ε'_N	ϱ'(1.7)	ε'_N	ϱ'(2.0)	ε'_N	ϱ'(4.0)
4	5.3E-2	1.3	3.9E-2	1.5	2.7E-2	1.9	2.0E-2	2.5
32	2.0E-2	1.4	8.2E-3	1.7	3.2E-3	2.0	3.6E-4	4.0
256	7.3E-3	1.4	1.7E-3	1.7	3.8E-4	2.0	5.6E-6	4.0
1024	3.6E-3	1.4	5.7E-4	1.7	9.0E-5	2.0	3.5E-7	4.0
$\nu = \frac{9}{10}$	$r = 1$		$r = 1.909$		$r = 2.818$		$r = 20$	
N	ε_N	ϱ(2.1)	ε_N	ϱ(4.3)	ε_N	ϱ(8.0)	ε_N	ϱ(8.0)
4	2.2E-3	1.7	7.3E-4	3.1	4.5E-4	4.7	5.9E-3	1.1
32	3.2E-4	2.0	1.2E-5	4.1	1.7E-6	7.1	1.6E-4	5.7
256	3.3E-5	2.1	1.5E-7	4.3	4.1E-9	7.5	3.9E-7	7.9
1024	7.3E-6	2.1	8.1E-9	4.3	7.2E-11	7.6	6.0E-9	8.0
N	ε'_N	ϱ'(1.1)	ε'_N	ϱ'(1.1)	ε'_N	ϱ'(1.2)	ε'_N	ϱ'(4.0)
4	2.5E-2	1.2	3.4E-2	0.7	3.4E-2	0.9	4.6E-2	1.0
32	3.5E-2	1.0	2.7E-2	1.1	2.0E-2	1.2	2.4E-3	3.9
256	3.0E-2	1.1	1.8E-2	1.1	1.1E-2	1.2	3.7E-5	4.0
1024	2.6E-2	1.1	1.4E-2	1.1	7.4E-3	1.2	2.3E-6	4.0

Table 5. Method 1; $\eta_1 = \frac{3-\sqrt{3}}{6}, \eta_2 = \frac{3+\sqrt{3}}{6}$

$\nu = -\frac{1}{4}$	$r = 1$		$r = 1.189$		$r = 1.378$		$r = 1.6$	
N	ε_N	ϱ(4.8)	ε_N	ϱ(6.4)	ε_N	ϱ(8.0)	ε_N	ϱ(8.0)
4	5.9E-4	4.5	3.3E-4	6.0	2.4E-4	7.5	3.4E-4	7.1
32	5.8E-6	4.7	1.3E-6	6.4	4.7E-7	8.0	7.0E-7	7.9
256	5.5E-8	4.8	5.2E-9	6.4	9.1E-10	8.0	1.4E-9	8.0
1024	2.4E-9	4.8	1.3E-10	6.4	1.4E-11	8.0	2.1E-11	8.0
N	ε'_N	ϱ'(2.4)	ε'_N	ϱ'(2.8)	ε'_N	ϱ'(3.3)	ε'_N	ϱ'(4.0)
4	1.1E-2	2.2	7.8E-3	2.6	5.7E-3	3.1	4.7E-3	3.9
32	8.4E-4	2.4	3.7E-4	2.8	1.6E-4	3.3	7.3E-5	4.0
256	6.3E-5	2.4	1.7E-5	2.8	4.6E-6	3.3	1.1E-6	4.0
1024	1.1E-5	2.4	2.2E-6	2.8	4.2E-7	3.3	7.2E-8	4.0
$\nu = 0$	$r = 1$		$r = 1.275$		$r = 1.550$		$r = 2$	
N	ε_N	ϱ(4.0)	ε_N	ϱ(5.9)	ε_N	ϱ(8.0)	ε_N	ϱ(8.0)
4	3.4E-3	3.8	1.6E-3	5.6	1.1E-3	7.7	1.9E-3	6.9
32	5.6E-5	4.0	8.4E-6	5.8	2.2E-6	8.0	3.9E-6	8.0
256	8.8E-7	4.0	4.2E-8	5.9	4.2E-9	8.1	7.6E-9	8.0
1024	5.5E-8	4.0	1.2E-9	5.9	6.4E-11	8.1	1.2E-10	8.0
N	ε'_N	ϱ'(2.0)	ε'_N	ϱ'(2.4)	ε'_N	ϱ'(2.9)	ε'_N	ϱ'(4.0)
4	5.9E-2	1.9	4.1E-2	2.3	2.8E-2	2.8	2.0E-2	3.7
32	7.8E-3	2.0	3.0E-3	2.4	1.2E-3	2.9	3.4E-4	4.0
256	9.8E-4	2.0	2.1E-4	2.4	4.6E-5	2.9	5.2E-6	4.0
1024	2.4E-4	2.0	3.6E-5	2.4	5.4E-6	2.9	3.3E-7	4.0
$\nu = \frac{1}{2}$	$r = 1$		$r = 1.533$		$r = 2.067$		$r = 4$	
N	ε_N	ϱ(2.8)	ε_N	ϱ(4.9)	ε_N	ϱ(8.0)	ε_N	ϱ(8.0)
4	2.7E-3	2.6	9.4E-4	4.6	5.7E-4	6.2	1.9E-3	4.8
32	1.3E-4	2.8	8.2E-6	4.9	1.2E-6	8.1	5.6E-6	7.7
256	5.8E-6	2.8	6.8E-8	4.9	2.1E-9	8.2	1.1E-8	8.0
1024	7.2E-7	2.8	2.8E-9	4.9	3.1E-11	8.2	1.8E-10	8.0
N	ε'_N	ϱ'(1.4)	ε'_N	ϱ'(1.7)	ε'_N	ϱ'(2.0)	ε'_N	ϱ'(4.0)
4	4.1E-2	1.2	3.0E-2	1.5	2.1E-2	1.9	1.5E-2	3.0
32	1.6E-2	1.4	6.4E-3	1.7	2.5E-3	2.0	2.6E-4	4.0
256	5.7E-3	1.4	1.3E-3	1.7	3.0E-4	2.0	4.1E-6	4.0
1024	2.9E-3	1.4	4.5E-4	1.7	7.1E-5	2.0	2.6E-7	4.0
$\nu = \frac{9}{10}$	$r = 1$		$r = 1.909$		$r = 2.818$		$r = 20$	
N	ε_N	ϱ(2.1)	ε_N	ϱ(4.3)	ε_N	ϱ(8.0)	ε_N	ϱ(8.0)
4	1.9E-3	1.8	6.3E-4	3.2	3.6E-4	6.0	6.2E-3	1.1
32	2.7E-4	2.0	8.9E-6	4.3	9.8E-7	7.8	1.6E-4	5.6
256	2.9E-5	2.1	1.1E-7	4.3	1.8E-9	8.2	3.7E-7	7.7
1024	6.3E-6	2.1	6.2E-9	4.3	2.6E-11	8.3	5.9E-9	7.9
N	ε'_N	ϱ'(1.1)	ε'_N	ϱ'(1.1)	ε'_N	ϱ'(1.2)	ε'_N	ϱ'(4.0)
4	2.5E-2	1.4	2.7E-2	0.9	2.7E-2	0.9	5.9E-2	1.0
32	2.8E-2	1.0	2.2E-2	1.1	1.6E-2	1.2	2.8E-3	4.2
256	2.4E-2	1.1	1.5E-2	1.1	8.9E-3	1.2	2.8E-5	4.3
1024	2.1E-2	1.1	1.1E-2	1.1	6.0E-3	1.2	1.7E-6	4.0

Theorem 2. [4] *Assume that* $p, q \in C^{m+1,\nu}(0,T]$, $K \in \mathcal{W}^{m+1,\nu}(\Delta_T)$, $m \in \mathbb{N} = \{1, 2, \ldots\}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}$, $\nu < 1$ *and that the parameters* η_j *are chosen so that the interpolatory quadrature approximation* $\int_0^1 \varphi(s)\mathrm{d}s \approx \sum_{j=1}^m A_j \varphi(\eta_j)$, *with*

appropriate weights $\{A_j\}$, is exact for all polynomials of degree m. Then there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ the error estimate

$$||u-y||_\infty \leq c \begin{cases} N^{-r(2-\nu)} & \textit{for} \quad 1 \leq r < \frac{m+1}{2-\nu}, \\ N^{-r(2-\nu)}(1+\log N) & \textit{for} \quad r = \frac{m+1}{2-\nu}, \\ N^{-m-1} & \textit{for} \quad r > \frac{m+1}{2-\nu} \end{cases}$$

holds, where c is a positive constant which is independent of N.

Numerical experiments in case $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{5}{6}$, when the corresponding quadrature formula is exact for all polynomials up to order 2, are presented in Table 4. Numerical results show that the convergence rate is much better in this case (compared with Table 1) and agrees well with the estimate of Theorem 2.

Remark 1. If we use Gaussian parameters $\left(\eta_1 = \frac{3-\sqrt{3}}{6},\ \eta_2 = \frac{3+\sqrt{3}}{6}\right)$ that are exact for polynomials of order $2m-1=3$, we do not get any further improvement in the convergence rate, although the actual errors are slightly smaller due to a smaller error coefficient. The corresponding numerical experiments are presented in Table 5.

5. METHOD 2

The second reformulation of problem {(1),(2)} is based on the integration of both sides of (1) over $(0,t)$. Using this and (2), Eq. (1) may be rewritten as a linear Volterra integral equation with respect to y:

$$y(t) = f_2(t) + \int_0^t K_2(t,s)y(s)\mathrm{d}s, \qquad t \in [0,T], \tag{13}$$

where

$$f_2(t) = y_0 + \int_0^t q(s)\mathrm{d}s, \quad K_2(t,s) = p(s) + \int_s^t K(\tau,s)\mathrm{d}\tau\,. \tag{14}$$

We look for an approximate solution u of Eq. (13) in $S_m^{(-1)}(\Pi_N^r)$, $m,N \in \mathbb{N}$: this approximation $u = u^{(N)} \in S_m^{-1}(\Pi_N^r)$ will be determined by the collocation method from the following conditions:

$$u_j(t_{jk}) = f_2(t_{jk}) + \int_0^{t_{jk}} K_2(t_{jk},s)u(s)\mathrm{d}s, \quad k=1,\ldots,m+1;\ j=1,\ldots,N. \tag{15}$$

Here, f_2 and K_2 are defined in (14), $u_j = u\big|_{[t_{j-1},t_j]}$ ($j = 1, \ldots, N$) is the restriction of $u \in S_m^{(-1)}(\Pi_N^r)$ to $[t_{j-1}, t_j]$, and the collocation points $\{t_{jk}\}$ are given by $t_{jk} = t_{j-1} + \eta_k(t_j - t_{j-1})$, where $\{t_j\}$ are the nodes of Π_N^r and $0 \leq \eta_1 < \ldots < \eta_{m+1} \leq 1$ is a fixed system of parameters which does not depend on j and N.

Theorem 3. [1] *Assume* (3) *and that the collocation points* (6), *where* $k = 1, \ldots, m+1$, *are used. Then for all sufficiently large* $N \in \mathbb{N}$ *and for every choice of parameters* $\{\eta_j\}$ $(0 \leq \eta_1 < \ldots < \eta_{m+1} \leq 1)$ *the collocation conditions* (15) *define a unique approximation* $u \in S_m^{(-1)}(\Pi_N^r)$ *to* y, *the solution of the Cauchy problem* $\{(1), (2)\}$. *Then the following error estimates hold*:
1) *if* $m < 1 - \nu$, *then*

$$\begin{aligned} ||u - y||_\infty &\leq c\, N^{-m-1} \quad && for \quad r \geq 1, \\ ||u' - y'||_\infty &\leq c\, N^{-m} \quad && for \quad r = 1, \\ ||u' - y'||_{\varepsilon,\infty} &\leq c_\varepsilon N^{-m} \quad && for \quad r > 1; \end{aligned}$$

2) *if* $m = 1 - \nu$, *then*

$$\begin{aligned} ||u - y||_\infty &\leq c \begin{cases} N^{-m-1}(1 + |\log N|) & for \quad r = 1, \\ N^{-m-1} & for \quad r > 1; \end{cases} \\ ||u' - y'||_\infty &\leq c\, N^{-m}(1 + |\log N|) \qquad for \quad r = 1, \\ ||u' - y'||_{\varepsilon,\infty} &\leq c_\varepsilon N^{-m} \qquad for \quad r > 1; \end{aligned}$$

3) *if* $m > 1 - \nu$, *then*

$$\begin{aligned} ||u - y||_\infty \quad &\leq c \begin{cases} N^{-r(2-\nu)} & for \quad 1 \leq r \leq \frac{m+1}{2-\nu}, \\ N^{-m-1} & for \quad r > \frac{m+1}{2-\nu}; \end{cases} \\ ||u' - y'||_\infty \quad &\leq c\, N^{-(1-\nu)} \qquad for \quad r = 1, \\ ||u' - y'||_{\varepsilon,\infty} \quad &\leq c_\varepsilon \begin{cases} N^{-r(1-\nu)} & for \quad 1 \leq r \leq \frac{m}{1-\nu}, \\ N^{-m} & for \quad r > \frac{m}{1-\nu}. \end{cases} \end{aligned}$$

Here the constants c *and* c_ε *are independent of* N *and*

$$\|u' - y'\|_{\varepsilon,\infty} = \max_{j=1,\ldots,N} \left(\max_{t \in [t_{j-1},t_j] \cap [\varepsilon,T]} |u_j'(t) - y'(t)| \right),\ 0 < \varepsilon < T.$$

The corresponding numerical results are presented in Table 6. This table is comparable to Table 4. As we can see, the agreement with theoretical estimates is very good for all values of r and ν and we may conclude that in case of the test equations, the error estimates of Theorem 3 correspond to the leading term of the errors $\|u - y\|_\infty$ and $\|u' - y'\|_\infty$, which are dominating even for small values of N. Moreover, it seems that the estimate for the derivative of the error,

$$||u' - y'||_\infty \le c \begin{cases} N^{-r(1-\nu)} & \text{for} \quad 1 \le r \le \frac{m}{1-\nu}, \\ N^{-m} & \text{for} \quad r > \frac{m}{1-\nu}, \end{cases}$$

may be valid.

Table 6. Method 2; $\eta_1 = \frac{1}{6}, \eta_2 = \frac{1}{2}, \eta_3 = \frac{5}{6}$

$\nu = -\frac{1}{4}$	$r = 1$		$r = 1.189$		$r = 1.378$		$r = 1.6$	
N	ε_N	ϱ(4.8)	ε_N	ϱ(6.4)	ε_N	ϱ(8.0)	ε_N	ϱ(8.0)
4	2.0E-4	4.4	1.1E-4	5.9	9.8E-5	8.1	1.4E-4	7.4
32	2.0E-6	4.7	4.6E-7	6.3	1.8E-7	8.1	2.9E-7	8.0
256	1.9E-8	4.8	1.8E-9	6.4	3.6E-10	8.0	5.6E-10	8.0
1024	8.3E-10	4.8	4.4E-11	6.4	5.6E-12	8.0	8.7E-12	8.0
N	ε'_N	ϱ'(2.4)	ε'_N	ϱ'(2.8)	ε'_N	ϱ'(3.3)	ε'_N	ϱ'(4.0)
4	1.7E-2	2.3	1.2E-2	2.8	8.8E-3	3.3	7.2E-3	3.9
32	1.3E-3	2.4	5.6E-4	2.8	2.5E-4	3.3	1.1E-4	4.0
256	9.4E-5	2.4	2.6E-5	2.8	6.9E-6	3.3	1.8E-6	4.0
1024	1.7E-5	2.4	3.3E-6	2.8	6.3E-7	3.3	1.1E-7	4.0
$\nu = 0$	$r = 1$		$r = 1.275$		$r = 1.550$		$r = 2$	
N	ε_N	ϱ(4.0)	ε_N	ϱ(5.9)	ε_N	ϱ(8.0)	ε_N	ϱ(8.0)
4	1.1E-3	3.8	5.1E-4	5.5	3.8E-4	7.5	7.1E-4	6.3
32	1.8E-5	4.0	2.7E-6	5.8	7.4E-7	8.0	1.6E-6	7.9
256	2.8E-7	4.0	1.3E-8	5.9	1.4E-9	8.0	3.1E-9	8.0
1024	1.8E-8	4.0	3.9E-10	5.9	2.3E-11	8.0	4.9E-11	8.0
N	ε'_N	ϱ'(2.0)	ε'_N	ϱ'(2.4)	ε'_N	ϱ'(2.9)	ε'_N	ϱ'(4.0)
4	9.1E-2	2.0	6.2E-2	2.4	4.3E-2	2.9	3.3E-2	3.9
32	1.1E-2	2.0	4.4E-3	2.4	1.7E-3	2.9	5.2E-4	4.0
256	1.4E-3	2.0	3.1E-4	2.4	6.8E-5	2.9	8.1E-6	4.0
1024	3.6E-4	2.0	5.3E-5	2.4	7.9E-6	2.9	5.1E-7	4.0
$\boldsymbol{\nu = \frac{1}{2}}$	$r = 1$		$r = 1.533$		$r = 2.067$		$r = 4$	
N	ε_N	ϱ(2.8)	ε_N	ϱ(4.9)	ε_N	ϱ(8.0)	ε_N	ϱ(8.0)
4	6.8E-4	2.5	2.4E-4	4.4	1.8E-4	7.3	8.4E-4	3.8
32	3.3E-5	2.8	2.1E-6	4.9	3.3E-7	8.1	2.3E-6	7.8
256	1.5E-6	2.8	1.8E-8	4.9	6.4E-10	8.0	4.7E-9	8.0
1024	1.9E-7	2.8	7.3E-10	4.9	1.0E-11	8.0	7.3E-11	8.0
N	ε'_N	ϱ'(1.4)	ε'_N	ϱ'(1.7)	ε'_N	ϱ'(2.0)	ε'_N	ϱ'(4.0)
4	6.3E-2	1.4	4.4E-2	1.7	3.0E-2	2.0	2.5E-2	3.3
32	2.2E-2	1.4	8.9E-3	1.7	3.5E-3	2.0	4.0E-4	4.0
256	7.9E-3	1.4	1.8E-3	1.7	4.1E-4	2.0	6.2E-6	4.0
1024	4.0E-3	1.4	6.2E-4	1.7	9.8E-5	2.0	3.9E-7	4.0
$\boldsymbol{\nu = \frac{9}{10}}$	$r = 1$		$r = 1.909$		$r = 2.818$		$r = 20$	
N	ε_N	ϱ(2.1)	ε_N	ϱ(4.3)	ε_N	ϱ(8.0)	ε_N	ϱ(8.0)
4	4.2E-4	2.3	1.2E-4	5.2	1.7E-4	7.9	3.5E-3	1.0
32	5.3E-5	2.0	1.8E-6	4.3	2.4E-7	8.6	8.8E-5	6.6
256	5.8E-6	2.1	2.3E-8	4.3	4.3E-10	8.1	1.6E-7	8.2
1024	1.3E-6	2.1	1.2E-9	4.3	6.8E-12	7.9	2.4E-9	8.1
N	ε'_N	ϱ'(1.1)	ε'_N	ϱ'(1.1)	ε'_N	ϱ'(1.2)	ε'_N	ϱ'(4.0)
4	4.5E-2	0.9	4.2E-2	1.1	3.8E-2	1.2	3.6E-2	1.0
32	4.0E-2	1.1	2.9E-2	1.1	2.1E-2	1.2	2.7E-3	3.9
256	3.2E-2	1.1	2.0E-2	1.1	1.2E-2	1.2	4.2E-5	4.0
1024	2.8E-2	1.1	1.5E-2	1.1	8.0E-3	1.2	2.6E-6	4.0

In Table 7 the Gaussian parameters are used. We can see that the agreement with the theoretical estimates is very good, and there is no further improvement in the convergence rate.

Table 7. Method 2; $\eta_1 = \frac{5-\sqrt{15}}{10}, \eta_2 = \frac{1}{2}, \eta_3 = \frac{5+\sqrt{15}}{10}$

$\nu = -\frac{1}{4}$	$r = 1$		$r = 1.189$		$r = 1.378$		$r = 1.6$	
N	ε_N	ϱ(4.8)	ε_N	ϱ(6.4)	ε_N	ϱ(8.0)	ε_N	ϱ(8.0)
4	2.2E-4	4.7	1.2E-4	6.3	1.0E-4	7.7	1.5E-4	7.1
32	2.0E-6	4.7	4.7E-7	6.4	2.1E-7	8.0	3.2E-7	7.9
256	1.9E-8	4.8	1.8E-9	6.4	4.0E-10	8.0	6.3E-10	8.0
1024	8.4E-10	4.8	4.4E-11	6.4	6.3E-12	8.0	9.8E-12	8.0
N	ε'_N	ϱ'(2.4)	ε'_N	ϱ'(2.8)	ε'_N	ϱ'(3.3)	ε'_N	ϱ'(4.0)
4	1.5E-2	2.3	1.1E-2	2.7	7.8E-3	3.2	6.4E-3	3.9
32	1.1E-3	2.4	5.0E-4	2.8	2.2E-4	3.3	1.0E-4	4.0
256	8.4E-5	2.4	2.3E-5	2.8	6.1E-6	3.3	1.6E-6	4.0
1024	1.5E-5	2.4	2.9E-6	2.8	5.6E-7	3.3	9.9E-8	4.0
$\nu = 0$	$r = 1$		$r = 1.275$		$r = 1.550$		$r = 2$	
N	ε_N	ϱ(4.0)	ε_N	ϱ(5.9)	ε_N	ϱ(8.0)	ε_N	ϱ(8.0)
4	1.1E-3	4.0	5.2E-4	5.8	4.2E-4	7.7	7.9E-4	6.6
32	1.7E-5	4.0	2.6E-6	5.9	8.3E-7	8.0	1.8E-6	7.9
256	2.7E-7	4.0	1.3E-8	5.9	1.6E-9	8.0	3.5E-9	8.0
1024	1.7E-8	4.0	3.8E-10	5.9	2.5E-11	8.0	5.5E-11	8.0
N	ε'_N	ϱ'(2.0)	ε'_N	ϱ'(2.4)	ε'_N	ϱ'(2.9)	ε'_N	ϱ'(4.0)
4	8.0E-2	2.0	5.5E-2	2.4	3.8E-2	2.9	2.9E-2	3.9
32	1.0E-2	2.0	3.9E-3	2.4	1.5E-3	2.9	4.6E-4	4.0
256	1.3E-3	2.0	2.8E-4	2.4	6.0E-5	2.9	7.2E-6	4.0
1024	3.2E-4	2.0	4.7E-5	2.4	7.0E-6	2.9	4.5E-7	4.0
$\nu = \frac{1}{2}$	$r = 1$		$r = 1.533$		$r = 2.067$		$r = 4$	
N	ε_N	ϱ(2.8)	ε_N	ϱ(4.9)	ε_N	ϱ(8.0)	ε_N	ϱ(8.0)
4	6.8E-4	2.8	2.2E-4	4.9	1.8E-4	7.4	8.7E-4	4.2
32	3.0E-5	2.8	1.9E-6	4.9	3.7E-7	8.0	2.6E-6	7.6
256	1.3E-6	2.8	1.6E-8	4.9	7.2E-10	8.0	5.2E-9	8.0
1024	1.6E-7	2.8	6.4E-10	4.9	1.1E-11	8.0	8.2E-11	8.0
N	ε'_N	ϱ'(1.4)	ε'_N	ϱ'(1.7)	ε'_N	ϱ'(2.0)	ε'_N	ϱ'(4.0)
4	5.5E-2	1.4	3.8E-2	1.7	2.7E-2	2.0	2.2E-2	3.2
32	2.0E-2	1.4	7.8E-3	1.7	3.1E-3	2.0	3.5E-4	4.0
256	7.0E-3	1.4	1.6E-3	1.7	3.6E-4	2.0	5.5E-6	4.0
1024	3.5E-3	1.4	5.5E-4	1.7	8.7E-5	2.0	3.4E-7	4.0
$\nu = \frac{9}{10}$	$r = 1$		$r = 1.909$		$r = 2.818$		$r = 20$	
N	ε_N	ϱ(2.1)	ε_N	ϱ(4.3)	ε_N	ϱ(8.0)	ε_N	ϱ(8.0)
4	5.2E-4	2.2	1.2E-4	4.7	1.3E-4	8.8	2.9E-3	1.0
32	4.8E-5	2.2	1.5E-6	4.3	2.6E-7	8.2	6.8E-5	6.5
256	4.8E-6	2.1	1.9E-8	4.3	4.9E-10	8.0	1.7E-7	7.9
1024	1.0E-6	2.1	1.0E-9	4.3	7.6E-12	8.0	2.7E-9	8.0
N	ε'_N	ϱ'(1.1)	ε'_N	ϱ'(1.1)	ε'_N	ϱ'(1.2)	ε'_N	ϱ'(4.0)
4	3.8E-2	0.9	3.7E-2	1.0	3.3E-2	1.2	3.8E-2	1.0
32	3.5E-2	1.1	2.6E-2	1.1	1.9E-2	1.2	2.5E-3	3.8
256	2.8E-2	1.1	1.7E-2	1.1	1.0E-2	1.2	3.7E-5	4.0
1024	2.5E-2	1.1	1.3E-2	1.1	7.0E-3	1.2	2.3E-6	4.0

However, if we use collocation parameters that are exact for polynomials of order $m+1$, it is possible to get superconvergence at the collocation points. Brunner et al. [8] have proved the following result (see Theorem 2.3 in [8]) for nonlinear Volterra integral equation.

Theorem 4. (adapted to the linear case) *Let the following conditions be fulfilled*:

(V1) *The kernel* $K(t,s)$ *is* $m'+\mu'+1$ *times* $(m',\mu' \in \mathbb{Z},\ m' \geq 1,\ 0 \leq \mu' \leq m'-1)$ *continuously differentiable with respect to* t, s *for* $t \in [0,T],\ s \in [0,t)$, *and satisfies* (4) *with* $i + j \leq m' + \mu' + 1,\ \nu' \in (-\infty, 1)$.

(V2) $f \in C^{m'+\mu'+1,\nu'}(0,T]$.

(V3) *The collocation points* (6) *are generated by the knots* $\eta_j,\ j = 1,\dots,m'$ *of a quadrature formula* $\int_0^1 \phi(\xi)\mathrm{d}\xi \approx \sum_{i=1}^m w_i\phi(\eta_i),\ 0 \leq \eta_1 < \ldots < \eta_{m'} \leq 1$, *which is exact for all polynomials of degree* $m' + \mu',\ 0 \leq \mu' \leq m' - 1$.

(V4) *The scaling parameter* $r = r(m',\nu',\mu') \geq 1$ *is subject to the restrictions*

$$\begin{aligned}
&r > \frac{m'}{1-\nu'},\ r \geq \frac{m'+1-\nu'}{2-\nu'} && if \quad 1-\nu' < \mu'+1,\\
&r > \frac{m'}{1-\nu'},\ r > \frac{m'+\mu'+1}{2-\nu'} && if \quad \mu'+1 \leq 1-\nu' < m',\\
&r \geq \frac{m'+\mu'+1}{2-\nu'},\ r > 1 && if \quad 1-\nu' = m',\\
&r \geq \frac{m'+\mu'+1}{2-\nu'} && if \quad 1-\nu' > m'.
\end{aligned} \tag{16}$$

Then the approximate solution $u \in S_{m'}^{(-1)}(\Pi_N^r)$ *of the equation*

$$y(t) = \int_0^t K(t,s)y(s)\mathrm{d}s + f(t), \quad 0 \leq t \leq T,$$

satisfies the error estimate

$$\max_{\substack{k=1,\dots,m;\\ j=1,\dots,N}} |u(t_{jk}) - y(t_{jk})| \leq cN^{-m'} \begin{cases} N^{-1} & if \quad \nu' < 0,\\ N^{-1}(1+\log N) & if \quad \nu' = 0,\\ N^{-(1-\nu')} & if \quad \nu' > 0, \end{cases} \tag{17}$$

where c *is a positive constant which is independent of* N.

For superconvergence at the collocation points for sufficiently large values of r, in case of Volterra integro-differential equations (1), we obtain the following result from Theorem 4.

Theorem 5. *Let the following conditions be fulfilled*:

1) $p \in C^{m+2}[0,T],\ q \in C^{m+1,\nu}[0,T],\ K \in \mathcal{W}^{m+2,\nu}(\Delta_T),\ m \in \mathbb{N}$, $-\infty < \nu < 1$.

2) *The collocation points* (6)*, where* $k = 1, \dots, m+1$*, are generated by the grid points* $t_j = T(j/N)^r$*,* $j = 0, \dots, N$*, and by the knots* η_j*,* $j = 1, \dots, m+1$*, of a quadrature approximation* $\int_0^1 \phi(s)\mathrm{d}s \approx \sum_{q=1}^{m+1} A_q \phi(\eta_q)$*,* $0 \le \eta_1 < \dots < \eta_{m+1} \le 1$*, with appropriate weights* $\{A_q\}$*, which is exact for all polynomials of degree* $m+1$*.*
3) *The scaling parameter* $r = r(m, \nu) \ge 1$ *satisfies the inequality* $r > \frac{m+1}{2-\nu}$*.*

Then, with the notation of Theorem (3)*, the error estimate*

$$\max_{k=1,\dots,m+1; j=1,\dots,N} |u(t_{jk}) - y(t_{jk})| \le cN^{-m-2} \tag{18}$$

holds, where c *is a positive constant which is independent of* N*.*

Proof. We consider the linear Volterra integral equation (13)

$$y(t) = f_2(t) + \int_0^t K_2(t,s) y(s)\mathrm{d}s, \quad t \in [0,T],$$

where

$$f_2(t) = y_0 + \int_0^t q(s)\mathrm{d}s, \qquad K_2(t,s) = p(s) + \int_s^t K(\tau, s)\mathrm{d}\tau.$$

It is easy to check that if $p \in C^{m+2}[0,T]$, $K \in \mathcal{W}^{m+2,\nu}(\Delta_T)$, then $K_2 \in \mathcal{W}^{m+2,\nu-1}(\Delta_T)$ and if $q \in C^{m+1,\nu}[0,T]$, then $f_2 \in C^{m+2,\nu-1}[0,T]$.

In our notation, the assumptions (V1)–(V3) of Theorem 4 are satisfied if $m' = m+1$, $\nu' = \nu - 1 < 0$, and $\mu' = 0$. It remains to check the restrictions on the scaling parameter $r = r(m,\nu) \ge 1$. Assume that $r = r(m,\nu) \ge 1$ satisfies the inequality $r > \frac{m+1}{2-\nu}$. We shall show that then it also satisfies the conditions (16).

We see that the first case $1 - \nu' < \mu' + 1$ (i.e. $2 - \nu < 1$) never holds.

For the second case $\mu' + 1 \le 1 - \nu' < m'$, i.e. $1 \le 2 - \nu < m+1$, i.e. $0 \le 1 - \nu < m$ we must show that

$$r > \frac{m'}{1-\nu'} = \frac{m+1}{2-\nu},$$

which is satisfied, and

$$r > \frac{m' + \mu' + 1}{2 - \nu'} = \frac{m+2}{3-\nu}. \tag{19}$$

For the last equality we find the difference

$$\frac{m+1}{2-\nu} - \frac{m+2}{3-\nu} = \frac{m+\nu-1}{(2-\nu)(2-\nu)},$$

which is greater than 0, because $m > 1-\nu$, i.e. $m+\nu-1 > 0$, and the denominator is positive as well. It means that the condition (19) is also satisfied.

If $1-\nu' = m'$, i.e. $2-\nu = m+1$, then $r \geq \frac{m'+\mu'+1}{2-\nu'} = \frac{1-\nu'+1}{2-\nu'} = 1$ and other restriction $r > 1$ gives us the condition $r > 1$, which is satisfied, since in our case $r > \frac{m+1}{2-\nu} = 1$.

If $1-\nu' > m'$, then $2-\nu' > m'+1$ and we get that $r \geq \frac{m'+\mu'+1}{2-\nu'} = \frac{m'+1}{2-\nu'}$ follows from the condition $r \geq 1$.

We have shown that the restrictions (16) are satisfied. Since $\nu' = \nu - 1 < 0$, we get from (17) the error estimate (18). $\square$

Numerical results about superconvergence are presented in Tables 8–10. The errors at the collocation points are denoted by

$$\delta_N = \{\max |u(t_{jk}) - y(t_{jk})| : k = 1, \dots, m+1;\ j = 1, \dots, N\}.$$

In Table 8 we can see that for sufficiently large values of r ($r > \frac{m+1}{2-\nu}$) the numerical results are in good accordance with the theoretical error estimate of Theorem 5. The notation $\varrho(?)$ in this table indicates that the theoretical convergence rate is unknown for the corresponding values of r and ν.

Table 8. Method 2; $\eta_1 = \frac{1}{4}, \eta_2 = \frac{1}{2}, \eta_3 = \frac{3}{4}$

$\nu = -\frac{1}{4}$	$r = 1$		$r = 1.189$		$r = 1.378$		$r = 1.6$	
N	δ_N	ϱ(?)	δ_N	ϱ(?)	δ_N	ϱ(16.0)	δ_N	ϱ(16.0)
4	2.3E-5	9.1	1.1E-5	12.6	1.3E-5	14.9	2.1E-5	13.2
32	2.8E-8	9.4	4.1E-9	14.2	3.4E-9	15.8	5.7E-9	15.8
256	3.3E-11	9.5	1.4E-12	14.3	8.5E-13	15.9	1.4E-12	16.0
1024	3.7E-13	9.5	6.9E-15	14.4	3.1E-15	16.8	4.4E-15	19.4
$\nu = 0$	$r = 1$		$r = 1.275$		$r = 1.550$		$r = 2$	
N	δ_N	ϱ(?)	δ_N	ϱ(?)	δ_N	ϱ(16.0)	δ_N	ϱ(16.0)
4	9.9E-5	6.4	3.4E-5	11.3	2.9E-5	15.3	7.1E-5	12.2
32	2.4E-7	7.7	1.8E-8	13.0	7.0E-9	16.1	1.6E-8	16.5
256	4.9E-10	8.0	7.0E-12	13.7	1.7E-12	16.0	3.8E-12	16.1
1024	7.8E-12	8.0	3.7E-14	13.8	6.7E-15	15.9	1.5E-14	16.0
$\nu = \frac{1}{2}$	$r = 1$		$r = 1.533$		$r = 2.067$		$r = 4$	
N	δ_N	ϱ(?)	δ_N	ϱ(?)	δ_N	ϱ(16.0)	δ_N	ϱ(16.0)
4	1.2E-4	5.4	3.1E-5	11.8	5.1E-5	13.2	3.7E-4	5.8
32	6.0E-7	6.0	1.5E-8	13.1	1.2E-8	16.0	1.6E-7	15.0
256	2.9E-9	5.9	6.2E-12	13.6	3.1E-12	16.0	4.1E-11	15.9
1024	8.6E-11	5.8	3.3E-14	13.7	1.1E-14	16.5	1.6E-13	16.3
$\nu = \frac{9}{10}$	$r = 1$		$r = 1.909$		$r = 2.818$		$r = 20$	
N	δ_N	ϱ(?)	δ_N	ϱ(?)	δ_N	ϱ(16.0)	δ_N	ϱ(16.0)
4	3.3E-4	2.7	5.4E-5	6.9	7.0E-5	9.4	1.9E-3	1.0
32	5.6E-6	4.3	3.0E-8	13.1	1.7E-8	16.4	3.3E-5	8.0
256	6.7E-8	4.5	1.2E-11	14.0	4.0E-12	16.0	9.9E-9	16.7
1024	3.3E-9	4.5	5.6E-14	14.4	5.5E-14	4.5	3.9E-11	15.6

By analysing the numerical results corresponding to smaller values of r ($r < \frac{m+1}{2-\nu}$) we can deduce the following conjecture:

Conjecture 1. *Let the conditions* 1 *and* 2 *of Theorem* 5 *be fulfilled. Then, with the notation of Theorem* 3*, the error estimate*

$$\max_{\substack{k=1,\dots,m+1;\\ j=1,\dots,N}} |u(t_{jk}) - y(t_{jk})| \leq c \begin{cases} N^{-r(3-\nu)} & for \quad 1 \leq r < \frac{m+2}{3-\nu}, \\ N^{-r(3-\nu)}(1+\log N) & for \quad r = \frac{m+2}{3-\nu}, \\ N^{-m-2} & for \quad r > \frac{m+2}{3-\nu} \end{cases} \tag{20}$$

holds, where c is a positive constant, which is independent of N.

To confirm the error estimate (20), Table 9 is presented, where the theoretical convergence rate corresponding to Conjecture 1 is typed in bold-face.

As we can see in Table 9, the observed errors are in good agreement with the estimate (20). Similarly to Table 1, if r is close to the value $\frac{m+2}{3-\nu} = 1.6$, after which the maximal convergence rate is achieved, the observed convergence rate is smaller than the predicted one, but approaches slowly to the predicted theoretical value.

In Table 10 we have used the Gaussian parameters. As we can see, the numerical experiments in this case are in good agreement with the error estimate (20) and do not give any further improvement in the convergence rate. In this table the theoretical convergence rates that do not follow from Theorem 5 are again typed in bold-face.

Table 9. Method 2; $\eta_1 = \frac{1}{4}, \eta_2 = \frac{1}{2}, \eta_3 = \frac{3}{4}$, and $\nu = \frac{1}{2}$

	$r=1$		$r=1.1$		$r=1.2$		$r=1.3$	
N	δ_N	ϱ(**5.7**)	δ_N	ϱ(**6.7**)	δ_N	ϱ(**8.0**)	δ_N	ϱ(**9.5**)
4	1.2E-4	5.4	8.7E-5	6.4	6.1E-5	7.7	4.3E-5	9.1
32	6.0E-7	6.0	2.4E-7	7.2	1.0E-7	8.5	4.6E-8	10.0
256	2.9E-9	5.9	7.1E-10	6.9	1.7E-10	8.2	5.1E-11	9.6
1024	8.6E-11	5.8	1.5E-11	6.8	2.6E-12	8.1	5.6E-13	9.5
	$r=1.4$		$r=1.5$		$r=1.6$		$r=1.7$	
N	δ_N	ϱ(**11.3**)	δ_N	ϱ(**13.5**)	δ_N	ϱ(**16.0**)	δ_N	ϱ(**16.0**)
4	3.6E-5	9.1	3.2E-5	10.9	3.1E-5	12.8	3.3E-5	13.9
32	2.6E-8	11.4	1.7E-8	12.7	1.3E-8	13.9	1.1E-8	14.8
256	1.8E-11	11.3	7.7E-12	13.1	4.3E-12	14.5	3.1E-12	15.3
1024	1.4E-13	11.3	4.5E-14	13.2	2.0E-14	14.7	1.3E-14	15.6
	$r=1.8$		$r=1.9$		$r=2.0$		$r=2.1$	
N	δ_N	ϱ(**16.0**)	δ_N	ϱ(**16.0**)	δ_N	ϱ(**16.0**)	δ_N	ϱ(16.0)
4	3.4E-5	14.8	3.9E-5	14.5	4.6E-5	13.7	5.4E-5	13.0
32	1.0E-8	15.3	1.1E-8	15.7	1.1E-8	15.9	1.3E-8	16.0
256	2.7E-12	15.8	2.6E-12	15.9	2.8E-12	16.0	3.2E-12	16.0
1024	1.0E-14	16.0	1.0E-14	16.2	1.1E-14	16.2	1.2E-14	16.6

Table 10. Method 2; $\eta_1 = \frac{5-\sqrt{15}}{10}, \eta_2 = \frac{1}{2}, \eta_3 = \frac{5+\sqrt{15}}{10}$

$\nu = -\frac{1}{4}$	$r = 1$		$r = 1.189$		$r = 1.378$		$r = 1.6$	
N	δ_N	ϱ(**9.5**)	δ_N	ϱ(**14.6**)	δ_N	ϱ(16.0)	δ_N	ϱ(16.0)
4	7.3E-6	9.2	3.1E-6	14.2	4.7E-6	14.3	7.8E-6	12.6
32	8.7E-9	9.5	1.0E-9	14.5	1.2E-9	15.8	2.2E-9	15.7
256	1.0E-11	9.5	3.4E-13	14.6	3.1E-13	15.8	5.6E-13	15.9
1024	1.1E-13	9.5	4.1E-15	5.7	4.8E-15	4.7	5.9E-15	6.5
$\nu = 0$	$r = 1$		$r = 1.275$		$r = 1.550$		$r = 2$	
N	δ_N	ϱ(**8.0**)	δ_N	ϱ(**14.2**)	δ_N	ϱ(16.0)	δ_N	ϱ(16.0)
4	3.6E-5	7.1	1.2E-5	12.6	1.9E-5	12.4	4.3E-5	9.8
32	7.9E-8	7.8	4.6E-9	14.0	5.6E-9	15.6	1.5E-8	15.2
256	1.6E-10	8.0	1.6E-12	14.2	1.4E-12	16.0	3.9E-12	15.9
1024	2.5E-12	8.0	8.1E-15	14.2	5.5E-15	15.8	1.5E-14	16.0
$\nu = \frac{1}{2}$	$r = 1$		$r = 1.533$		$r = 2.067$		$r = 4$	
N	δ_N	ϱ(**5.7**)	δ_N	ϱ(**14.2**)	δ_N	ϱ(16.0)	δ_N	ϱ(16.0)
4	3.8E-5	5.8	6.9E-6	14.0	1.6E-5	12.9	1.2E-4	5.9
32	1.9E-7	5.8	2.0E-9	14.8	3.7E-9	16.4	5.1E-8	15.4
256	9.7E-10	5.7	6.7E-13	14.4	8.7E-13	16.1	1.2E-11	16.2
1024	3.0E-11	5.7	3.3E-15	14.3	5.3E-15	10.8	5.1E-14	15.0
$\nu = \frac{9}{10}$	$r = 1$		$r = 1.909$		$r = 2.818$		$r = 20$	
N	δ_N	ϱ(**4.3**)	δ_N	ϱ(**16.0**)	δ_N	ϱ(16.0)	δ_N	ϱ(16.0)
4	1.5E-4	3.5	2.4E-5	15.7	9.1E-5	10.0	2.5E-3	1.0
32	2.1E-6	4.4	4.5E-9	17.0	2.1E-8	16.3	4.0E-5	8.8
256	2.3E-8	4.5	9.9E-13	15.8	4.6E-12	16.6	1.2E-8	15.6
1024	1.1E-9	4.5	4.8E-14	2.2	6.4E-14	5.1	4.6E-11	16.6

6. COMPARISON OF METHODS 1 AND 2

Theoretical estimates and numerical experiments show that, in terms of uniform convergence, Method 2, with arbitrary collocation parameters, for computing approximate solution, as well as an approximation for the derivative of the solution, is equivalent to Method 1 if the conditions of Theorem 2 are satisfied.

An advantage of Method 2 is that if we use a special choice of collocation parameters, it is possible to obtain faster convergence (superconvergence) at the collocation points.

Considering that in the case of Method 2 with $u \in S_m^{(-1)}(\Pi_N)$, the complexity of implementation and computation time are comparable to those of Method 1 with $v \in S_m^{(-1)}(\Pi_N)$ $(u \in S_{m+1}^{(0)}(\Pi_N))$, Method 1 seems to be preferable to Method 2 if the assumptions of Theorem 2 hold.

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Teoreetiliste veahinnangute optimaalsus lineaarse nõrgalt singulaarse Volterra integro-diferentsiaalvõrrandi lahendamisel splain-kollokatsioonimeetoditega

Inga Parts

On vaadeldud kahte splain-kollokatsioonimeetodit Volterra integro-diferentsiaalvõrrandi lahendamiseks, toodud ära vastavad koonduvusteoreemid ja sooritatud hulgaliselt numbrilisi eksperimente teoreetiliste hinnangute optimaalsuse kontrollimiseks. Teoreetiliste tulemuste puudumise korral on numbriliste eksperimentide baasil püstitatud hüpotees meetodi koonduvuskiiruse kohta.