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On the description of stochastic systems

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Abstract. A set-up of the systemic description of a (stochastic) system, particular states of which form (on the lowest determination level of the system state) a population of random events, is presented. Elementary and hierarchic stochastic systems are considered and the structure of their description is formulated. As an application of the conception, the set-up of the description of liquid media motion is considered.

Key words: probability, system, liquid media.

1. INTRODUCTION

The generic notion of the description of any object (or phenomenon) behaving stochastically is a *random event*. The randomness of an event expresses its systemic property revealed in a population of events, each of which could have happened instead of the given event under the same conditions, specified as the object's state fixed in terms reflecting the quality of the object as an entirety. The object, particular states of which are random events, is further called a *stochastic system.* There are two different levels of presentation of the state of a stochastic system – the level of a stochastic system as an entirety and the level of a random event. The state of a stochastic system as an entirety organizes the population of random events by providing the population with a probability distribution. A population, characterized by a probability distribution, is called a *statistical ensemble*. The properties of a statistical ensemble as well as of the probability distribution are reflected by the system quality as an entirety. Particular descriptions of a stochastic system, formulated in terms characterizing the system as an entirety, its particular states, and organization of its particular states as random events expressed by the probability distribution, form the subject of systemic description. The systemic description treats a stochastic system from the point of view of all its particular descriptions and links connecting the particular descriptions.

As different formation conditions of the population of random events lead to different probability distributions, the determination of those conditions must be included into the determination of the probability distribution. Unfortunately, this obvious requirement is often forgotten. As an example, the statistical hydromechanics [1] asserts that the probability distribution for a velocity field includes (in the case of incompressible fluid) all statistical information about the flow field. This assertion is deficient. The statistical properties of the turbulent flow field, and as a consequence, the properties of the probability distribution depend also on the scales of the motions treated as turbulent. The determination of the motion scales is included into the determination of the formation conditions of probability distributions. In particular, the choice of different scales of motion may be used to reveal different properties of the turbulent flow field. Locally isotropic and homogeneous turbulence [2] and large-scale turbulence, characterized by the property of rotational anisotropy [3–5], serve as examples. The parameters, determining the formation conditions of probabilistic characteristics of an object or phenomenon under consideration, are typically missing not only in statistical hydromechanics but also in the theory of games [6], financial mathematics [7,8], theory of algorithms for signal and image processing [9], etc.

Inclusion of parameters fixing the formation conditions of the probability distribution into its determination is essential not only for specification of the information embodied into the notion "probability distribution". It becomes unavoidable, for example, when considering stochastic fields with coherent structures [10–12]. The existence of coherent structures indicates a probable ergodicity problem. If coherent structures violate the ergodicity assumption, the parameters characterizing the coherent structures must be treated as parameters determining the probability distribution, but not as characteristics of random events. The determination of the property of rotational anisotropy [3–5], caused by a preferred orientation of rotation of large-scale eddies (as specific coherent structures in a turbulent motion field), is an example.

A stochastic system, the formation conditions of which are determined uniquely, is henceforth called an *elementary stochastic system.* The dependence of the properties of a stochastic system on specific conditions of its formation allows formulation of different elementary systemic descriptions of an object or phenomenon [4]. Unlike the elementary stochastic system, the *hierarchic stochastic system* forms under hierarchy of conditions of probabilities formation. The systemic description of a hierarchic stochastic system considers particular systemic descriptions founded on considering elementary stochastic systems as contractions of a unique systemic description. Different particular systemic descriptions of a hierarchic stochastic system can be formulated for revealing different sets of details. All such systemic descriptions are considered to be equivalent as containing the same information, revealed within different systemic descriptions in different details. This property of systemic description is called *metatheoretical invariance.*

A set-up of the description of stochastic systems in the sense determined above, which is the main task of this paper, is applied to a set-up of systemic description of the hydrodynamic situation in the final section of the paper. The aspects of the approach as a generic classification scheme for different theories, formulated to describe different aspects of the hydrodynamic situation and as a recipe for the construction of new particular theories, are demonstrated. The formulated systemic description of the hydrodynamic situation not only organizes particular hydrodynamic theories and theories connected to hydrodynamics into a unique description, but serves also as a basis for the harmonization of theories involved into the systemic description and as an essential basis for the organization of leaning of the subject.

The tendency to formalize the organization of the knowledge in different fields of science and to formulate unique principles for building up the “taxonomy” of theories that are related to a particular field of knowledge is common in many fields of science like mathematics and computer science [13], and biology [14] (where it is based on the category theory). Unlike these approaches, the present paper finds not only the structure of such “taxonomy”, determined as a fixed systemic description, but also equivalency of different “taxonomies” (at least in case of stochastic systems), formulated as different systemic descriptions applied to the same object, phenomenon or field of knowledge by formulating connections between them.

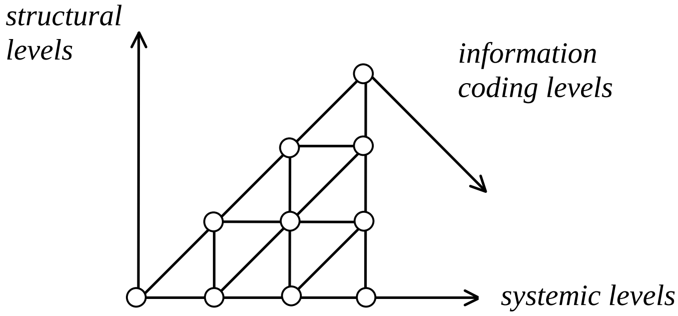
2. TERMINOLOGY AND STRUCTURE OF THE SYSTEMIC DESCRIPTION OF AN ARBITRARY SYSTEM

2.1. Notions connected to system presentation

Sign – characteristic, variable, measure, etc. conveying certain information about the state of a system.

Code – a complete set of signs the quantitative determinacy of which unequivocally fixes the state of a system. A code is defined with a precision up to an arbitrary transformation of the signs defining it, as long as the information they convey remains invariant.

Code grid – the structure



(1)

organizing the set of codes of systemic description. The nodes in code grid (1) correspond to fixed codes and the lines between the nodes to connections between the codes.

The following notions are used in (1):

structural level – level of representation of a system by its elements of definite kind;

systemic level – level of organization of system elements;

information coding level – level characterizing the type of organization of system elements; each successive information coding level expresses an order among system elements, expressed in terms of the preceding code on the same structural level.

2.2. Notions connected to systemic description

Node theory – the theory, formulated on a fixed code in code grid (1). Let $\boldsymbol{a} = \{a, b, ...\}$ denote an arbitrary code $(a, b, ...$ – signs of the code $\boldsymbol{a}$). The node theory, formulated on the code $\boldsymbol{a}$, is expressed as

$$\frac{\partial \boldsymbol{a}}{\partial t} = \mathcal{H}\,\boldsymbol{a},$$

where $\partial \boldsymbol{a}/\partial t = \{\partial a/\partial t,\ \partial b/\partial t, ...\}$ and $\mathcal{H}$ is an operator acting on signs of the code $\boldsymbol{a}$ ($\mathcal{H}\boldsymbol{a} = \{\mathcal{H}a_1, \mathcal{H}a_2, ...\}$). It ties a system state determined in terms of signs of the code $\boldsymbol{a}$ with their time derivatives.

Link theory – the theory establishing correspondence between two or more node theories.

Recoding theory – the link theory, which establishes correspondence between two node theories, formulated on “adjacent” codes in the code grid. If $\boldsymbol{a}$ and $\boldsymbol{A}$ denote such “adjacent” codes, while the systemic or structural level corresponding to $\boldsymbol{A}$ is higher with respect to the structural level corresponding to $\boldsymbol{a}$, the recording theory is formulated in the following symbolic form:

$$\boldsymbol{A} = Q_1\,\boldsymbol{a}, \qquad \frac{\partial \boldsymbol{A}}{\partial t} = Q_1 \frac{\partial \boldsymbol{a}}{\partial t},$$

where Q_1 is the operator connecting the system state fixed in terms of signs of the code $\boldsymbol{A} = \{A, B, ...\}$ $(A, B, ...$ are the signs of the code $\boldsymbol{A}$) with its states fixed in terms of signs of the code $\boldsymbol{a}$, and Q_2 denotes the operator connecting the time derivatives of signs of the codes $\boldsymbol{a}$ and $\boldsymbol{A}$.

S*ystemic theory* – the theory treating the full set of node theories related to a fixed decomposition level of a system and connected by link (recoding) theories, as a system of node theories. The number of structural levels of system representation determines the *dimension of systemic description*.

Extension and contraction of a systemic description – the procedures leading to an increase or decrease in the dimension of systemic description.

Metatheory – the description level of a system postulating equivalency of all of its systemic descriptions, linked by procedures of their extension or contraction.

2.3. Some principles of systemic description

(i) The properties of a system, expressed in different codes (i.e. treated within different node theories), do not exclude but complement each other. Some of the properties, such as determinacy and stochasticity, continuity, and discreteness, may prove mutually exclusive within one of the node theories.

(ii) The completeness of a systemic description can be attained only through using all node and link theories. If some particular theory predicted by the systemic description is absent, one can try constructing the respective theory.

(iii) All systemic descriptions of a system, differing in the character of decomposition of its states, are considered equivalent on the meta-theoretical level. The possibility of the formation of different systemic descriptions for an object or phenomenon ascribes "systemic dimension" to systemic descriptions. While different systemic descriptions realize the description of a system on different levels of its complexity, then the metatheoretical level of description turns the description invariant with respect to a specific choice of systemic description.

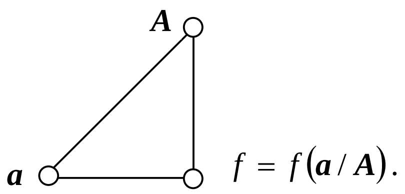
3. STOCHASTIC SYSTEMS

Let us discuss the class of *stochastic systems.* Consider the lower information coding level of systemic description (1). A state of a system, fixed by an arbitrary code on this description level, is treated as a random event realizable in the conditions determined in terms of codes of higher structural levels and as the state which determines the conditions for the formation of sets of random events fixed in terms of codes of lower structural levels. All descriptions (node theories) corresponding to the same systemic level have the same *predictability horizon*, defined as the time during which the fixed system state uniquely determines its states in the future. For this reason we shall call the systemic levels of a stochastic system as *determination levels*.

When formulating descriptions of stochastic systems we assume that there exists a mutually unambiguous correspondence between the system states determined on the codes of the lowest information coding levels and of points of the spaces determined as the spaces of possible quantitative measures of signs of these codes. We shall term these spaces as *phase spaces*. Using the notion of phase space makes it possible to characterize the states by using such metric notions as distance, proximity, etc. It should be underlined that system states indistinguishable in the phase space of any fixed structural level are distinguishable in a phase space of a lower structural level.

3.1. An elementary stochastic system

The code grid of an elementary stochastic system has the form



(2)

The signs of the code $\boldsymbol{A}$ fix the state of the system on a higher structural level as an entirety. The signs of the code $\boldsymbol{a}$ fix the system state, if characterized by the system state determined on the code $\boldsymbol{A}$, as a random event. The code $f = f(\boldsymbol{a}/\boldsymbol{A})$ fixes the state of the system as the probability distribution of system states fixed on signs of the code $\boldsymbol{a}$ and formed under conditions fixed on signs of the code $\boldsymbol{A}$.

The complete systemic description of an elementary stochastic system contains three node theories denoted as "$\boldsymbol{a}$", "$\boldsymbol{A}$", and "f", describing the behaviour of the system in terms of the codes $\boldsymbol{a}$, $\boldsymbol{A}$, and $f = f(\boldsymbol{a}/\boldsymbol{A})$, and three recoding theories, denoted as "$\boldsymbol{a}; \boldsymbol{A}$", "$\boldsymbol{a}; f$", and "$\boldsymbol{A}; f$", binding the pairs of node theories.

The code grid (2) stresses the systemic nature of any statistical description. Such a description presumes at least three particular descriptions (descriptions of events given as a unique specimen, description of the formation conditions of the stochastic system, and description of probability distribution) and links between them.

3.1.1. Node theories of the systemic description of an elementary stochastic system

Formulation of the theories "$\boldsymbol{A}$" and "$\boldsymbol{a}$" is founded on qualitative determinacy of a system as entirety and its particular states. The predictability horizon of the theory "$\boldsymbol{A}$" (as of the node theory corresponding to the highest determination level) is not bounded, while the predictability horizon of the theory "$\boldsymbol{a}$" (denoted as $T_{\boldsymbol{a}}$) is bounded. It determines a relatively short time duration starting from the initial state fixed on the code $\boldsymbol{a}$. Due to limitation of the predictability horizon $T_{\boldsymbol{a}}$, the system state fixed on the code $\boldsymbol{a}$ at some time instant determines the motion uniquely only in a small time interval. During this time interval the phase trajectories, close at an initial time instant, can neither touch nor intersect. When the time duration exceeds this interval, the motion becomes not determined by its initial state fixed on the code $\boldsymbol{a}$ and obtains random character in time.

The theory "f" determines the system state in terms of the probability distribution $f = f(\boldsymbol{a}/\boldsymbol{A})$. The theory "$f$" considers the parameters of the formation of the probability distribution, determined in terms of signs of the code $\boldsymbol{A}$, as *hidden variables.* (The sense of these parameters as determining the system state on a higher structural level is ascribed to them within systemic description.)

3.1.2. Recoding theories

The recoding theory "$\boldsymbol{a}; \boldsymbol{A}$" establishes the connection between the node theories "$\boldsymbol{a}$" and "$\boldsymbol{A}$" by the integral relations

$$\boldsymbol{A}(t) = \int \tilde{\boldsymbol{A}}(\boldsymbol{a})\, p d\boldsymbol{a}, \qquad \frac{\partial}{\partial t}\boldsymbol{A} = \int \frac{\partial \tilde{\boldsymbol{A}}}{\partial \boldsymbol{a}} \frac{d\boldsymbol{a}}{dt}\, p d\boldsymbol{a}, \tag{3}$$

where $\partial/\partial \boldsymbol{a} = \{\partial/\partial a,\ \partial/\partial b, ...\}$, $d\boldsymbol{a} = dadb...$, $\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}(\boldsymbol{a})$ denotes a finite-dimensional set of functions determined on the set of signs of the code $\boldsymbol{a}$ and $p = p(\boldsymbol{a}/[t - T/2, t + T/2])$ denotes the probability distribution determined on the Poincaré sections formed for realizations $\boldsymbol{a} = \boldsymbol{a}(t')$, where $t' \in [t - T/2, t + T/2]$, $T >> T_{\boldsymbol{a}}$. The relations (3) bind signs of the codes $\boldsymbol{a}$ and $\boldsymbol{A}$ as well as their time derivatives $d\boldsymbol{a}/dt$ and $\partial \boldsymbol{A}/\partial t$. For $\tilde{\boldsymbol{A}} = \boldsymbol{a} = \boldsymbol{a}(t)$ the second relation in (3) reduces to commutability of averaging and time derivative operators. The determination of $\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}(\boldsymbol{a})$ and of the duration of the time interval for the formation of the Poincaré sections have to be solved bearing in mind the quality of the system that is described on the level of the code $\boldsymbol{A}$.

Unlike the theory "$\boldsymbol{a}; \boldsymbol{A}$", deducing the probability distribution $p = p(\boldsymbol{a}/[t - T/2, t + T/2])$ from the real phase trajectory given in an only specimen, the theory "$\boldsymbol{a}; f$" considers the probability distribution $f = f(\boldsymbol{a}/\boldsymbol{A})$ as formed as a result of the divergence of phase trajectories for the time duration exceeding the predictability horizon of the description on the level of the code $\boldsymbol{a}$. The formation in time of a variety of virtual phase trajectories (within which one trajectory is real but does not differentiate from the others) determines a set of phase points at each fixed time instant. The set of *virtual states* corresponding to the phase points has the following properties: (i) the number of virtual states keeps growing unboundedly in time, (ii) the ratio of the number of virtual states with phase points in any given subdomain of the phase space to the total number of virtual states tends to a limit, and (iii) any subset of virtual states transforms in time into a set with a structure indistinguishable from the initial one.

The properties (i) and (ii) legitimize the probabilistic description of the set of virtual states determined on the code $\boldsymbol{a}$ as random events and justify the description of the situation by the probability distribution $f = f(\boldsymbol{a}/\boldsymbol{A})$. The property (iii) declares the conservation of a system's quality, expressed as conservation of $f = f(\boldsymbol{a}/\boldsymbol{A})$ for the motion of phase points,

$$\frac{\partial f}{\partial t} = -\frac{\partial f}{\partial \boldsymbol{a}} \frac{d\boldsymbol{a}}{dt}. \tag{4}$$

The condition (4) binds $\partial f/\partial t$ and $\partial \boldsymbol{a}/\partial t$ and, together with the foundation of specific properties of $f = f(\boldsymbol{a}/\boldsymbol{A})$, forms the essence of the theory "$\boldsymbol{a}$; f".

The theory "$\boldsymbol{A}$; f" establishes the connection between the node theories "$\boldsymbol{a}$" and "f" by the integral relations

$$\boldsymbol{A} = \int \boldsymbol{A}^*(\boldsymbol{a}) f d\boldsymbol{a}, \qquad \frac{\partial \boldsymbol{A}}{\partial t} = \int \boldsymbol{A}^*(\boldsymbol{a}) \frac{\partial}{\partial t} f \, d\boldsymbol{a}, \tag{5}$$

where $\int F(\boldsymbol{a})\, d\boldsymbol{a} \equiv \int_a \int_b \ldots \int F(a,b...)\, dadb...$, with $F(\boldsymbol{a})$ denoting an arbitrary function from signs of the code $\boldsymbol{a}$, and $\boldsymbol{A}^* = \boldsymbol{A}^*(\boldsymbol{a})$ (like $\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}}(\boldsymbol{a})$ in the theory "$\boldsymbol{a}; \boldsymbol{A}$") denotes a finite-dimensional set of functions determined on the basis of a set of signs of the code $\boldsymbol{a}$. When expanding $\boldsymbol{A}^*(\boldsymbol{a})$ in the first expression in (5) into Taylor series by signs of the code $\boldsymbol{a}$, this relation becomes equivalent to determination of signs of the code $\boldsymbol{A}$ as a set of probabilistic moments of the distribution function $f = f(\boldsymbol{a}/\boldsymbol{A})$. As the node theories "$\boldsymbol{A}$" and "$f$" belong to the same determination level, they can be considered as expressing the same quality on different levels of the given structural representation of the system. Determination of this quality itself cannot be solved within the theory "$\boldsymbol{A}$; f". It only declares the type of connection between the node theories "$\boldsymbol{A}$" and "f".

3.1.3. Systemic description of an elementary stochastic system

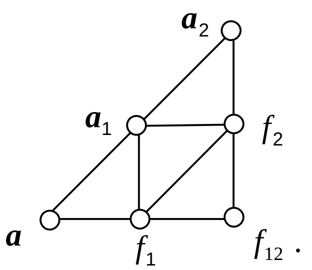
The systemic description of an elementary stochastic system coordinates the node and recoding theories by the following restrictions:

$$\boldsymbol{A}^* \equiv \tilde{\boldsymbol{A}}, \qquad f = p. \tag{6}$$

The restrictions (6) harmonize particular descriptions included into the systemic description and turn the description, despite variance of particular descriptions, into entirety. The essence of the systemic description as harmonizing particular descriptions imitates the objectivity of the described object or phenomenon. An object or phenomenon can be studied by means of different particular theories. All these particular theories have specific questions to answer, and answers to the same question set up by different particular theories must coincide.

3.2. A stochastic system of the second order

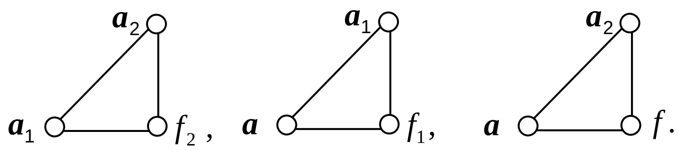
The code grid of a stochastic system of the second order has the form



(7)

In (7), $f_1 = f_1(\boldsymbol{a}/\boldsymbol{a}_1)$ and $f_2 = f_2(\boldsymbol{a}_1/\boldsymbol{a}_2)$ are probability densities of states fixed on the levels of the codes $\boldsymbol{a}$ and $\boldsymbol{a}_1$ formed in the conditions fixed on the levels of the codes $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$; $f_{12} = f_{12}[f_1(\boldsymbol{a}/\boldsymbol{a}_1)/\boldsymbol{a}_2]$ is probability density for f_1, formed in the conditions fixed on the level of the code $\boldsymbol{a}_2$.

The description (7) allows of the next three contractions to the level of an elementary stochastic system:



(8)

In (8),

$$f = f(\boldsymbol{a}/\boldsymbol{a}_2) = \int f_1 f_{12} df_1, \tag{9}$$

where $\int f(\boldsymbol{a}/\boldsymbol{a}_2)\, d\boldsymbol{a} = 1$. In accordance with (9), the probability distribution $f = f(\boldsymbol{a}/\boldsymbol{a}_2)$ is defined as the distribution f_1 averaged by the distribution f_{12}.

According to the definition of the elementary stochastic system, the links between the codes $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}$ are determined as follows:

$$\boldsymbol{a}_1 = \int \tilde{\boldsymbol{a}}_1(\boldsymbol{a})\, f_1(\boldsymbol{a}/\boldsymbol{a}_1)\, d\boldsymbol{a}$$

and

$$\boldsymbol{a}_2 = \int \tilde{\boldsymbol{a}}_2(\boldsymbol{a})\, f_1(\boldsymbol{a}/\boldsymbol{a}_2)\, d\boldsymbol{a} = \int \tilde{\boldsymbol{a}}_2^*(\boldsymbol{a}_1)\, f_2(\boldsymbol{a}_1/\boldsymbol{a}_2)\, d\boldsymbol{a}_1, \tag{10}$$

where $\tilde{\boldsymbol{a}}_1(\boldsymbol{a})$, $\tilde{\boldsymbol{a}}_2^*(\boldsymbol{a}_1)$, and $\tilde{\boldsymbol{a}}_2(\boldsymbol{a})$ are given sets of functions on the signs of their arguments, while the probability distributions $f_1 = f(\boldsymbol{a}/\boldsymbol{a}_1)$, $f_2 = f(\boldsymbol{a}_1/\boldsymbol{a}_2)$, and $f = f(\boldsymbol{a}/\boldsymbol{a}_2)$ are interlinked by the relation

$$f(\boldsymbol{a}/\boldsymbol{a}_2) = \int f_2\, f_1 d\boldsymbol{a}_1. \tag{11}$$

The link between $\tilde{\boldsymbol{a}}_2^*(\boldsymbol{a}_1)$ and $\tilde{\boldsymbol{a}}_2(\boldsymbol{a})$

$$\tilde{\boldsymbol{a}}_2^*(\boldsymbol{a}_1) = \int \tilde{\boldsymbol{a}}_2(\boldsymbol{a}) f_1 d\boldsymbol{a}$$

follows from (10) and (11).

Defining the averaging operators

$$\mathcal{P}_1 = \int \ldots f_1 d\boldsymbol{a}, \quad \mathcal{P}_2 = \int \ldots f_2 d\boldsymbol{a}_1, \quad \mathcal{P} = \int \ldots f d\boldsymbol{a},$$

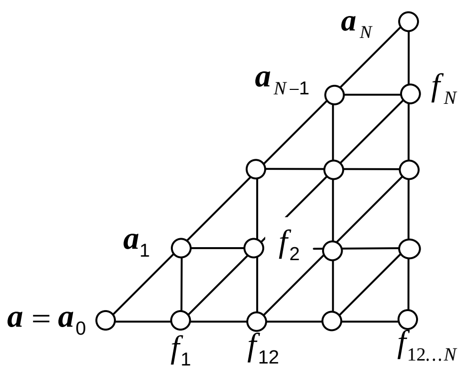
we have for $\mathcal{P}$

$$\mathcal{P} = \mathcal{P}_2 \mathcal{P}_1. \tag{12}$$

The comparison of the first and the third part of the description (8) shows that the state determined on the level of the code $\boldsymbol{a}_2$ can be decomposed into random events of two different kinds, determined on the codes $\boldsymbol{a}$ and $\boldsymbol{a}_1$. Correspondingly, the quality of the system described on the code $\boldsymbol{a}_2$ can be found on two different probability distributions $f = f(\boldsymbol{a}/\boldsymbol{a}_2)$ and $f_2 = f(\boldsymbol{a}_1/\boldsymbol{a}_2)$. This comparison points to the fact that the theory "$\boldsymbol{a}_2$" must afford the formulation invariant with regard to the character of decomposition of its state to the states fixed on a lower structural level as random events. The comparison of the second and the third part of the description (8) demonstrates that the behaviour of a system on the level of the code $\boldsymbol{a}$ can serve for the explanation of its different qualities on a higher level of determinacy of system behaviour. This comparison shows that the properties of a system revealed on the level of the code $\boldsymbol{a}$ can never determine unambiguously the system qualities described on higher levels of determinacy of its behaviour. Let us consider now the turbulence problem. The first comparison claims the possibility of formulating turbulence mechanics on the basis of qualitative determinacy of the turbulent level of motion organization without decomposing it down to its realizations in terms of a non-averaged flow field described by the Navier–Stokes equation. The second comparison affirms that the quality of turbulence cannot be deduced from realizations of the medium's behaviour described by the Navier–Stokes equation only.

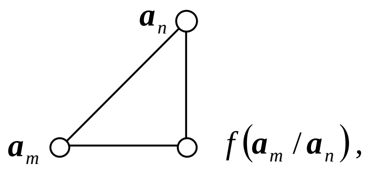
3.3. Hierarchic stochastic systems

The code grid of a hierarchic stochastic system with an arbitrary number of structural levels N has the form



, (13)

where $f_n = f(\boldsymbol{a}_{n-1} / \boldsymbol{a}_n), \ldots, f_{m\ldots n} = f(f_{m\ldots n-1} / \boldsymbol{a}_n)$ $(m < n = 1, \ldots, N)$. The description of a system with the code grid (13) allows of the following contractions down to the description on the level of the elementary system:



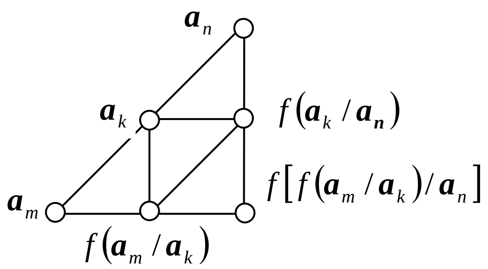
where $f(\boldsymbol{a}_m / \boldsymbol{a}_n)$ $(n > m = 0, 1, ..., N-1)$ are determined as

$$f(\boldsymbol{a}_m / \boldsymbol{a}_n) = \int \ldots \int f_{m+1} \cdots f_{m+1\ldots n} df_{m+1} \ldots df_{m+1\ldots n-1}$$

and signs of the code $\boldsymbol{a}_n$ are expressed through the signs of the code $\boldsymbol{a}_m$ as

$$\boldsymbol{a}_n = \int \tilde{\boldsymbol{a}}_n(\boldsymbol{a}_m) f(\boldsymbol{a}_m / \boldsymbol{a}_n) d\boldsymbol{a}_m.$$

Let us consider now the following contractions of the description (13) to systemic descriptions of the second order:



(14)

,

where $\boldsymbol{a}_m$, $\boldsymbol{a}_k$, and $\boldsymbol{a}_n$ $(m < k < n \le N)$ denote the codes of arbitrarily chosen structural levels in (13). Applying the relation (12) to the systemic description (14), we have

$$\mathcal{P}_{n;m} = \mathcal{P}_{n;k}\mathcal{P}_{k;m}, \tag{15}$$

where, for arbitrary p and q, $\mathcal{P}_{p;q}$ defines operators $\int \ldots f(\boldsymbol{a}_q / \boldsymbol{a}_p) d\boldsymbol{a}_q$. Denoting the operators $\mathcal{P}_{n;n-1}$ and $\mathcal{P}_{n;1}$ as $\mathcal{P}_n$ and $\mathcal{P}[n]$, we get from (15) that $\mathcal{P}_{n;m} = \mathcal{P}_n\mathcal{P}_{n-1}...\mathcal{P}_{m+1}$ and $\mathcal{P}[n] = \mathcal{P}_n\mathcal{P}_{n-1}...\mathcal{P}_1$, while the evident rule

$$\mathcal{P}[n]\,\mathcal{P}[m] = \mathcal{P}[m]\,\mathcal{P}[n] = \mathcal{P}[n]$$

holds. Applying the operators $\mathcal{P}[n]$ to signs of $\boldsymbol{a}$ and denoting $\mathcal{P}[n]a$ and $a[n-1] - a[n]$ as $a[n]$ and $a[n]'$, we have

$$a[n]'[n]=0,\quad (a[m]b[n])[n]=a[n]b[n]. \tag{16}$$

Using (16) it is easy to see that the signs of the code $\boldsymbol{a}$ and the probabilistic moments $(a[m]b[m])[n]$ have the following representations:

$$a=\sum_{k=1}^{n}a[k]'+a[n] \tag{17}$$

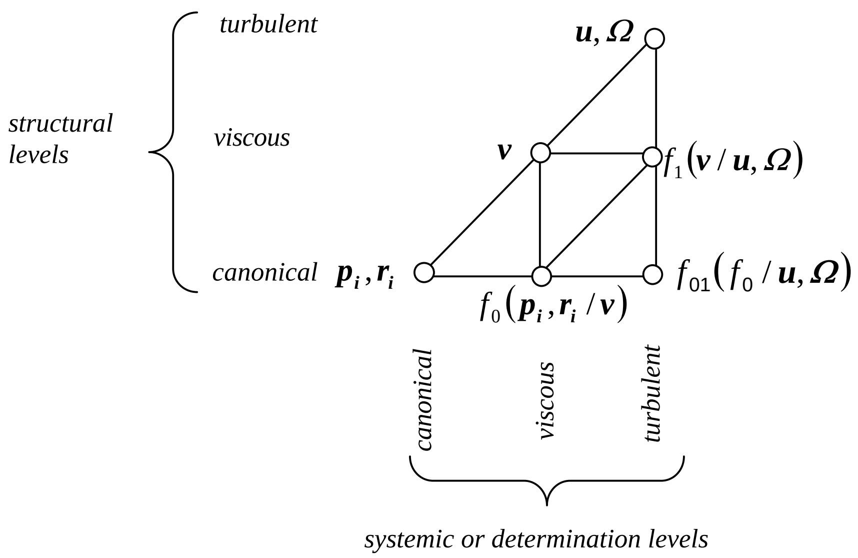
and

$$(a[m]b[m])[n]=a[n]b[n]+\sum_{k=m+1}^{n}(a[k]'b[k]')[n]. \tag{18}$$

The quantities on the right side of (17) represent different variability constituents of a revealed by using a set of averaging operators $\mathscr{P}[1],\ldots,\mathscr{P}[n]$, and the quantities on the right side of (18) represent different structural components of $(a[m]b[m])[n]$. Let us stress that according to (18) only components in (17) of the same variability prove mutually correlative, which distinguishes the nature of interaction on the same structural level from the nature of interaction between different structural levels.

4. THE SET-UP OF SYSTEMIC DESCRIPTION: THE MOTION OF LIQUID MEDIA

In this section the set-up of the systemic description with respect to an arbitrary stochastic system will be applied to the description of mechanics of motions of liquid media. Considering canonical, viscous, and turbulent levels of the structural and systemic organization of the medium, the code grid of the systemic description is expressed as



(19)

The node theories corresponding to the nodes of the lowest information coding level in (19) represent the following mechanical theories: Hamilton mechanics [15] (the code – canonical variables $\boldsymbol{p}_i$ and $\boldsymbol{r}_i$, denoting respectively the momentum and coordinate of the ith particle; $i = 1, \ldots, N$, where N is the number of particles); classical thermo-hydrodynamics [16,17] (the code – flow velocity $\boldsymbol{v}$ and the signs of thermodynamics code); turbulence mechanics [3–5] (the code – flow velocity $\boldsymbol{u} = \overline{\boldsymbol{v}}$ and internal rotation velocity of turbulent continuum $\boldsymbol{\Omega} = \overline{\boldsymbol{v}' \times \partial \boldsymbol{e} / \partial s}$, where $\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{u}$ and $\boldsymbol{e} = \boldsymbol{v}'/v'$; the overbar denotes averaging with the probability distribution $f_1 = f_1(\boldsymbol{v}/\boldsymbol{u}, \boldsymbol{\Omega})$).

The node theories based on the probability distribution functions $f_0 = f_0(\boldsymbol{p_i}, \boldsymbol{r_i}/\boldsymbol{v})$ and $f_1 = f_1(\boldsymbol{v}/\boldsymbol{u}, \boldsymbol{\Omega})$ are classified as kinetic theory [18] and statistical hydromechanics [1].

The prediction of the node theory based on the code $f_{01} = (f_0/\boldsymbol{u}, \boldsymbol{\Omega})$, missing within classical statistical theories, is one of the advantages of the formulated systemic description.

In addition to the six node theories the code grid (19) includes nine recoding theories. These theories discuss the problems like stability, formation of chaos, problems of self-organization (including the problems of coherent structures) and others, connected to the liquid medium motion. The formulation of recoding theories turns the set of node theories into the system of theories. In this system the node theories obtain specific systemic properties revealed through harmonization of their formulations. Not one theory in this system can be replaced by other theories.

The set-up of the systemic description of the hydrodynamic situation is not restricted by the description corresponding to the code grid (19). The extension of the description (19) by decomposing the turbulent level into sublevels of different variability reveal the “systemic degrees of freedom” of systemic descriptions, as well as it serves as the key for revealing such important features of turbulent motion as cascading processes between different scales of motion [3–5]. Moreover, extending the description (19) by introducing physical structural levels forms the systemic description common in mechanics (dealing with the macroscopic behaviour of the medium) and physics (dealing with the microscopic and quantum levels of the medium). Such a systemic description can be treated as an alternative for the desired unitary field theory. It replaces aspiration for the formulation of a universal field theory, from which all other field theories follow as its particular cases, to achieve the integrity of description on the level of a set of particular theories organized according to systemic principles.

5. CONCLUSIONS

The notion *stochastic system* is a generic notion for treating a wide class of complex natural objects and phenomena. Usually this notion is applied to an object or phenomenon behaving randomly, the description of which requires application of statistical methods [6–9]. Such a treatment considers statistical characteristics as information obtained from data interpreted as a set of random events. The conditions of the formation of probability characteristics are determined by the conditions of collecting data in this situation. The variability of these conditions, as well as the absence of a unique set of notions determining these conditions with adequate accuracy, largely devalues the obtained results. Particularly it leads to the impossibility of repeating the procedure of collecting data. The situation becomes uniquely determined if the conditions for forming probabilities are fixed on characteristics, which are invariant with respect to realizations of the particular states of the described object or phenomenon as random events. The set-up of the description of stochastic systems suggested in this paper considers just such a situation.

Besides founding a theoretical basis for formulating specific systemic descriptions of stochastic systems in different fields of knowledge (as the description (19), for example), the formulated description has many other advantages. It can predict new particular theories (like the theory "$f_{01}(f_0/\boldsymbol{u},\boldsymbol{\Omega})$" in (19)) and creates additional criteria for formulating particular theories. For instance, the possibility of the contraction of the systemic description (19) to an arbitrary node theory of a lower information coding level makes it impossible to deduce, for example, the quality of turbulence described within the theory "$\boldsymbol{u},\boldsymbol{\Omega}$" from the quality described within any other node theory in (19). Such a quality must be determined independently of the quality described within other node theories and, in particular, of the quality described within classical thermo-hydrodynamics.

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Stohhastiliste süsteemide kirjeldusest

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Töös käsitletakse stohhastilisi süsteeme. Stohhastilised süsteemid defineeritakse kui süsteemid, mille olekud on määratletud juhuslike sündmustena. Formuleeritakse stohhastiliste süsteemide süsteemikirjeldamise üldstruktuur, mida seejärel illustreeritakse hüdrodünaamilise keskkonna süsteemikirjelduse formuleerimise näitel.