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Relationship between join and betweenness geometries

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Abstract. A treatment of join geometry was elaborated by Prenowitz and Jantosciak in a voluminous monograph of 1979 (*Join Geometries. A Theory of Convex Sets and Linear Geometry*), which in most part deals with the theory of convex sets but touches also upon linear geometry and the betweenness relation. The latter relation was taken as the only basic notion (besides the notion of point) by the Estonian mathematicians J. Sarv, J. Nuut, and A. Tudeberg (Humal) in their treatment of the foundations of geometry in the 1930s. A solid betweenness geometry as a theory of betweenness models was worked out by the author of the present paper in 1964 but it appeared in publications not widely available. On the basis of the 1979 monograph, the author analyses the relationship between these two geometries. First, betweenness geometry is recapitulated, and then the more general interimity and betwixtness geometries are introduced. It is proved that a betweenness geometry is at the same time an ordered join geometry, and conversely, an exchanged join geometry is a betwixtness geometry, but the more special ordered join geometry coincides with betweenness geometry. In higher than two dimensions the latter is Desarguesian and leads to a convex region in a linear space over an ordered skew field.

Key words: join geometry, betweenness model, convex region, Desarguesian space.

1. INTRODUCTION

After having about fifty years ago completed his investigations into betweenness geometry (see [1,2], based on [3–5]) and carried on with problems of differential geometry, the author recently stumbled upon an interesting monograph by W. Prenowitz and J. Jantosciak *Join Geometries. A Theory of Convex Sets and Linear Geometry* (see [6]). The present paper is the author's reaction to that event.

The betweenness relation has fascinated the investigators for a long time. Already C. F. Gauss in his letter to F. Bolyai (6 March 1832; see [7], p. 222) pointed to the absence of betweenness postulates in Euclid's treatment. Elimination of this defect was started fifty years later by Pasch [8]. Further development of the logical foundation of synthetic geometry in the 19th century (through the works of G. Peano, F. Amodeo, G. Veronese, G. Fano, F. Enriques, and M. Pieri) led to Hilbert's fundamental *Grundlagen der Geometrie* [9], where the betweenness relation is subjected to the *axioms of connection and of order* (I 1–7, II 1–5 of Hilbert's list), called by Schur [10] the *projective axioms* of geometry.

In the first decades of the 20th century axiomatics of the betweenness relation was investigated in the U.S. by Moore [11] and Veblen [12] in the framework of these projective axioms. They indicated also some redundancy in Hilbert's axiomatics, which was taken into consideration by Hilbert in the following editions (e.g. in the seventh edition of [9]). In addition, the standpoint was developed that the lines and planes can be considered as sets of points and that special axioms of connection are expedient only for lines (not for planes, because all requisites for them can be deduced). Note that this standpoint was not accepted by Hilbert in the following editions of his *Grundlagen*, but was afterwards adopted in the U.S. by Huntington [13–16], who in 1926 gave an elaborated system of axioms for the betweenness relation, but only in dimension 1, i.e. for the case of a line.

This standpoint was developed further in Estonia, first by Nuut [17] for dimension one (as a geometrical foundation of real numbers) and afterwards by Sarv [3] for an arbitrary dimension n. Extending the Moore–Veblen approach, Sarv proposed a self-dependent axiomatics for the betweenness relation, so that all axioms of connection, including also those concerning the lines, became the consequences. This self-dependent axiomatics was simplified and then perfected by Nuut [4] and Tudeberg (from 1936 Humal) [5]. As a result an extremely simple axiomatics was worked out for the n-dimensional geometry using only two basic concepts: "point" and "between".

The author of the present paper developed in [1] a comprehensive theory of the *models of betweenness*, based on this axiomatics. At the same time it was proved in [2] that in dimension >2 this model reduces to a convex domain in n-dimensional linear space over an ordered skew field. Later Pimenov [18] (in Appendix: Local betweenness relation) called the perfected axiomatics the *Humal–Lumiste axiomatics* and its model in dimension 2, when the above result cannot be used, the *Lumiste plane*. As a whole, the theory of these models, including also the Huntington–Nuut theory for dimension 1, can be called the *betweenness geometry*.

Approximately at the same time, Rubinshtein [19–21] developed (together with Rutkovskij) a theory of *axial structures*, which is tightly connected with betweenness geometry and uses some of its results (with exact references to [1,2]).

Independently also another approach, independent of the axioms of connection, was evolved. In [22], Schur tried to work out a part of geometry based on the basic concepts of "point" and "line segment" (Ger. *Strecke*). This approach was elaborated by Prenowitz [23] (see also [24,25]). The segment was considered as the "join" of its endpoints, and so the *join operation* was introduced in the set of points. In the monograph [6] a complete *join geometry* of these *join spaces* was developed.

The aim of the present paper is to investigate the relationship between the join and betweenness geometries. In join geometry we can rest on [6]. The essential part of it is summarized here in Section 2 (and also afterwards). Since the publications about the betweenness geometry are not widely available (a great deal of them are written in Estonian, namely [1,3], or in Russian, e.g. [2,18]), we have to recapitulate here in Sections 4 and 5 the outlines of this theory, relatively little known at present. Meanwhile, in Section 3 the earlier betweenness geometry is separated into some parts having in mind the later join geometry. So the *interimity models* and *betwixtness geometry*[1] are introduced separately.

The main topic is treated in Sections 6 and 7. It is shown that the betweenness geometry is at the same time the ordered join geometry. Conversely, the exchanged join geometry is a betwixtness geometry, but the more special ordered join geometry is a betweenness geometry.

In Section 8 a relationship with the projective geometry is established and the Desarguesian theorem is proved together with its converse. Finally, in Section 9, the Main Theorem is proved, asserting that in higher than two dimension the betweenness geometry (ordered join geometry) is Desarguesian and leads to a convex region in a linear space over an ordered skew field.

2. JOIN SPACE AND JOIN GEOMETRY

Following [6], let us consider the pair $(S, \cdot)$ of a set S and an operation $\cdot$, which assigns to any ordered pair (a, b) of elements of S a subset of S, denoted by $a \cdot b$ and called the *join of a and b*. For any pair (A, B) of subsets of S the set $A \cdot B$ determined by

$$A \cdot B = \bigcup_{a \in A, b \in B} a \cdot b,$$

is called the *join of A and B*.

The pair $(S, \cdot)$ above is called a *join system* (see [6], Sections 2.2 and 2.3) if

$$\mathbf{J1}: \ a \cdot b \neq \emptyset; \ \ \mathbf{J2}: \ a \cdot b = b \cdot a; \ \ \mathbf{J3}: \ (a \cdot b) \cdot c = a \cdot (b \cdot c); \ \ \mathbf{J4}: \ a \cdot a = a,$$

where in **J3** $(a \cdot b) \cdot c$ is, of course, the join of $a \cdot b$ and c. Further, the subset $a/b = \{x | b \cdot x \supset a\}$ is called the *extension of a from b*, and let $A \approx B$ mean that A and B have a nonempty intersection, i.e. that they have a common element.

The join system is called a *join space* (see [6], Section 5.1) if, moreover,

$$\mathbf{J5}: \ a/b \neq \emptyset; \quad \mathbf{J6}: \ a/b \approx c/d \Rightarrow a \cdot d \approx b \cdot c; \quad \mathbf{J7}: \ a/a = a.$$

[1] Here the word “betwixt” has been in mind (which, according to dictionaries, is now archaic except in the expression *betwixt and between*), as well as the word “interim”.

The theory of join spaces, called the *join geometry*, is developed in the monograph [6]. Mainly the properties of convex sets, in particular of linear sets, and of convex (resp. linear) hulls are considered.

Here a set A is called a *convex* set if $A \supset x, y$ implies $A \supset x \cdot y$ (see [6], Section 2.9). A convex set A for which $A \supset x, y$ implies $A \supset x/y$ is called a *linear* set (see [6], Section 6.2). The least linear set which contains a given set A is called the *linear hull* of this given set and denoted by $< A >$ ([6], Section 6.8). The linear hull of two distinct a and b is called the *line* $< ab >$. It is established in [6] (Section 6.10, (2)) that

$$< ab > \supset a \cdot b \cup a/b \cup b/a \cup a \cup b.$$

If **J1–J7** are complemented by

$$\mathbf{E} : (c \subset < ab >) \wedge (c \neq a) \Rightarrow (< ac > = < ab >),$$

then the join geometry is called an *exchange join geometry* (see [6], Section 11.1).

In such a geometry the linear hull of three a, b, c, not in the same line, is called a *plane* $< abc >$ (see [6], Section 11.6).

In a join geometry the *betweenness* relation is introduced by the following definition (see [6], Section 4.23): suppose $x \subset a \cdot b$ and $a \neq b$; then we say x is *between* a and b, and write (axb).

Finally the following *order postulate* is added in [6], Section 12.1:

O. For every three distinct a, b, c of a line either $a \subset b \cdot c$, or $b \subset a \cdot c$, or $c \subset a \cdot b$, i.e. at least one is between the two others.

If **O** is added to **J1–J7**, then the join geometry is called the *ordered join geometry*. In this geometry (see [6], Section 12.3)

$$< ab > = a \cdot b \cup a/b \cup b/a \cup a \cup b.$$

It is established in [6], Section 12.2 that an ordered join geometry is an exchanged join geometry, but the converse is not valid: there exist examples of exchanged join geometries which are not ordered join geometries (see also [23], pp. 62–68).

3. BETWEENNESS GEOMETRY AND RELATED MODELS

The independent concept of betweenness model was introduced more than 40 years ago in [1,2], following [3–5]. Having in mind the join geometry, it is more convenient to separate the definition into some parts as follows.

Let S be a set, and let there be given a subset B in $S \times S \times S$ (i.e. a *ternary relation* for S). Further (abc) will mean that $(a, b, c) \in \mathbf{B}$ and then b is said to be *between* a and c. Moreover, let us denote

$$\langle abc\rangle = (abc) \vee (bca) \vee (cab); \qquad [abc] = \langle abc\rangle \vee (a = b) \vee (b = c) \vee (c = a). \quad (1)$$

The triplet (a, b, c) is said to be *correct* if $\langle abc\rangle$, and *collinear* if $[abc]$. The subset $\{x|(axb)\}$ is called an *interval* ab with ends a and b.

Let us start with a preparatory concept. The pair $(S, \mathbf{B})$ is called an *interimity model* and $\mathbf{B}$ an *interimity relation* if

$$\mathbf{B1} : (a \neq b) \Rightarrow \exists c, (abc); \;\; \mathbf{B2} : (abc) = (cba);$$

$$\mathbf{B3} : (abc) \Rightarrow \neg(acb); \;\; \mathbf{B4} : \langle abc\rangle \wedge [abd] \Rightarrow [cda]; \;\; \mathbf{B5} : (a \neq b) \Rightarrow \exists c, \neg[abc].$$

The basic concept will be introduced by the following definition: if in an interimity model $(S, \mathbf{B})$, in addition,

$$\mathbf{B6} : \neg[abc] \wedge (abd) \wedge (bec) \Rightarrow \exists f, ((afc) \wedge (def)),$$

then this $(S, \mathbf{B})$ is called a *betweenness model* and $\mathbf{B}$ is said to be a *betweenness relation* (see [1,2]).

A subsidiary concept gives now the following definition: if in an interimity model $(S, \mathbf{B})$ **B6** is replaced by

$$\mathbf{B\bar{6}} : \neg[abc] \wedge (abd) \wedge (aec) \Rightarrow \exists f, ((bfc) \wedge (dfe)),$$

then this $(S, \mathbf{B})$ is called a *betwixtness model* and $\mathbf{B}$ is said to be a *betwixtness relation* (see the footnote[1] above).

The connecting instrument for the betweenness and betwixtness models is the so-called *Pasch postulate*

$$\begin{aligned} \mathbf{P} : \; & \neg[abc] \wedge (bec) \wedge (d \in P_{abc}) \wedge (d \notin L_{bc}) \wedge (a \notin L_{de}) \\ & \Rightarrow \exists f, (f \in L_{de}) \wedge [(afb) \vee (afc)], \end{aligned}$$

where $L_{ab} = \{x|[xab]\}$ is a *line* determined by a, b, $a \neq b$, and $P_{abc} = Q_a \cup Q_b \cup Q_c$ with noncollinear a, b, c and $Q_a = L_{ab} \cup L_{ac} \bigcup_{x \in bc} L_{ax}$ is a *plane* determined by these a, b, c (and obviously not depending on their reordering).

It will be proved in the present paper that every betweenness model is a join space with an exchange ordered join geometry. On the other hand, every join space with exchanged join geometry is a betwixtness model. As a corollary, every betweenness model is also a betwixtness model. This last result can be established even directly: for every betweenness model also $\mathbf{B\bar{6}}$ holds and, moreover, also the Pasch postulate **P** is valid.

4. THE INTERIMITY MODEL

It is natural to start with the interimity model.

Lemma 1. *If in an interimity model* (abc)*, then* a, b, c *are three distinct points.*

Proof. Indeed, **B3** excludes $b = c$, and together with **B2** excludes also $b = a$. Finally, $a = c$ is impossible as well, because if $c = a$, then $b \neq a$ and due to **B5** $\exists d, \neg[abd]$, but on the other hand, $(abc) \Rightarrow \langle acb \rangle$ and $(a = c) \Rightarrow [acd]$, and these together imply due to **B4** that $[dba] = [abd]$, but this contradicts $\neg[abd]$ and finishes the proof.

If a triplet a, b, c is correct, i.e. $\langle abc \rangle$, then due to Lemma 1 here a, b, c are three different points and due to **B2, B3** only one of them is between the two others. Recall that if $[abc]$, then a, b, c are said to be collinear. It is obvious that correctness and collinearity of any three $a,\ b,\ c$ does not depend on their order, i.e.

$$\langle abc \rangle = \langle bca \rangle = \langle cab \rangle, \qquad [abc] = [bca] = [cab]. \tag{2}$$

Lemma 2. *In an interimity model let a, b, c be collinear, i.e. $[abc]$ and so* (1) *holds. Here only the following four possibilities occur*:

$$1)\ (a = b) \vee (b = c) \vee (c = a), \quad 2)\ (abc), \quad 3)\ (bca), \quad 4)\ (cab).$$

Each of them excludes the three others.

Proof. The first possibility follows from Lemma 1. Due to **B2, B3**, $(abc) = (cba) \Rightarrow \neg(cab)$, $(abc) \Rightarrow \neg(acb) = \neg(bca)$. Due to the same Lemma 1, $(abc) \Rightarrow \neg[(a = b) \vee (b = c) \vee (c = a)]$.

Lemma 3. *In an interimity model there hold*

$$\neg[abc] \wedge \langle abd \rangle \Rightarrow \neg[acd], \tag{3}$$

$$\neg[abc] \wedge [abd] \wedge [adc] \Rightarrow (a = d), \tag{4}$$

$$(abc) \wedge (bcd) \Rightarrow \langle abd \rangle, \tag{5}$$

$$\neg[abc] \wedge (adb) \wedge (aec) \Rightarrow d \neq e. \tag{6}$$

Proof. Let us suppose for (3), by reductio ad absurdum, that $[acd]$. Then due to (1) and **B4**, $\langle abc \rangle \wedge [acd] = \langle acb \rangle \wedge \neg[acd] \Rightarrow [bda] = [abd]$, but this is impossible.

For (4), $\neg[abc] \Rightarrow (a \neq b)$, $\neg[abc] \wedge [adc] \Rightarrow (b \neq d)$; now by reductio ad absurdum,

$$[abd] \wedge (a \neq b) \wedge (b \neq d) \wedge (a \neq d) \Rightarrow \langle abd \rangle = \langle adb \rangle$$

and then due to **B4** $\langle adb \rangle \wedge [adc] \Rightarrow [bca] = [abc]$, but this is impossible.

For (5), due to Lemma 1, $(abc) \wedge (bcd) \Rightarrow (a \neq b) \wedge (b \neq d)$. Also $d \neq a$, because otherwise, due to **B2**, $(bcd) = (bca) = (acb)$ and now, due to **B3**, $\neg(abc)$, which is impossible. Further, due to (1), $(abc) \wedge (bcd) = \langle bca \rangle \wedge [bcd]$, and now due to **B4** $[adb]$, which is, due to (1), equivalent to $[abd]$, but this together with $(a \neq b) \wedge (b \neq d) \wedge (d \neq d)$ implies $\langle abd \rangle$, as needed.

For (6), $(adb) \Rightarrow [abd]$, and now, by reductio ad absurdum, if one supposes $d = e$, then $(aec) = (adc) \Rightarrow [adc]$, and (4) would yield $a = d$. On the other hand, due to (1), $(adb) \Rightarrow a \neq d$, which gives a contradiction. This finishes the proof.

For a line the following assertions can be proved, which show that in an interimity model the points a, b are not some specific points of a line L_{ab}, but can be exchanged by every two of its different points c, d. Indeed, there holds

Lemma 4. *If $c \in L_{ab}$ and $c \neq a$, then $L_{ac} = L_{ab}$.*

Proof. This is obvious if $c = b$. Otherwise $[abc] \wedge (a \neq b) \wedge (b \neq c) \wedge (c \neq a) \Rightarrow \langle abc \rangle$ and, due to **B4**, $\langle abc \rangle \wedge [abx] \Rightarrow [cxa]$, thus $x \in L_{ab} \Rightarrow x \in L_{ac}$. But also $\langle acb \rangle \wedge [acy] \Rightarrow [bya]$, thus $y \in L_{ac} \Rightarrow y \in L_{ab}$.

Using this lemma two times, one obtains

Theorem 5. *If in an interimity model two different points c, d belong to a line L_{ab}, then $L_{cd} = L_{ab}$.*

Otherwise, a line is uniquely determined by any two of its different points. Recall that in the definition of a line L_{ab} due to (1) $[xab] = (xab) \vee (abx) \vee (bxa) \vee (x = a) \vee (x = b)$ (note that here $a = b$ is excluded). Hence a and b divide the remaining part of L_{ab} into three subsets: 1) $ab = \{x | (axb)\}$ (note that, due to **B2**, $ab = ba$), 2) $a/b = \{x | (xab)\}$, and 3) $b/a = \{x | (abx) = (xba)\}$. Here ab is called the *interval* with ends a and b; further, a/b will be called its *extension over an end a*. It follows that $L_{ab} = ab \cup (a/b) \cup (b/a) \cup a \cup b$, i.e. a line L_{ab} is a union of an interval, its ends, and its extensions over both ends.

Note that up to now only **B1**–**B4** are used and, in an extreme case, S can consist only of the points of one single line L_{ab}. Further let also **B5** be taken along. Here $\neg[abc]$ means that a, b, c are three noncollinear points, i.e. three different points, not belonging to one line. If a, b, c are noncollinear, then they are said to be the *vertices*, the intervals bc, ca, ab the *sides* (opposite to a, b, c, respectively) of the *triangle* $\triangle abc$, which is considered as the union of all of them. Here a/b and b/a are the *extensions of the side ab*, and $ab \cup a \cup b$ is the *closed side*.

Note that the subset $Q_a = L_{ab} \cup L_{ac} \bigcup_{x \in bc} L_{ax}$ in the definition of a plane P_{abc} can be now interpreted as the union of points on the lines, which are determined by a vertex a of the triangle $\triangle abc$ and the points of its opposite closed side. The plane P_{abc} itself can be interpreted as the union of the points on the lines, which are determined by any of the vertices and the points of its opposite closed side of a triangle $\triangle abc$.

The theory of interimity models is rather poor if one is not willing to add to **B1**–**B5** some new postulate. Some possibilities were indicated above, which will lead to the betweenness or betwixtness geometries. The added postulates allow now some interpretations. For instance, the Pasch postulate says that if a line contains a point d of the plane of a triangle $\triangle abc$, which does not belong to a side or its extension (e.g. $d \notin L_{bc}$), and intersects a side (e.g. bc in e), and does not contain any of vertices, then this line (e.g. L_{de}) intersects also one of the other two sides (correspondingly in f).

Below analogous interpretations will be given also for **B6** and $\mathbf{B\bar{6}}$.

Remark. The interimity models are in interesting relationship with the *geometry of geodesics*, developed in [26] as a theory of G-spaces.

A G-space is a metric space, i.e. a set G with $\rho : G \times G \to \mathbf{R}^+$ satisfying a) $\rho(x,y) = 0 \Leftrightarrow x = y$, b) $\rho(x,y) + \rho(y,z) \geq \rho(x,z)$, c) $\rho(x,y) = \rho(y,x)$, which is (i) finitely compact, i.e. the bounded infinite subsets in G have limit points, and (ii) convex in the sense of Menger: $(x \neq y) \Rightarrow \exists z, \rho(x,z) + \rho(z,y) = \rho(x,y)$.

Moreover, for every $a \in G$ there must exist a real number $r > 0$ so that in the set $\{x|\rho(a,x) < r\}$ there exists a point z distinct from the points x and y with $\rho(x,y) + \rho(y,z) = \rho(x,z)$, and if here for z_1, z_2 there would be $\rho(y,z_1) = \rho(y,z_2)$, then $z_1 = z_2$.

The betweenness relation can be introduced in a G-space by

$$(xzy) \Longleftrightarrow [\rho(x,z) + \rho(z,y) = \rho(x,y)] \wedge (x, y, z \text{ are three different points}).$$

Here **B2** follows directly from c). Also **B3** is satisfied. Indeed, if together with (xzy) one supposes (xyz), then at the same time $\rho(x,z) + \rho(z,y) = \rho(x,y)$ and $\rho(x,y) + \rho(y,z) = \rho(x,z)$, but this due to c) would lead to $\rho(y,z) = 0$, thus to $y = z$ (see a)), which is impossible, because x, y, z must be different. If the G-space is not one-dimensional, i.e. neither a straight line nor a circle (see [23], §9), then also **B5** is satisfied.

With **B4** the situation is more complicated. Here only a part of it holds in general. In [26], §6 it is proved as (6.6) that $(wxy) \wedge (wyz) \Leftrightarrow (xyz) \wedge (wxz)$ (this follows easily from b)), but **B4** as a whole cannot be satisfied in general.

Finally, for **B1** the statements (7.4) and (8.5) of [26] are substantial. According to these for every point p there exists a positive real number ρ_p such that in the sphere $S(p, \rho_p)$ **B1** holds. Consequently, **B1** holds everywhere if $\rho_p = \infty$, i.e. if geodesics are straight lines.

5. THE BETWEENNESS MODEL

An interimity model will turn into a betweenness model if one adds **B6** to **B1–B5**. The concept of a triangle, introduced in Section 3, allows us to interpret this **B6** in the following way.

The premise $\neg[abc]$ means that there exists a triangle $\triangle abc$. The other premises $(abd) \wedge (bec)$ mean that there are $d \in b/a$ and $e \in bc$, where the side bc and extension b/a have a common endpoint b.

Note that here the premises of $\mathbf{B\bar{6}}$ differ from those of **B6** only by the fact that $e \in bc$ is replaced by $e \in ac$ and so bc is changed by the side ab, which does not have a common endpoint with the extension b/a.

It is remarkable that also $\mathbf{B\bar{6}}$ is valid in a betweenness model. To prove this, first some lemmas are to be established.

Lemma 6. *In a betweenness model*

$$\neg[abc] \wedge (afb) \wedge (bdc) \wedge (cea) \Rightarrow \neg[def], \tag{7}$$

i.e. there does not exist a line intersecting all three sides of a triangle $\triangle abc$*.*

Proof. From (6) it follows that d, e, f are all distinct. Due to Lemma 1 also a, f, b are all distinct, like c, e, a. As a consequence, a, f, e are all distinct. Further, $\neg\langle afe\rangle$ because otherwise there would be, due to **B4**, $\langle afe\rangle \wedge (cea) \Rightarrow \langle aef\rangle \wedge [aec] \Rightarrow [fca]$, $(afb) \wedge [fca] \Rightarrow \langle afb\rangle \wedge [afc] \Rightarrow [bca]$, which is impossible now. Using permutations, and also (1), one obtains $\neg[afe] \wedge \neg[bdf] \wedge \neg[ced]$.

Finally, reductio ad absurdum will be used. So, let us suppose $[def]$. Then due to (1),

$$\langle def\rangle = (def) \vee (efd) \vee (fde).$$

Here it is sufficient to consider the last case when (fde), because the other two differ only by a permutation. Due to **B6**,

$$\neg[afe] \wedge (afb) \wedge (fde) \Rightarrow \exists p, (ape) \wedge (bdp);$$

due to **B4**,

$$(ape) \wedge (aec) \Rightarrow \langle aep\rangle \wedge [aec] \Rightarrow [pca] = [cap],$$

and similarly

$$(bdp) \wedge (bdc) \Rightarrow \langle bdp\rangle \wedge [bdc] \Rightarrow [pcb] = [cpb].$$

Now due to (4),

$$\neg[cab] \wedge [cap] \wedge [cpb] \Rightarrow (c = p),$$

thus $(ape) \wedge (c = p) \Rightarrow (ace) \Rightarrow \neg(aec)$, but this contradicts (cea). Hence the supposition is impossible and (7) holds, indeed. This finishes the proof.

Lemma 7. *If for ab and cd*, $c \in ab$ *and* $b \in cd$, *then* $b \in ad$ *and* $c \in ad$; *otherwise*,

$$(acb) \wedge (cbd) \Rightarrow (abd) \wedge (acd). \tag{8}$$

Proof. First the part $(acb) \wedge (cbd) \Rightarrow (acd)$ will be proved as follows.

Due to Lemma 1, **B5**, and **B1**, $(acb) \Rightarrow (a \neq c) \Rightarrow \exists e, \neg[ace]$ and $\neg[ace] \Rightarrow (c \neq e) \Rightarrow \exists f, (cef)$. Due to (1) and (3), $(acb) \wedge \neg[ace] \Rightarrow \langle cab\rangle \wedge \neg[cae] \Rightarrow \neg[cbe]$ and $(cef) \wedge \neg[ceb] \Rightarrow \neg[cfb]$. Further, due to **B2** and **B6**, $\neg[bcf] \wedge (bca) \wedge (cef) \Rightarrow \exists g, (bgf) \wedge (aeg)$ and $\neg[cbf] \wedge (cbd) \wedge (bgf) \Rightarrow \exists h, (chf) \wedge (dgh)$.

Again, due to (1) and (3), $\langle aeg\rangle \wedge \neg[aeg] \Rightarrow \neg[abc] = \neg[acg]$. On the other hand, (5) gives $(acb) \wedge (cbd) \Rightarrow \langle acd\rangle$ and now due to (3), $\langle acd\rangle \wedge \neg[acg] \Rightarrow \neg[adg] = \neg[dga]$. So, due to **B6**, $\neg[dga] \wedge (dgh) \wedge (gea) \Rightarrow \exists i, (dia) \wedge (hei)$.

It remains to show that $i = c$. First, due to **B4**, $(cef) \wedge (chf) \Rightarrow \langle cfe\rangle \wedge [cfh] \Rightarrow [ceh]$; similarly, $(hei) \wedge [ceh] \Rightarrow \langle ehi\rangle \wedge [ehc] \Rightarrow [eic]$ and $(dia) \wedge \langle acd\rangle \Rightarrow \langle adi\rangle \wedge [adc] \Rightarrow [aic]$. Now, (4) gives $\neg[cae] \wedge [cai] \wedge [sie] \Rightarrow (c = i)$, thus $(dia) \wedge (c = i) \Rightarrow (dca) \Rightarrow (acd)$, indeed.

The remaining part follows easily: $(acb) \wedge (cbd) \Rightarrow (dbc) \wedge (bca) \Rightarrow (dba) \Rightarrow (abd)$. This finishes the proof.

Lemma 8. *If for ab and ad, $c \in ab$ and $b \in ad$, then $b \in cd$ and $c \in ad$; otherwise*

$$(acb) \wedge (abd) \Rightarrow (cbd) \wedge (acd). \tag{9}$$

Proof. First let us prove the part

$$(acb) \wedge (abd) \Rightarrow (acd). \tag{9'}$$

By the arguments used above one can find e and f, so that $\neg[abe] \wedge (eaf)$, and then g and h, so that $(bge) \wedge (fcg)$ and $(ahe) \wedge (dgh)$. Due to (7), $\neg[abe] \wedge (acb) \wedge (ahe) \wedge (bge) \Rightarrow \neg[cgh]$, and due to (3), $\langle gcf\rangle \wedge \neg[gch] \Rightarrow \neg[gfh]$. Now, **B6** gives $\neg[hgf] \wedge (hgd) \wedge (gcf) \Rightarrow \exists i, (haf) \wedge (dci)$. But here $i = a$ can be established in the same way as in the previous proof. Thus $(dci) \wedge (i = a) \Rightarrow (dca) = (acd)$, indeed.

It remains to prove the other part $(acb) \wedge (abd) \Rightarrow (cbd)$. Here $b \neq c$ and $b \neq d$, but also $c \neq d$, because otherwise $(abd) = (abc)$ and, due to **B3**, $\neg(acb)$, which is impossible.

Further, $(acb) \wedge (abd) \Rightarrow \langle bac\rangle \wedge [bad]$, and this, due to **B4**, gives $[cdb]$ for three different c, d, b, thus $\langle cdb\rangle = (cdb) \vee (dbc) \vee (bcd)$. It remains to show that here only the middle alternative can occur; the other two lead to contadictions.

For $(cdb) = (bdc)$ this follows easily: from the part already proved $(bdc) \wedge (bca) \Rightarrow (bda) = (adb) \Rightarrow \neg(abd)$.

For $(bcd) = (dcb)$ this is not so easy. Because of **B5, B1**, $(acb) \Rightarrow (a \neq b) \Rightarrow \exists e, \neg(abe) \Rightarrow (e \neq a) \Rightarrow \exists f, (eaf)$. Now, due to (3), $\langle abd\rangle \wedge \neg[abe] \Rightarrow \neg[ade]$ and $\langle bad\rangle \neg[bae] \Rightarrow \neg[bde]$. From **B6** now $\neg[eab] \wedge (eaf) \wedge (acb) \Rightarrow \exists g, ((egb) \wedge (fcg))$. From the part already proved, $(acb) \wedge (abd) \Rightarrow (acd)$, and from **B6**, $\neg[eac] \wedge (eaf) \wedge (acd) \Rightarrow \exists h, ((ehd) \wedge (fch))$. Now, due to **B4**, $(fch) \wedge (fcg) \Rightarrow \langle fch\rangle \wedge [cfg] \Rightarrow [chg]$, but, on the other hand, due to (7), $\neg[bde] \wedge (bcd) \wedge (bge) \wedge (dhe) \Rightarrow \neg[cgh]$. A contradiction occurs here and this finishes the proof of (9).

Lemma 9. *If $c, d \in ab$, then either $c \in ad$, or $d \in ac$, or $c = d$; otherwise*

$$(acb) \wedge (adb) \Rightarrow (acd) \vee (adc) \vee (c = d). \tag{10}$$

Proof. Due to **B4**, $(acb) \wedge (adb) \Rightarrow \langle abc \rangle \wedge [abd] \Rightarrow [cda] = (cda) \vee (dac) \vee (acd) \vee (c = d) \vee (d = a) \vee (a = c)$. Due to Lemma 1, here $d = a$ and $a = c$ are impossible. But also (dac) is impossible, because from (8) it would follow that $(dac) \wedge (acb) \Rightarrow (dcb) \wedge (dab)$ and, due to **B2, B3**, $(dab) = (bad) \Rightarrow \neg(bda) = \neg(adb)$, which gives a contradiction.

Lemma 10. *Likewise*

$$(abc) \wedge (abd) \Rightarrow (acd) \vee (adc) \vee (c = d). \tag{11}$$

Proof. Due to **B4**, $(abc) \wedge (abd) \Rightarrow \langle abc \rangle \wedge [abd] \Rightarrow [cda] = (cda) \vee (dac) \vee (acd) \vee (c = d) \vee (d = a) \vee (a = c)$. Due to Lemma 1, here $d = a$ and $a = c$ are impossible. But also (dac) is impossible, because from (9) it would follow that $(cba) \wedge (cad) \Rightarrow (bad)$, and, due to **B2, B3**, $(bad) = (dab) \Rightarrow \neg(dba) = \neg(abd)$, which gives a contradiction.

Note that in (10) the premise is, due to **B2**, symmetric with respect to a, b. Thus

$$(acb) \wedge (adb) \Rightarrow (bcd) \vee (bdc) \vee (c = d). \tag{10$'$}$$

Here $(acd) \wedge (bcd)$ is impossible because it leads, due to **B2**, to $(dca) \wedge (dcb)$ and this, due to (11), to $(dab) \vee (dba) \vee (a = b)$, which contradicts (adb) (see **B1, B2** and Lemma 1). Thus

$$(acb) \wedge (adb) \Rightarrow [(acd) \wedge (bdc)] \vee [(adc) \wedge (bcd)] \vee (c = d), \tag{10$''$}$$

where each component in the conclusion excludes the other two, which is easy to control.

Now we are able to prove

Theorem 11. *Every betweenness geometry is also a betwixtness geometry, i.e. if* **B1–B6** *hold, then also* $\mathbf{B\bar{6}}$ *holds.*

Proof. It must be proved that $\neg[abc] \wedge (abd) \wedge (aec) \Rightarrow \exists f, ((bfc) \wedge (dfe))$. Due to Lemma 1 and **B1**, $(aec) \Rightarrow (a \neq e) \Rightarrow \exists g, (eag)$. Due to **B6**, $\neg[ead] \wedge (eag) \wedge (abd) \Rightarrow \exists h, ((ehd) \wedge (gbh))$. Now from **B2** and (8) it follows that $(aec) \wedge (eag) = (cea) \wedge (eag) \Rightarrow (ceg)$ and again, due to **B6**, $\neg[ced] \wedge (ceg) \wedge (ehd) \Rightarrow \exists i, ((cid) \wedge (ghi))$. From (9) and **B2** it follows that $(gbh) \wedge (ghi) \Rightarrow (bhi) = (ihb)$ and again, due to **B6**, $\neg[cib] \wedge (cid) \wedge (ihb) \Rightarrow \exists f, ((cfb) \wedge (dhf))$. Now, due to **B2** and (11), $(dhf) \wedge (ehd) = (dhf) \wedge (dhe) \Rightarrow (dfe) \vee (def) \vee (f = e)$, and from **B6** it follows, due to $\neg[abc] \wedge (abd) \wedge (bfc)$, that here only (dfe) is possible. This finishes the proof.

The following theorem can be proved now as well.

Theorem 12. *The interval ab is not empty but is an infinite subset.*

Proof. Here $a \neq b$ and, due to **B5**, $\exists c, \neg[abc]$, thus $b \neq c$. Due to **B1**, $\exists d, (bcd)$, thus $\langle bcd \rangle$. Now, due to (3), $\langle bcd \rangle \wedge \neg[bca] \Rightarrow \neg[bda]$, thus $(a \neq d)$. The same **B1** gives $\exists e, (ade)$. Here b, d, a must be noncollinear, because b, d, c are correct and otherwise, because of **B4**, b, c, a would be collinear, which is now impossible. Due to **B6**, $\neg[adb] \wedge (ade) \wedge (dcb) \Rightarrow \exists f, ((ecf) \wedge (afb)) \Rightarrow f \in ab$.

The same argument gives $\exists g, (agf)$ and, due to Lemma 1, $g \neq f$. But from (8), $(agf) \wedge (afb) \Rightarrow (agb)$, thus also $g \in ab$. So this can be continued until infinity. This finishes the proof.

Theorem 13. *For a triangle $\triangle abc$, the subset $\{x | \exists y, (byc) \wedge (axy)\}$ does not depend on the reordering of vertices a, b, c.*

Proof. Here b, c can be interchanged, due to **B2**. Thus only the interchanging of a, b is to be considered.

Due to (3), $\neg[abc] \wedge (byc) \Rightarrow \langle cby \rangle \wedge \neg[cba] \Rightarrow \neg[cya]$. Now, due to **B6**, $\neg[cya] \wedge (cyb) \wedge (yxa) \Rightarrow \exists z, ((cza) \wedge (bxz))$. Here a, b are interchanged, indeed. This finishes the proof.

It is natural to call the subset considered in Theorem 13 the *interior* of the triangle $\triangle abc$. Here any permutation of a, b, c is admissible.

The interpretation of the Pasch postulate **P** can be detailed as follows. Its premises mean that there is a line L_{ed}, which is determined by a point e of a side bc of the triangle $\triangle abc$ and a point d of the plane P_{abc} of this triangle, and does not contain any of its vertices. The assertion is that this line must intersect at least one of the other two sides in a point f. (Both of them cannot intersect, because this is excluded by Lemma 6.) Briefly: if a line in a plane of a triangle intersects one side and does not contain any of vertices, then it intersects one of the other two sides (but not both of them).

Theorem 14. *In the betweenness geometry the Pasch postulate is valid.*

Proof. Since $P_{abc} = Q_a \cup Q_b \cup Q_c$, the point d belongs to one of Q_a, Q_b, Q_c. If $d \in Q_c$, then either $d \in L_{ca}$, or $d \in L_{cb}$, or $d \in L_{cx}$, where $x \in ab$. In the first two cases one can use **B6**, or $\mathbf{B\bar{6}}$, for $\triangle abc$ to obtain the needed point f, or simply $f = d$. In the third case one can use the same **B6**, or $\mathbf{B\bar{6}}$, for $\triangle axc$, or $\triangle xbc$.

Note that in **P** the vertices a and b can be interchanged. So it suffices to consider the possibility $d \in Q_a$. The case $d \in L_{ca}$ was analysed above, but $d \in L_{ba}$ is impossible now (then L_{ed} would contain a and b). Thus only the case remains where $d \in L_{ax}$ with $x \in bc$. If $d \in a/x$ or $d \in x/a$, then **B6** or $\mathbf{B\bar{6}}$ can be used for $\triangle axb$. If $d \in ax$, then **B6** can be used for $\triangle axb$ to obtain h, so that $(ahb) \wedge (cdh)$ and now again **B6** can be used for $\triangle ahc$ or $\triangle hbc$ to obtain f in ac or bc, respectively. This finishes the proof.

Also the following holds.

Theorem 15. *In an interimity geometry, the Pasch postulate* **P** *yields* **B6**, *i.e. one can obtain a betweenness geometry adding* **P**, *instead of* **B6**, *to* **B1**–**B5**.

Proof. Let us consider the premises of **B6**. Here (abd) means, due to definitions of Q_a and P_{abc}, that $d \in P_{abc}$. Moreover, $d \notin L_{bc}$, because otherwise, if $d \in L_{bc}$, one would have, due to Lemma 4, that $L_{bc} = L_{bd}$, but on the other hand, here $d \in L_{ab}$, thus $L_{ab} = L_{bd}$, hence there would be $L_{ab} = L_{bc}$, which contradicts the premise $\neg[abc]$. Similarly $a \notin L_{de}$ due the same argument. It follows that the premises of **P** are satisfied. The same argument as above gives that here (afb) is impossible and only (def) can occur. This finishes the proof.

Theorems 15 and 11 together give that in an interimity geometry **P** yields also $\mathbf{B\bar{6}}$. This can be proved, of course, also directly using the same argument as above.

Let us stop here temporarily the treatment of betweenness geometry and turn to our main topic, to the relationship between betweenness and join geometries.

6. FROM BETWEENNESS GEOMETRY TO EXCHANGE ORDERED JOIN GEOMETRY

If one wants to proceed from betweenness geometry to join geometry, one has to introduce in $(S, \mathbf{B})$ first a join operation.

For any two different points $a, b \in S$ let the join $a \cdot b$ be defined as the interval ab, i.e. if $a \neq b$, then $a \cdot b = ab$; moreover, let $a \cdot a = a$.

Theorem 16. *A betweenness model $(S, \mathbf{B})$ with this join operation turns to be a join space with exchange ordered join geometry.*

Proof. One has to show that here **J1–J7** are satisfied. **J1** (for $b = a$) and **J4** follow directly from the definition of $a \cdot a$, and **J2** from **B2**. **J1** for $a \neq b$ follows directly from Theorem 12.

If in **J3** a, b, c are noncollinear, then $(a \cdot b) \cdot c = \{x | \exists y, (ayb) \wedge (yxc)\}$ and, due to Theorem 13, this does not depend on the reordering of a, b, c, so that **J3** for this case holds.

If $a = b = c$, then both sides of **J3** are simply a. If $a = b \neq c$, then $(a \cdot a) \cdot c = a \cdot c = ac = \{y | (ayc)\}$ and $a \cdot (a \cdot c) = \{x | \exists y, (axy) \wedge (ayc)\}$, but here, due to $(9')$, $(axy) \wedge (ayc) \Rightarrow (axc)$, so that $a \cdot (a \cdot c) = \{x | (axc)\} = ac$ as well. Thus **J3** is satisfied. In the remaining cases $a \neq b = c$ and $a = c \neq b$ the control is similar.

This shows that $(S, \mathbf{B})$ is at least a join system.

Further, in **J5–J7** $a/b = \{x | b \cdot x \supset a\}$ is now $a/b = \{x | (bax)\}$ for $a \neq b$ and $a/a = \{x | a \cdot x \supset a\} = a$. Here **J5** follows immediately from **B1**, and **J7** is satisfied trivially. It remains to prove that also **J6** holds. For this, the following cases are to be considered.

If $a = b$ and $c = d$, then $a/b = a/a = a$, $c/d = c/c = c$, thus $a/b \approx c/d$ in **J6** means that $a = c$, so that $a = b = c = d$ and $a \cdot d = b \cdot c = a$, thus $a \cdot d \approx b \cdot c$, indeed.

If $a = b$, but $c \neq d$, then $a/b = a/a = a$ and $a/b \approx c/d$ means that $a \in c/d$, which is equivalent to (acd), thus $a \in L_{cd}$ and, due to Lemma 1, a, c, d are all different. Hence, due to Lemma 4, $L_{ad} = L_{ac}$ and either $ad \subset ac$ or $ac \subset ad$, so that $a \cdot d \approx b \cdot c = a \cdot c$, indeed.

In general, $a \neq b$ and $b \neq d$. Now $a/b \approx c/d$ means that $\exists x, (xab) \wedge (xcd)$. Here two subcases are to be treated separately.

If x, b, c are noncollinear, then, due to Theorem 11 and $\mathbf{B\bar{6}}$, $\neg[xcb] \wedge (xcd) \wedge (xab) \Rightarrow \exists y, (cyb) \wedge (dya)$, thus $y \in a \cdot d$ and $y \in b \cdot c$, so $a \cdot d \approx b \cdot c$, indeed.

If x, b, c are collinear, then $b \in L_{xc}$, moreover $d \in L_{xc}$, so that $x, b, c, d \in L_{xc}$. Due to Lemma 4, $L_{xb} = L_{xc}$ and since (xab), also a belongs to the same line. Here, due to $x \in a/b$ and $x \in c/d$, there exist several possibilities for the allocation of a, b, c, d on this line. For each of them it can be shown that $a \cdot d \approx b \cdot c$.

All this shows that $(S, \mathbf{B})$ is a join space. Moreover, its geometry is an exchange join geometry. Indeed, Lemma 4 shows that here **E** is satisfied, because L_{ab} in betweenness geometry and l_{ab} in join geometry are the same subsets, as is seen from their decompositions in Sections 3 and 2, respectively.

Also **O** is here satisfied. Indeed, three distinct a, b, c of a line are collinear, thus $[abc] = (abc) \vee (bca) \vee (cab)$ holds, but this is exactly $b \subset a \cdot c$, or $c \subset b \cdot a = a \cdot b$, or $a \subset c \cdot b = b \cdot c$. This finishes the proof.

7. FROM EXCHANGE JOIN GEOMETRY TO BETWIXTNESS AND BETWEENNESS GEOMETRIES

Let now the converse be investigated. So let one have a join space $(S, \cdot)$ with the exchange join geometry, i.e. **J1–J7** and **E** hold.

In join geometry the points $a_1, ..., a_m$ are called *linearly dependent* if $a_i \subset < a_1, ..., a_{i-1}, a_{i+1}, ..., a_n >$ for some $i, 1 \leq i \leq m$. The set $\{a_1, ..., a_n\}$ of linearly independent points for which $< a_1, ..., a_n >= S$ is called the *basis* of exchange join geometry and $n - 1$ is called its *dimension* (see [6], Section 11.6).

The betweenness relation $(...)$ in join geometry is introduced as follows (see Section 1 above):

$$(abc) = (a \neq c) \wedge b \subset a \cdot c.$$

Theorem 17. *The betweenness relation turns a join space $(S, \cdot)$, with exchange join geometry and dimension > 1, into a betwixtness model.*

Proof. Here **B1** and **B2** follow immediately from **J1** and **J2**. To establish that **B3**: $(abc) \Rightarrow \neg(acb)$ holds, let us use reductio ad absurdum. So let together with (abc) also (acb) hold; in terms of join geometry, at the same time $a \neq c$, $b \subset a \cdot c$, and $c \subset a \cdot b$. From this, by eliminating b and using **J3, J4**, one would get $c \subset a \cdot (a \cdot c) = (a \cdot a) \cdot c = a \cdot c$, so $c \subset a \cdot c$, but this contradicts Theorem 4.9 in [6], which asserts that if $a \neq c$, then $a \cdot c \not\supset a, c$. (The proof of this assertion by reductio ad absurdum is simple: suppose $a \cdot a \supset c$; then due to **J2** $c \cdot a \supset c$, thus $a \subset c/c = c$ by **J7**, that is $a = c$, but this contradicts the supposition.)

For **B4** we have first to interpret $\langle abc \rangle = (abc) \vee (bca) \vee (cab)$. This means that a, b, c are all different and $(b \subset a \cdot c) \vee (c \subset b \cdot a) \vee (a \subset c \cdot b)$. Thus

$[abd] = (b \subset a \cdot d) \vee (d \subset b \cdot a) \vee (a \subset d \cdot b) \vee (a = b) \vee (b = d) \vee (d = a)$. Since in **B4** $a \neq b$, but a/b in both join and betweenness geometries is the same set (indeed, $a/b = \{x | b \cdot x \supset a\} = \{x | (xab)\}$; cf. Theorem 10), also $< ab >$ and L_{ab} is the same set, as follows from their common decomposition $ab \cup (a/b) \cup (b/a) \cup a \cup b$. Hence in **B4** $\langle abc \rangle \wedge [abd]$ is equivalent to $(c, d \subset < ab >) \wedge (c \neq a)$. Now due to **E** $< ac >=< ab >$ and since $d \in L_{ab}$ yields $d \in L_{ac}$, due to L_{ab}=$< ab >=< ac >$=L_{ac}, so $[dac] = [cda]$, as is needed in **B4**.

B5 is a consequence from the assumption that dimension of the join space is > 1.

Finally, $\mathbf{B\bar{6}}$ follows from **J6**. Indeed, $\neg[abc] \wedge (abd) \wedge (aec)$ means in join geometry that $a \subset b/d$ and $a \subset e/c$, so that $b/d \approx e/c$. Due to **J6** now $b \cdot c \approx d \cdot e$, thus f exists so that $f \subset b \cdot c$ and $f \subset d \cdot e$. Remembering here the definition of *between* in join geometry, one sees that $(bfc) \wedge (dfe)$, as is needed for $\mathbf{B\bar{6}}$. This finishes the proof.

Recall that among exchanged join geometries there are ordered join geometries.

Theorem 18. *Every ordered join geometry is also a betweenness geometry.*

Proof. An ordered join geometry, as well as an exchanged join geometry, is also a betwixtness geometry, as was just established in the previous theorem. In [6], Section 12.23, Exercise 2, it is asserted that in ordered join geometry, moreover, the Pasch postulate **P** is valid. (Here it can be noted that Pasch postulate is the same for the join space and the interimity model because the line is the same, as is established above, and likewise the plane is the same. Indeed, the decomposition $P_{abc} = Q_a \cup Q_b \cup Q_c$ in interimity model holds also for ordered join geometry, as follows from Theorem 12.20 of [6], namely from its particular case for $< a_1, a_2, a_3 >$.) Further, it is easy to prove that **P** yields **B6**. Indeed, the premises $\neg[abc] \wedge (abd) \wedge (bec)$ of **B6** say that d and e satisfy the conditions stated in premises of **P**. So also the assertion of **P** is valid. But (afb) is here impossible, because then $d \subset < ab >$ and $f \subset < ab >$ would hold in this join geometry, thus due to Theorem 11.1 in [6] one would have $< de >=< ab >$, which contradicts a premise of **P**. This finishes the proof.

Note that there exist examples of exchange join geometries which are not ordered join geometries. One such simple example is given in [6], Section 12.1, for dimension 1, but there are indicated also other examples in dimension n; one of these can be found in [23], pp. 62–68. Among the last examples there exist also the betwixtness geometries which are not betweenness geometries. This shows that although Pasch postulate yields both **B6** and $\mathbf{B\bar{6}}$ as was established above, $\mathbf{B\bar{6}}$ does not yield **B6**.

At the present time betwixtness geometry is not as profoundly developed as betweenness geometry.

8. LINES AND PLANES IN A BETWEENNESS 3-SPACE

Theorem 14 shows that in a betweenness geometry one can use all concepts and results of an exchange ordered geometry, as derived in [6]. (Note that many of them are given alredy in [1,2].) In particular, a subset is *linear* if it is closed under extension ([6], Section 6.3). In $L =< a_1, ..., a_{n+1} >$ the points $a_1, ..., a_{n+1}$ are linearly independent if no fewer than $n+1$ of them generate L (in the sense that L is their linear hull). Then they form a *basis* of L and n is called the *dimension* of L; the denotation $n = d(L)$ will be used ([6], Section 11.6). So every line $L = L_{ab}$ has dimension 1, every plane $L = P_{abc}$ has dimension 2. An L of dimension 3 is called a 3-*space*.

Theorem 5 above has the following generalization (given in [6] as Theorem 11.8):

Theorem 19. *Let $a_1, ..., a_{n+1}$ be linearly independent. Then there is a unique linear subset of dimension n which contains $a_1, ..., a_{n+1}$, namely $< a_1, ..., a_{n+1} >$.*

For $n = 2$ and $n = 3$ this was established earlier in [1] (as Theorems 18 and 29, respectively).

Let us consider further a 3-space L and prove the following assertion (see [1], Theorem 30).

Theorem 20. *If two planes of a* 3*-space L have a common point p, then they have one more common point q, $q \neq p$, thus a common line L_{pq}. If these planes do not coincide, then all of their common points belong to this line L_{pq}.*

Proof. Let the first plane be determined as $< pab >$ and the other as $< pcd >$. On the first points e and f can be taken so that $(bpf) \wedge (afe)$. Then the 3-space considered is $L =< abce >$ and is determined by the tetrahedron with vertices a, b, c, e, edges ab, bc, ca, ae, be, ce, and faces abc, abe, bce, cae; here, e.g., $abc = (a \cdot b) \cdot c$ (using join geometry notations) is the interior of $\triangle abc$ and due to Theorem 13 (or **J3**) does not depend on the reordering of a, b, c. The opposite edges and faces are defined as usual, i.e. not having a common vertex.

For d there are, with respect to the tetrahedron above, the following four possibilities: d is collinear with 1) two vertices, or 2) one vertex and one point of some of its opposite edge, or 3) one vertex and one point of its opposite face, or 4) one point of an edge and other point of the opposite edge. (This follows from [6], Theorem 12.20, which for $n = 4$ is the nearest generalization of the statements above that $L_{ab} = ab \cup (a/b) \cup (b/a) \cup a \cup b$ and $P_{abc} = Q_a \cup Q_b \cup Q_c$; see also [1], Theorem 2.)

For the first possibility, one of the vertices a, b, e is q. For the second possibility, q is one of the points of the edges ab, ae or be.

In the third possibility, the vertices must be considered separately. For c the point of its opposite face abe, collinear with c, d, is indeed the desired q. For b, let the point of ace, which is collinear to b, d, be denoted by g. Now

$\neg[bpd] \wedge (bpf) \wedge (bgd)$, thus, due to $\mathbf{B\bar{6}}$, $\exists h, (phd) \wedge (fhg)$. For g, as a point of the interior ace of $\triangle ace$, there exists i so that $(aie) \wedge (cgi)$. Now $\neg[igf] \wedge (igc) \wedge (ghf)$, and due to **B6** $\exists q, (iqf) \wedge (chq)$. This q is the other point needed. For the remaining two vertices the situation is analogous.

In the fourth possibility, let d be collinear to points u and v of egdes ac and be, respectively. Here $L_{pv} \subset P_{abe}$ and either $a \subset L_{pv}$, then $q = a$, or, in view of Pasch postulate, L_{pv} intersects one of the other two edges ab and ae. Let it intersect ae in w, so that (vpw). Now $\neg[vuw] \wedge (vud) \wedge (vpw)$ and, due to $\mathbf{B\bar{6}}$, $\exists f, (ufw) \wedge (dfp)$; further, $\neg[auw] \wedge (auc) \wedge (ufw)$ and thus, due to **B6**, $\exists q, (aqw) \wedge (cfq)$. This q is now the point needed.

The other pairs of opposite edges can be reduced to this previous case by reordering a, b, e.

Together with points p and q, both planes above contain also the line L_{pq}. If we suppose that these planes have a common point outside this L_{pq}, then these planes would coincide due to Theorem 19, which is impossible. This finishes the proof.

The set of all lines through a fixed point o is called a *bundle* of lines; o is its *centre*. The planes through o are called the *bundle planes*.

Due to Theorem 20 every two different bundle lines determine a bundle plane containing theses lines, and two different bundle planes in a 3-space intersect in a bundle line. Hence the bundle of lines in a 3-space turns to be a *projective plane*, interpreting its lines and planes as the "points" and (straight-)"lines". Then L_{oa} and P_{oab} will be denoted by A and AB, respectively. The analogue of a "triangle" is then a *trihedron angle* $\triangle ABC$, its "vertices" A, B, C are then the *edge lines*. The bundle planes through two different edge lines of a tetrahedron angle are then called the *face planes* AB, BC, and CA.

Theorem 21 (the Desarguesian theorem). *If between the edge lines of two tetrahedron angles $\triangle ABC$ and $\triangle A'B'C'$ of a bundle of lines* (*with centre o*) *in a* 3-*space there is a one-to-one correspondence $A \to A'$, $B \to B'$, $C \to C'$, such that the bundle planes AA', BB', and CC' intersect in a bundle line $L_{od} = D$, then the intersected lines $AB \cap A'B'$, $BC \cap B'C'$, and $CA \cap C'A'$ of the corresponding face planes belong to a bundle plane.*

Proof. Let a, a', d be chosen on $L_{oa} = A$, $L_{oa'} = A'$, $L_{od} = D$ so that (ada'). Further, let b, b' be chosen on $L_{ob} = B$, $L_{ob'} = B'$ so that (dbb'), and c, c' on $L_{oc} = C$, $L_{oc'} = C'$ so that (dcc'), but $c \notin P_{abd}$. Here $P_{abc} \neq P_{a'b'c'}$. Moreover, $\neg[a'b'd] \wedge (a'da) \wedge (dbb')$. Due to **B6**, $\exists p, (a'pb') \wedge (abp)$.

Also $\neg[dbc] \wedge (dbb') \wedge (dc'c)$. Due to $\mathbf{B\bar{6}}$, $\exists q, (bqc) \wedge (b'qc')$. Similarly, $\neg[adc] \wedge (ada') \wedge (dcc')$. Due to $\mathbf{B\bar{6}}$, $\exists r, (arc) \wedge (a'c'r)$.

Now $AB \cap A'B' = L_{op}$, $BC \cap B'C' = L_{oq}$, $CA \cap C'A' = L_{or}$. Here p, q, r are common points of two different planes P_{abc} and $P_{a'b'c'}$, therefore they belong to a line, thus L_{op}, L_{oq}, L_{or} belong to a bundle plane, indeed. This finishes the proof.

Also the converse holds.

Theorem 22 (the converse Desarguesian theorem). *If between the edge lines of two tetrahedron angles $\triangle ABC$ and $\triangle A'B'C'$ of a bundle of lines* $($*with centre* $o)$ *in a* 3*-space there is a one-to-one correspondence $A \to A'$, $B \to B'$, $C \to C'$, such that the intersect lines $AB \cap A'B'$, $BC \cap B'C'$, and $CA \cap C'A'$ of the corresponding face planes belong to a bundle plane, then the bundle planes AA', BB', and CC' intersect in a bundle line $L_{od} = D$.*

Proof. Let $AA' \cap BB' = D = L_{od}$. It must be established that $CC' \supset D$. To this end, let $CD \cap C'A'$ be denoted by C_1. It suffices to prove that $C_1 = C'$.

Now for the trihedron angles $\triangle ABC$ and $\triangle A'B'C_1$ the premises of Theorem 21 are satisfied. Thus the intersected lines $AB \cap A'B'$, $BC \cap B'C_1$, and $CA \cap C_1A' = C'A'$ of the corresponding face planes belong to a bundle plane. But now on the same bundle plane lies also $BC \cap B'C'$. This means that $B'C'$ and $B'C_1$ intersect BC on the same bundle line and hence coincide. It follows that also $C_1 = C'$. This finishes the proof.

Theorem 23. *If in a* 3*-space among the points $a, a', b, b', c, c', d, d'$ any three are noncollinear and* 1) $b' \in P_{aa'b}$, 2) $c, d \notin P_{aa'b}$, 3) $c' \in P_{aa'c} \cap P_{bb'c}$, 4) $d' \in P_{aa'd} \cap P_{bb'd}$, 5) $P_{aa'b} \cap P_{cc'd} \neq \emptyset$*, then c, d, c', d' belong to a plane.*

Proof. If $L_{aa'}$ and $L_{bb'}$ intersect in a point o, then $P_{aa'c} \cap P_{bb'c} \ni o$. Due to Theorem 20 the intersection line $P_{aa'c} \cap P_{bb'c}$ goes through o. The same holds also for $P_{aa'd} \cap P_{bb'd}$ and so the assertion is valid.

If $L_{aa'} \cap L_{bb'} = \emptyset$, one can choose p so that (abp) and $b' \in L_{bb'}$ so that $(a'b'p)$. Due to premise 5) and Theorem 20, $P_{aa'b} \cap P_{cc'd}$ is a line, on which points e and e' can be chosen so that $\exists q, (bqe') \wedge (b'qe)$. Due to **B6**, $\exists r, (a're) \wedge (pqr)$.

Let us consider, in the bundle of lines with centre c, the trihedrons determined by a, b, e' and by a', b', e, respectively. The bundle planes $P_{caa'}$, $P_{cbb'}$, and $P_{cee'}$ intersect in a bundle line $L_{cc'}$. Due to Theorem 21 the intersection lines of the corresponding face planes belong to a bundle plane. It follows that $L_{ae'} \ni r$ and so, in the bundle of lines with centre d, the corresponding face planes of trihedrons, determined by a, b, e' and by a', b', e, intersect in lines belonging to the bundle plane P_{dpq}. Now, due to Theorem 22, the bundle planes of the corresponding edge lines, among them also $P_{dee'}$, intersect in the bundle line $L_{dd'}$. But this $P_{dee'}$ contains both $L_{cc'}$ and $L_{dd'}$. This finishes the proof.

9. LINEARLY ORDERED SKEW FIELDS AND COORDINATES

A well-known construction allows us to introduce coordinates in the projective space as points-symbols, and to define the addition and multiplication operations for these symbols, using the Desarguesian postulate, so that as a result a skew field is obtained (see [27], also [28] and [29], Ch. 20). By this and Theorems 21–23 one can introduce the coordinates from a linearly ordered skew field into the betweenness geometry (and due to Theorems 16 and 18 also into the ordered join geometry) so that the considered model (join space) is isomorphic to a convex region of a linear space over an linearly ordered skew field.

What follows shows shortly how to realize this programme.

The bundle of lines with centre o in a 3-space was considered above as a projective plane, where L_{oa}, L_{ob}, ... are interpreted as the "points" $A, B, ...$, and P_{oab} is interpreted as a (straight) "line" AB. In [27], Ch. VI, §5, two constructions are given.

Let on a "line" AB three "points" O, E, U be given so that U is different from A, B. The "points" P, Q can be chosen so that P, Q, U are "collinear" (i.e. belong to a "line").

I. In general, let R, S be chosen so that R, P, A are "collinear", Q, R, O are "collinear", S, Q, B are "collinear", and R, S, U are "collinear". Then T_I on AB, which is "collinear" with P and S, is interpreted as $T_I = A + B$.

In [27], Ch. VI, §§5, 7, it is proved that the allocation of T depends only on O, U, A, B and does not depend on the choice of P, Q, "collinear" with U. It is also established that $A + B = B + A$ and that the "points" of "line" AB, excluding U, with respect to this "+" constitute a commutative group. (Note that if we turn the projective plane into an affine plane with "improper points U, P, Q", the above construction turns to the parallel transport of the segment $[OB]$, so that O coincides with A, i.e. to the classical addition of segments.)

II. In general, let R, S be chosen so that R, P, A are "collinear" and S, Q, B are "collinear" as above, but now Q, R, E are "collinear" and R, S, O are "collinear". Then T_{II} on AB, which is "collinear" with P and S is interpreted as $T_{II} = A \cdot B$.

In [27], Ch. VI, §§5, 7, it is proved that the allocation of T_{II} depends only on O, E, U, A, B and does not depend on the choice of P, Q, "collinear" with U, also that with respect to "+" and "·" the "points" of "line" AB, excluding U, constitute a skew field. Here O and E are in the role of neutral elements, i.e. of null and unit, respectively. It is established as well that if one alters the allocation of O, E, U on AB, the new skew field is isomorphic to the previous one.

Now the coordinates from the skew field can be introduced into a betweenness space of dimension > 2 as follows.

Let first a 3-space be considered. There exist four linearly independent points a_0, a_1, a_2, a_3. One can choose a point e which does not belong to any of four planes, determined by some three of them.

Considering the bundle of lines with centre a_i, $i \in \{1, 2, 3\}$, and denoting $L_{a_i a_0} = O_i$, $L_{a_i a_j} = U_k$, where the indices i, j, k have three different values, one can take $P_{a_0 a_i a_j} \cap P_{e a_i a_k}$ in the role of E_{jk} and introduce on the "line" of "collinear" O_i, E_{jk}, U_k, excluding U_k, the structure of a skew field K_{jk}. Here K_{jk} and K_{ik} are isomorphic, as is shown in [27], Ch. VI, §§6, 8, where the isomorphism is denoted by T^{jk}_{ik}; also K_{kj} and K_{jk} are isomorphic with isomorphism $T^{jk}_{kj} = H_{jk}$. Thus there exists a skew field which is isomorphic to all of them and which is called in [27] the *skew field K of this geometry.*

Let x be a point not belonging to $P_{a_1a_2a_3}$. Then $L_{a_ix_k} = P_{a_0a_ia_j} \cap P_{xa_ia_k}$ is a "point" $X_{i,k}$ of the "line" O_iE_k, which does not coincide with U_k and thus is an element of K. Here actually $X_{i,k}$ does not depend on k, i.e. $X_{i,j} = X_{i,k} = X_i$, as is shown in [1] using Theorem 23.

These X_1, X_2, X_3, as elements of K, represent in a 3-space the *coordinates* of the point x, not belonging to $P_{a_1a_2a_3}$, with respect to the *frame* $\{a_0a_1a_2a_3; e\}$.

In the betweenness model (equivalently, in a join space with ordered join geometry) also of dimension $n > 3$ by means of analogous construction one can introduce the coordinates from a skew field K with respect to a frame $\{a_0a_1...a_n; e\}$, where the points $a_0, a_1, ..., a_n$ are linearly independent, and e is linearly independent with every n of them.

Note that the projective part of this for bundles of lines with centres $a_1, ..., a_n$ can be found in [27], Ch. VI, §8, where also the following is proved.

Theorem 24. *Every projective space of dimension* n (*either with* $n > 2$ *or the Desarguesian theorem holds*) *can be represented in the form of* $P_n(K)$, *which is a set of points being in bijection with the equivalence classes in* $K^{n+1} \setminus \{0\}$, *where* $K^{n+1} = \{(x_0, x_1, ..., x_n)\}$, $x_i \in K$, $i \in \{0, 1, ..., n\}$, $\{0\} = (0, 0, ..., 0)$ *and equivalence is determined by* $(x'_i) \sim (x_i) \Longleftrightarrow \exists\lambda,\ \ x'_i = \lambda x_i$.

To be more concrete, let us return to a 3-space, considering it with respect to a frame $\{a_0a_1a_2a_3; e\}$. The coordinates above X_1, X_2, X_3 for a point x are connected with projective coordinates x_0, x_1, x_2, x_3 for bundles of lines with centres a_1, a_2, a_3 by $X_i = x_i : x_0$, $i \in \{1, 2, 3\}$. Now to the points x of the plane $P_{a_1a_2a_3}$ (these were left out above, but in projective coordinates they are determined by $x_0 = 0$) one can ascribe the symbols $x_i/0$, where x_i are the last three projective coordinates of a point of L_{a_0x}. In [27], Ch. VI, §8, Theorem III, it is proved that three points a, b, x with projective coordinates, respectively, (a_0, a_1, a_2, a_3), (b_0, b_1, b_2, b_3), (x_0, x_1, x_2, x_3), are collinear if and only if the rank of a 3×4-matrix of these coordinates is less than 3. For different a and b this means that (x_α) is a linear combination of linearly independent (a_α) and (b_α), i.e. there exist $\lambda, \mu \in K$ such that $x_\alpha = \lambda a_\alpha + \mu b_\alpha$, $\alpha \in \{0, 1, 2, 3\}$. For $X_i = x_i/x_0$, $i \in \{1, 2, 3\}$, this gives $X_i = \bar\lambda A_i + \bar\mu B_i$, where $\bar\lambda = \lambda a_0/(\lambda a_0 + \mu b_0)$, $\bar\mu = \mu b_0/(\lambda a_0 + \mu b_0)$, $A_i = a_i/a_0$, and $B_i = b_i/b_0$. Here $\bar\lambda + \bar\mu = 1$ so that $X_i = \bar\lambda A_i + (1 - \bar\lambda)B_i$. For $A = (1, 0, 0) = a_1$ and $B = (0, 0, 0) = a_0$ this gives $X = (\bar\lambda, 0, 0)$. In betweenness geometry (and also in ordered join geometry) the line $L_{a_1a_0}$ (the line $< a_1a_0 >$, respectively) is a linearly ordered set of points. Hence the skew field K of this geometry is a linearly ordered skew field, and x with coordinates $X_i = \bar\lambda A_i + (1 - \bar\lambda)B_i$ is between the points a and b with coordinates, respectively, A_i and B_i if and only if $0 < \bar\lambda < 1$ in this K.

Note that there can be triples (X_1, X_2, X_3) which do not determine any point in the betweenness geometry. Namely, the lines, determined by (X_2, X_3), (X_3, X_1), and (X_1, X_2) in bundles of lines with centres, respectively, a_1, a_2, and a_3, need not intersect in a point x, but only belong to a plane for every pair (and so determine a new object, a so-called *ideal* or *non-proper* point). Hence, one can obtain instead of the whole K^3 only a region of it, which for the betweenness geometry must be convex, of course. (For join geometry it is noted, e.g. in [6], Section 2.9.)

In general, for a betweenness geometry (ordered join geometry) of dimension $n > 3$ the result is the same, only in the deduction above $i \in \{1, ..., n\}$. All this can be summarized as follows.

Main Theorem. *A betweenness model* $($*join space with ordered join geometry*$)$ *of dimension* $n \geq 3$ *is isomorphic to a convex region of a linear space* K^n *over a linearly ordered skew field* K*, where the betweenness is determined as above.*

Remarks

1. The Main Theorem is formulated for a betweenness model in [1] with a sketch of proof. For an ordered join geometry it is probably new, as far as we know; at least we cannot find it in the monograph [6].
2. The betweenness geometries (ordered join geometries) of dimension $n \geq 3$, for whose bundles of lines the Pappus theorem is valid, correspond to the case when in the Main Theorem K is commutative, i.e. reduces to an ordered field (see [27], Ch. V, §8).
3. The betweenness planes (in [18] called *Lumiste planes*), have not been investigated sufficiently up to now. At least the Main Theorem above does not hold for $n = 2$, in general, because there exist non-Desarguesian planes. One such example, given in [30], is described in [9] (1930), §23, and [27], Ch. VI, §2. Another example is given in [26], §12: a paraboloid $z = xy$ in Euclidean E^3, where (abc) for its points means that b is between a and c on the geodesic line through the latter two (see also [31]).
4. Due to [27], Ch. VI, §9, Theorem 1, non-Desarguesian Lumiste planes are also non-Pappian.

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Ühenduvuse ja vahelsuse geomeetria vahekord

Ülo Lumiste

Prenowitzi ja Jantosciaki mahukas monograafia aastast 1979 ühenduvuse geomeetriast käsitleb põhiosas kumerate hulkade geomeetriat, kuid puudutab ka lineaargeomeetriat ja vahelsuse relatsiooni. Viimane oli (koos punkti mõistega) võetud eesti matemaatikute J. Sarve, J. Nuudi ja A. Tudebergi (Humala) poolt 1930. aastail arendatud geomeetria aluste ainsaks põhimõisteks. Sellel alusel töötas käesoleva artikli autor 1964. aastal välja soliidse vahelsuse geomeetria kui vahelsuse mudelite teooria, kuid sellal leidis see avaldamist ainult vähese rahvusvahelise levikuga väljaannetes. Nüüd, mil talle sattus kätte 1979. aasta monograafia, käsitleb ta artiklis nende kahe geomeetria vahekorda. Uuesti on antud vahelsuse geomeetria lühitutvustus vajalikus ulatuses, kusjuures eelnevalt on välja arendatud selle alaosad. Põhiosas on tõestatud, et vahelsuse geomeetria on ühtlasi järjestatud ühenduvuse geomeetria, ja vastupidi: vahetuslik ühenduvuse geomeetria langeb kokku vahelsuse geomeetria ühe alaosaga, kuid spetsiaalsem järjestatud ühenduvuse geomeetria kogu vahelsuse geomeetriaga. Ühtlasi on näidatud, et viimases, kõrgema kui kahe mõõtme juhul, kehtib Desargues'i teoreem ning seetõttu on vastav mudel isomorfne kumera hulgaga samamõõtmelises lineaarses ruumis üle teatava, täielikult järjestatud kaldkorpuse.