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Decomposition of discrete-time nonlinear control systems

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Abstract. The goal of the paper was to extend the results on the decomposition of the state equations of continuous-time nonlinear systems into the discrete-time domain. The results on accessible–nonaccessible decomposition mimic those of the continuous-time case. Decomposition is carried out in the vector space of differential one-forms. The results on observable–unobservable decomposition are not carried over to the discrete-time domain, in general, since the observable space cannot always be locally spanned by exact one-forms whose integrals would define the observable state coordinates. We conjecture that for reversible systems the observable space is integrable.

Key words: nonlinear systems, discrete-time systems, decomposition, accessibility, observability, vector spaces, one-forms, algebraic methods.

1. INTRODUCTION

Accessibility (controllability) and observability are fundamental properties of control systems. For certain applications it will be useful to have system representations in which the nonaccessible and unobservable state variables can be clearly distinguished. Decomposition plays an important role, for example, in the realization problem. If the realization algorithm [1] is applied to an input–output equation, the resulting state equations are observable, but not necessarily accessible. To get a minimal realization, one may apply the algorithm from [2] to extracting its minimal realization whenever possible. Minimal realizability conditions in [2] are stated in terms of partial derivatives of the input–output equation, and the algorithm requires inversion of several nonlinear maps.

Although the approach of [2] is direct, it relies on the input–output equation and is therefore not an intrinsic, coordinate-free solution to the decomposition problem.

For a continuous-time nonlinear system the decomposition into accessible–nonaccessible and observable–unobservable subsystems has been carried out both via differential geometric [3,4] and algebraic methods [5,6]. We will extend the results of [5,6] to the discrete-time domain where the decomposition is carried out in the vector space of differential one-forms, and then the state coordinates of the decomposed subsystems can be found by integrating the corresponding vector spaces of differential one-forms. Although the results on the accessible–nonaccessible decomposition mimic those of the continuous-time case, the results on the observable–unobservable decomposition cannot be extended, in general, to the discrete-time case. The reason is that the observable space of one-forms can be nonintegrable, which means that it cannot be spanned by exact one-forms, whose integrals would define the observable state coordinates.

2. ALGEBRAIC FRAMEWORK

Consider a discrete-time single-input single-output nonlinear system Σ described by the equations

$$\begin{aligned} x(t+1) &= f(x(t), u(t)), \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where $u \in U \subset I\!R$ is the input variable, $y \in Y \subset I\!R$ is the output variable, $x \in X$, an open subset of $I\!R^n$, is the state variable, $f : X \times U \to X$ and $h : X \to Y$ are the real analytic functions. A system Σ is called *reversible* if the state transition map $f(\cdot, u)$ is a diffeomorphism of X, for each $u \in \mathcal{U}$. Reversible systems arise naturally through the sampling of continuous-time systems.

In order to be able to use mathematical tools from the algebraic framework of differential one-forms, we assume that the following assumption holds for system (1) throughout the paper:

Assumption 1. $f(x, u)$ is generically a submersion, i.e. generically

$$\operatorname{rank} \frac{\partial f(x,u)}{\partial(x,u)} = n.$$

We follow the notation of [7]. Let $\mathcal{K}$ denote the field of meromorphic functions in a finite number of variables $\{x(0), u(t),\ t \geq 0\}$. The forward-shift operator $\delta : \mathcal{K} \to \mathcal{K}$ is defined by $\delta\zeta(x(t), u(t)) = \zeta(f(x(t), u(t)), u(t+1))$. Under Assumption 1 the pair $(\mathcal{K}, \delta)$ is a difference field [7], and up to an isomorphism, there exists a unique difference field $(\mathcal{K}^*, \delta^*)$, called the *inversive closure* of $(\mathcal{K}, \delta)$, such that $\mathcal{K} \subset \mathcal{K}^*$, $\delta^* : \mathcal{K}^* \to \mathcal{K}^*$ is an automorphism and the restriction of δ^* to $\mathcal{K}$ equals δ. In [7] an explicit construction of $(\mathcal{K}^*, \delta^*)$ is given. By abuse of notation, hereinafter we assume that the inversive closure $(\mathcal{K}^*, \delta^*)$ is given and use the same symbol to denote the difference field $(\mathcal{K}, \delta)$ and its inversive closure.

Over the field $\mathcal{K}$ one can define a difference vector space $\mathcal{E} := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi \mid \varphi \in \mathcal{K}\}$. The operator δ induces a forward-shift operator $\Delta : \mathcal{E} \to \mathcal{E}$ by

$$\sum_i a_i \mathrm{d}\varphi_i \to \sum_i \delta a_i \mathrm{d}(\delta\varphi_i), a_i, \varphi_i \in \mathcal{K}.$$

We will say that $\omega \in \mathcal{E}$ is an exact one-form if $\omega = \mathrm{d}F$ for some $F \in \mathcal{K}$. A one-form ν for which $\mathrm{d}\nu = 0$ is said to be closed. It is well known that exact forms are closed, while closed forms are only locally exact.

Theorem 1 (Frobenius). *Let* $V = \mathrm{span}_{\mathcal{K}}\{\ \omega_1,\ \ldots,\ \omega_r\}$ *be a subspace of* $\mathcal{E}$. V *is closed if and only if*

$$\mathrm{d}\omega_i \wedge \omega_1 \wedge \ldots \wedge \omega_r = 0, \ \text{ for any } i = 1, \ldots, r.$$

3. THE ACCESSIBLE SPACE

The relative degree r of a one-form $\omega \in \mathcal{E}$ is defined to be the least integer such that $\Delta^r\omega \notin \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$. If such an integer does not exist, we set $r = \infty$.

A sequence of subspaces $\{\mathcal{H}_k\}$ of $\mathcal{E}$ is defined by

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x(0)\}, \\ \mathcal{H}_{k+1} &= \{\omega \in \mathcal{H}_k \mid \Delta\omega \in \mathcal{H}_k\},\ k \geq 1, \end{aligned} \tag{2}$$

and proved to be invariant under the state space diffeomorphism [7]. It is clear that sequence (2) is decreasing. Denote by k^* the least integer such that

$$\mathcal{H}_1 \supset \cdots \supset \mathcal{H}_{k^*} \supset \mathcal{H}_{k^*+1} = \mathcal{H}_{k^*+2} = \cdots =: \mathcal{H}_\infty.$$

Obviously, $\mathcal{H}_k$ contains the one-forms whose relative degree is equal to k or higher than k.

The subspace $\mathcal{H}_\infty$ contains the one-forms with infinite relative degree so that these one-forms will never be influenced by the input of the system. For system (1), $\mathcal{H}_\infty$ is a nonaccessible subspace, and the factor space $\mathcal{X}_a := \mathcal{X}/\mathcal{H}_\infty$ such that $\mathcal{X}_a \oplus \mathcal{H}_\infty = \mathcal{X}$ precisely describes the accessible part of the system, where $\mathcal{X}$ denotes $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\}$.

Theorem 2 [7]. *The following statements are equivalent*:
1. *Nonlinear system* (1) *is strongly accessible*;
2. $\mathcal{H}_\infty = \{0\}$;
3. $\mathcal{X}_a = \mathcal{X}$.

Although $\mathcal{H}_k$ are, in general, not completely integrable, i.e. they do not admit the basis which consists only of closed forms, the limit $\mathcal{H}_\infty$ turns out to be completely integrable [7]. There exist locally r functions, say $\zeta_1, \ldots, \zeta_r$, with infinite relative degree so that $\mathcal{H}_\infty = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\zeta_1, \ldots, \mathrm{d}\zeta_r\}$.

Since $\mathcal{H}_\infty$ is invariant under applying forward shift operator, one has in particular

$$\begin{aligned} \zeta_1(t+1) &= f_1(\zeta_1(t), \ldots, \zeta_r(t)), \\ &\vdots \\ \zeta_r(t+1) &= f_r(\zeta_1(t), \ldots, \zeta_r(t)). \end{aligned}$$

Proposition 1. *The accessible subspace $\mathcal{X}_a$ of system* (1) *is completely integrable.*

Proof. Since the state space $\mathcal{X}$ and nonaccessible subspace $\mathcal{H}_\infty$ are completely integrable, so is the subspace $\mathcal{X}_a$. $\square$

Now, choosing $\zeta_{r+1}, \ldots, \zeta_n$, from $\mathcal{X}_a = \mathrm{span}_{\mathcal{K}}\{\,\mathrm{d}\zeta_{r+1}, \ldots, \mathrm{d}\zeta_n\}$, we have

$$\begin{aligned} \zeta_1(t+1) &= f_1(\zeta_1(t), \ldots, \zeta_r(t)), \\ &\vdots \\ \zeta_r(t+1) &= f_r(\zeta_1(t), \ldots, \zeta_r(t)), \\ \zeta_{r+1}(t+1) &= f_{r+1}(\zeta(t), u(t)), \\ &\vdots \\ \zeta_n(t+1) &= f_n(\zeta(t), u(t)), \end{aligned}$$

where the variables $\zeta_1, \ldots, \zeta_r$ are nonaccessible and the variables $\zeta_{r+1}, \ldots, \zeta_n$ are accessible.

Example 1. Consider the system [8]

$$\begin{aligned} x_1(t+1) &= x_1(t)(x_3^2(t)+1)^2, \\ x_2(t+1) &= x_2(t)(x_3^2(t)+1)^3, \\ x_3(t+1) &= x_3(t)+u(t). \end{aligned}$$

Straightforward computation shows that

$$\mathcal{H}_3 = \mathcal{H}_\infty = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}[x_1^3(t)/x_2^2(t)]\}.$$

Now, choosing $\zeta_1(t) = x_1^3(t)/x_2^2(t)$ and the other coordinates as $\zeta_2(t) = x_2(t)$, $\zeta_3(t) = x_3(t)$, we get the state equations

$$\begin{aligned} \zeta_1(t+1) &= \zeta_1(t), \\ \zeta_2(t+1) &= \zeta_1(t)(\zeta_3^2(t)+1)^2, \\ \zeta_3(t+1) &= \zeta_3(t)+u(t), \end{aligned}$$

where the nonaccessible state ζ_1 is clearly separated out.

4. THE OBSERVABLE SPACE

We use the notation $\mathbf{u}_j = [u(0), \ldots, u(j)]$ for $j \geq 0$. For each $x_0 \in I\!R^n$ and each control sequence $\mathbf{u}_{k-1}$ we use $x(k, x_0, \mathbf{u}_{k-1})$ to denote the solution of (1) at time k starting at $x(0) = x_0$ and produced by the input sequence $\mathbf{u}_{k-1}$.

Also, we define

$$H_{n-1}(x_0, \mathbf{u}_{n-2}) = \begin{bmatrix} h(x(0, x_0)) \\ h(x(1, x_0, \mathbf{u}_0)) \\ \vdots \\ h(x(n-1, x_0, \mathbf{u}_{n-2})) \end{bmatrix}.$$

Definition 1 (Observability rank condition). *System* (1) *is said to be locally single experiment observable if*

$$\mathrm{rank}_{\mathcal{K}} \frac{\partial H_{n-1}(x_0, \mathbf{u}_{n-2})}{\partial x_0} = n.$$

Condition (1) is known as the observability rank condition and in the special case of the linear systems it reduces the standard Kalman observability criterion.

Define the difference output spaces $\mathcal{Y}^k$, $\mathcal{Y}$, and $\mathcal{U}$ as follows:

$$\begin{aligned} \mathcal{Y}^k &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y(t),\ \ 0 \leq t \leq k\}, \\ \mathcal{Y} &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y(t),\ \ t \geq 0\}, \\ \mathcal{U} &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u(t),\ \ t \geq 0\}. \end{aligned}$$

To define the observable space in analogy with the continuous-time case, we introduce the chain of subspaces

$$0 \subset \mathcal{O}_0 \subset \mathcal{O}_1 \subset \mathcal{O}_2 \ldots \subset \mathcal{O}_k \ldots, \qquad (3)$$

where $\mathcal{O}_k := \mathcal{X} \cap (\mathcal{Y}^k + \mathcal{U})$ is called the *observability filtration.*

Definition 2. *The subspace* $\mathcal{X} \cap (\mathcal{Y} + \mathcal{U})$ *is called the observable space of system* (1).

The observable space can be computed as the limit of the observability filtration (3). This limit will be denoted by $\mathcal{O}_\infty$ and obviously we have

$$\mathcal{O}_\infty = \mathcal{X} \cap (\mathcal{Y} + \mathcal{U}).$$

The unobservable space of system (1), denoted by $\mathcal{X}_{\bar{O}}$, is defined as a subspace of $\mathcal{X}$, which satisfies

$$\mathcal{X}_{\bar{O}} \cong \mathcal{X}/\mathcal{O}_\infty, \quad \mathcal{X}_{\bar{O}} \oplus \mathcal{O}_\infty = \mathcal{X},$$

where the factor-space $\mathcal{X}/\mathcal{O}_\infty$ provides a precise description of the system unobservable states.

Proposition 2. *The following statements are equivalent*:
1. *The observability rank condition is satisfied*;
2. $\mathcal{X} \subset \mathcal{Y} + \mathcal{U}$;
3. $\mathcal{O}_\infty = \mathcal{X}$;
4. $\mathcal{X}_{\bar{O}} = \{0\}$.

Unfortunately, unlike in the continuous-time case [3–6] for discrete-time systems, in general, $\mathcal{O}_\infty$ is not closed. Consider, for example, the counterexample:

$$\begin{aligned} x_1(t+1) &= u(t), \\ x_2(t+1) &= x_3(t), \\ x_3(t+1) &= x_1(t) + x_2(t)u(t), \\ y(t) &= x_3(t). \end{aligned} \tag{4}$$

The observability filtration is as follows:

$$\begin{aligned} \mathcal{O}_1 &= \mathrm{sp}_\mathcal{K}\{\mathrm{d}x_3(t)\}, \\ \mathcal{O}_2 &= \mathcal{O}_3 = \ldots = \mathcal{O}_\infty = \mathrm{sp}_\mathcal{K}\{\mathrm{d}x_3(t), \mathrm{d}x_1(t) + u(t)\mathrm{d}x_2(t)\}, \end{aligned}$$

and it is easy to check by Frobenius Theorem that $\mathcal{O}_2 = \mathcal{O}_\infty$ is not completely integrable. Of course, this is not to say that for most systems $\mathcal{O}_\infty$ is not integrable. Consider another 3rd-order state-affine system

$$\begin{aligned} x_1(t+1) &= x_1(t), \\ x_2(t+1) &= x_3(t) + u(t)x_2(t), \\ x_3(t+1) &= x_2(t), \\ y(t) &= x_3(t). \end{aligned} \tag{5}$$

For this system $\mathcal{O}_\infty = \mathrm{sp}_\mathcal{K}\{\mathrm{d}x_2(t), \mathrm{d}x_3(t)\}$, which is obviously integrable. Note, however, that unlike (4), system (5) is reversible. Still, reversibility is not a necessary condition for $\mathcal{O}_\infty$ to be invertible. $\mathcal{O}_\infty$ can be integrable for some nonreversible systems as is demonstrated by the following example:

$$\begin{aligned} x_1(t+1) &= x_1(t), \\ x_2(t+1) &= u(t), \\ x_3(t+1) &= x_2(t) + x_3(t)u(t), \\ y(t) &= x_3(t). \end{aligned}$$

Here, $\mathcal{O}_\infty = \mathrm{sp}_\mathcal{K}\{\mathrm{d}x_2(t), \mathrm{d}x_3(t)\}$.

An interesting open question is if one can find the subclasses with closed $\mathcal{O}_\infty$. In the continuous-time case, for analytic systems $\mathcal{O}_\infty$ is closed. The continuous-time case is simpler than the discrete-time one, due to the time-reversibility of differential equations. It is an open question if for reversible systems $\mathcal{O}_\infty$ is closed or not. At the moment, we are not aware of any examples of reversible systems with $\mathcal{O}_\infty$ being not integrable.

If $\mathcal{O}_\infty$ is closed, and therefore, has locally an exact basis $\{\mathrm{d}\zeta_1, \dots, \mathrm{d}\zeta_r\}$, one can complete the set $\{\zeta_1, \dots, \zeta_r\}$ to a basis $\{\zeta_1, \dots, \zeta_r, \zeta_{r+1}, \dots, \zeta_n\}$ of $\mathcal{X}$. Then, in these coordinates, the system reads as

$$\begin{array}{rcl} \zeta_1(t+1) & = & f_1(\zeta_1(t), \dots, \zeta_r(t), u(t)), \\ & \vdots & \\ \zeta_r(t+1) & = & f_r(\zeta_1(t), \dots, \zeta_r(t), u(t)), \\ \zeta_{r+1}(t+1) & = & f_{r+1}(\zeta(t), u(t)), \\ & \vdots & \\ \zeta_n & = & f_n(\zeta(t), u(t)), \\ y(t) & = & h(\zeta_1(t), \dots, \zeta_r(t)). \end{array}$$

5. CONCLUSIONS

Using the algebraic formalism based on the classification of the differential one-forms, we have carried out the decomposition of the state space of the discrete-time nonlinear control system into accessible–nonaccessible and observable–unobservable subsystems. Although the results on accessible–nonaccessible decomposition mimic those obtained for the continuous-time case, the results on observable–unobservable decomposition cannot be extended to the discrete-time case, in general. The reason is that the observable space of differential one-forms may not be integrable. In this case the subspace cannot be spanned by exact one-forms, whose integrals define the observable state coordinates. We conjecture that the observable space is integrable for reversible discrete-time systems.

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Diskreetsete mittelineaarsete juhtimissüsteemide dekomponeerimine

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Artikli eesmärgiks on sarnaselt pidevate süsteemidega dekomponeerida diskreetseid mittelineaarseid juhtimissüsteeme kirjeldav olekumudel juhitavaks/mittejuhitavaks ja jälgitavaks/mittejälgitavaks alamsüsteemiks. Tulemused juhitava/mittejuhitava lahutuse kohta imiteerivad pidevate süsteemide jaoks saaduid. Lahutus on leitud üksvormide vektorruumis üle meromorfsete funktsioonide korpuse. Tulemused jälgitava/mittejälgitava lahutuse kohta üldjuhul diskreetsetele süsteemidele üle ei kandu. Põhjus on selles, et vaadeldav üksvormide alamruum ei ole üldjuhul täielikult integreeruv, st tema baasi ei ole lokaalselt võimalik esitada eksaktsete diferentsiaalvormide abil, mille integraalid defineeriksid vaadeldavad olekukoordinaadid. Tuginedes näidetele ja asjaolule, et pidevad süsteemid on ajas pööratavad, on püstitatud hüpotees, et ajas pööratavate diskreetsete mittelineaarsete süsteemide vaadeldav alamruum on täielikult integreeruv.