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General solution of a system of differential equations modelling a class of exactly-solvable potentials. Part II: extended results

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Abstract. The complete, closed-form solution of a system of coupled differential equations introduced by Ge et al. (*Phys. Rev. A*, 2000, **62**, 052110–052117) and representing a set of potentials for which shift operators can be constructed is given. The general solution obtained can be used to perform a systematic search for new exactly-solvable potentials. This note is an extension of the paper published in *Proc. Estonian Acad. Sci. Phys. Math.* (2001, **50**, 1, 42–48).

Key words: exactly-solvable potential, Schrödinger equation, shift operator.

1. INTRODUCTION

Nowadays the classification of solvable models in quantum mechanics attracts much attention. Especially for the nonrelativistic Schrödinger equation numerous methods have been elaborated to track down exact solutions. These solutions are interesting in themselves as models of actual physical situations, and, furthermore, they are successfully used for elaborating approximate and qualitative methods. Besides common analytical methods, like the use of a suitable trial function [1,2] or mappings between Schrödinger equations for different potentials [3,4], especially algebraic methods have been found to be powerful tools for finding large classes of solvable potentials. Such algebraic methods include supersymmetric quantum mechanics [5−7], other factorization and intertwining techniques [8−10], and the use of shift operators: by giving a reference state of a physical system, the potential is determined up to a constant, and shift operators lead to the full spectrum of the system [11,12]. However, to come to the spectrum, one at least needs to know a reference state and the shift operators. Recently, for obtaining shift operators of a general physical system, a new method has been proposed [13,14] that works without the knowledge of any state. The quantum system is represented by a general Hamiltonian H and momentum operator P depending on a potential V and some unknown functions. By imposing certain constraints on the commutators $[H, Q]$ and $[H, P]$, where Q and P are coordinate and momentum operators, shift operators of H can be obtained, which in turn can yield ground state wavefunctions of H. Examples presented in [13] and [14] include potentials of harmonic/anharmonic oscillator type, Coulomb, Pöschl–Teller and more. However, the technical problem that comes with the above method lies in the constraints on $[H, Q]$ and $[H, P]$, a system of coupled differential equations that has to be solved first, before any shift operator can be obtained. In [13,14] this system is solved only exemplarily, leading to the potentials mentioned above. As the authors mention in [13], they did not exhaust all possibilities of solving the constraints, which would possibly lead to more complicated solvable potentials. In this note we address this problem by giving an explicit solution of the system of constraints. We obtain expressions for all classes of potentials for which shift operators can be obtained by the above method. The potentials we compute depend on at most one arbitrary function, its integral and derivatives.

The difference between this work and the previous paper [15] is the following: the present investigation of the constraints is much more extensive. We distinguish more cases and give far more simplified expressions for the solutions than in [15]. In Sec. 2 we derive the constraints; Sec. 3 and its subsections contain the computation of the solutions.

2. DERIVATION OF THE CONSTRAINTS

Consider a quantum system with the Hamiltonian H, coordinate operator Q, and momentum operator P. Suppose the following relations are fulfilled [16−19]:

$$[H, Q] = \Theta_1 \, Q + \Pi_1 \, P, \tag{1}$$

$$[H, P] = \Theta_2 \, Q + \Pi_2 \, P, \tag{2}$$

where $\Theta_i, \ \Pi_i$ may depend on the Hamiltonian H and some constants. Then we can construct raising and lowering operators for the Hamiltonian H by a method introduced in [13]. Furthermore, we are able to find the ground state wavefunction and the corresponding ground state energy from the lowering operator constructed.

Let us assume H and P to have the following general form:

$$H = X(x)\frac{\partial^2}{\partial x^2} + V(x), \tag{3}$$

$$P = Y(x)\frac{\partial}{\partial x} + Z(x), \tag{4}$$

where X, Y, Z, and the potential V are arbitrary functions. Let us point out here that it depends on the choice of X and V whether H has a discrete spectrum or not. Since both X and V shall turn out to be involved functions and hard to analyse in detail, we omit to impose further a-priori restrictions on them. Instead, as in [13], we just assume X and V to be chosen in such a way that H possesses a discrete spectrum.

Computing $[H,P]$ we get

$$\begin{aligned}[H,P] &= \big(2X(x)Y'(x) - X'(x)Y(x)\big)\frac{\partial^2}{\partial x^2} \\ &\quad + \big(X(x)Y''(x) + 2X(x)Z'(x)\big)\frac{\partial}{\partial x} + X(x)Z''(x) - Y(x)V'(x).\end{aligned}$$

Let us set for convenience

$$X(x)Y''(x) + 2X(x)Z'(x) = \alpha Y(x), \tag{5}$$

$$2X(x)Y'(x) - X'(x)Y(x) = (\beta Q(x) + \gamma)X(x), \tag{6}$$

where $\alpha,\ \beta$, and γ denote complex numbers. We made the choices (5) and (6) such that in conjunction with another setting for Q (following below) the commutator $[H,P]$ takes a form close to the desired one (2). Of course it is possible to replace (5), (6) and the following settings for Q by different ones. As was said before, the goal is just to bring the commutator $[H,P]$ into the form (2). Now the commutator $[H,P]$ takes the form

$$\begin{aligned}[H,P] = (\beta Q(x) + \gamma)H + \alpha P - (\beta Q(x) + \gamma)V(x) - \alpha Z(x) - Y(x)V'(x) \\ + X(x)Z''(x).\end{aligned}$$

To simplify the last expression, we set

$$Q(x) = -(\beta Q(x) + \gamma)V(x) - \alpha Z(x) - Y(x)V'(x) + X(x)Z''(x),$$

which can be solved for Q:

$$Q(x) = \frac{1}{1+\beta V(x)}(X(x)Z''(x) - \gamma V(x) - \alpha Z(x) - Y(x)V'(x)). \tag{7}$$

Finally, the commutator of H and P reads

$$[H,P] = Q(x)(\beta H + 1) + \alpha P + \gamma H. \tag{8}$$

In the same way we consider $[H,Q]$:

$$[H,Q(x)] = 2X(x)Q'(x)\frac{\partial}{\partial x} + X(x)Q''(x).$$

Setting

$$X(x)Q'(x) = \lambda Y(x), \tag{9}$$

$$-2\lambda Z(x) + X(x)Q''(x) = \nu Q(x) + \tau, \tag{10}$$

where $\lambda,\ \nu$, and τ are complex numbers, we finally come to

$$[H, Q(x)] = 2\lambda P + \nu Q(x) + \tau. \tag{11}$$

We see that (8) and (11) are not exactly of the desired forms (2) and (1), in particular, the quantities γH and τ do not appear in the latter. But, as mentioned in [13], by a suitable redefinition of the operators Q and P, both γH and τ, can be absorbed in certain cases.

However, the constraints to be solved are given by Eqs. (5)–(7), (9), and (10).

3. SOLUTION OF THE CONSTRAINTS AND EXAMPLES

Let us summarize the constraints:

$$-\gamma X(x) - \beta Q(x)X(x) - Y(x)X'(x) + 2X(x)Y'(x) = 0, \tag{12}$$

$$-\lambda Y(x) + X(x)Q'(x) = 0, \tag{13}$$

$$-\alpha Y(x) + 2X(x)Z'(x) + X(x)Y''(x) = 0, \tag{14}$$

$$-\tau - \nu Q(x) - 2\lambda Z(x) + X(x)Q''(x) = 0, \tag{15}$$

$$Q(x) + \gamma V(x) + \beta Q(x)V(x) + \alpha Z(x) + Y(x)V'(x) - X(x)Z''(x) = 0. \tag{16}$$

Our purpose is now to solve these equations explicitly (if possible). We are particularly interested in obtaining an explicit result for the potential V, because from the five constraints it is not possible to see which potentials are compatible with them. We distinguish three cases, that is $\beta = 0$ and $X'(x) = 0$ (first case); $\beta = 0$, $X'(x) \neq 0$, and $\gamma \neq 0$ (second case); $\beta = 0$, $X'(x) \neq 0$, and $\gamma = 0$ (third case); and the most general fourth case, $\beta \neq 0$ and $X'(x) \neq 0$.

3.1. Case 1: $\beta = 0$ and $X'(x) = 0$

This setting is considered in a few examples in [13], yielding potentials V like the harmonic oscillator and the radial harmonic oscillator. The constraints (12)–(16) simplify as follows (set $X(x) = X_0$):

$$-\gamma + 2Y'(x) = 0, \tag{17}$$

$$-\lambda Y(x) + X_0 Q'(x) = 0, \tag{18}$$

$$-\alpha Y(x) + 2X_0 Z'(x) + X_0 Y''(x) = 0, \tag{19}$$

$$-\tau - \nu Q(x) - 2\lambda Z(x) + X_0 Q''(x) = 0, \tag{20}$$

$$Q(x) + \gamma V(x) + \alpha Z(x) + Y(x)V'(x) - X_0 Z''(x) = 0. \tag{21}$$

We further assume $\lambda \neq 0$, because otherwise Eq. (18) would yield Q to be constant, which does not make much sense. Equation (17) yields immediately

$$Y(x) = \frac{\gamma}{2}\, x + Y_0. \tag{22}$$

Inserting the last result into Eq. (18) gives Q:

$$\begin{aligned} -\lambda \left(\frac{\gamma}{2}\, x + Y_0 \right) + X_0 Q'(x) &= 0 \\ \Rightarrow Q(x) &= \frac{\lambda\gamma}{4X_0}\, x^2 + \frac{\lambda Y_0}{X_0}\, x + Q_0. \end{aligned} \tag{23}$$

Using (22), we determine Z by Eq. (19):

$$Z(x) = \frac{\alpha\gamma}{8X_0}\, x^2 + \frac{\alpha Y_0}{2X_0}\, x + Z_0. \tag{24}$$

Before we compute the potential V, we have to fulfill Eq. (15) that represents an interrelation between Q and Z. Inserting (23), (24) and solving for Z, we get

$$Z(x) = \frac{1}{2\lambda} \left(-\frac{\nu\lambda\gamma}{4X_0}\, x^2 - \frac{\nu\lambda Y_0}{X_0}\, x - \tau - \nu Q_0 + \frac{\lambda\gamma}{2} \right). \tag{25}$$

This expression and (24) must be the same. It is easy to see that on setting

$$\alpha = -\nu, \tag{26}$$

$$Z_0 = \frac{-\tau - \nu Q_0}{2\lambda} + \frac{\gamma}{4} \tag{27}$$

expressions (24) and (25) coincide. Now we can consider Eq. (21) to get V. Solving for V yields

$$\begin{aligned} V(x) = &\exp\left(-\gamma \int \frac{1}{Y(x)}\, dx \right) \\ &\times \left(V_0 - \int \frac{\exp\left(\gamma \int \frac{1}{Y(x)}\, dx \right) \, (Q(x) + \alpha\, Z(x) - X_0\, Z''(x))}{Y(x)}\, dx \right). \end{aligned} \tag{28}$$

Using (23), (24), (26), and (27), the last expression can be simplified to

$$\begin{aligned} V(x) = &\ [16\, V_0\, X_0\, \lambda + x\ (4\, Y_0 + x\, \gamma) \\ &\times \left(- \left((8\, Q_0\, X_0 + x\ (4\, Y_0 + x\, \gamma)\ \lambda)\ \left(2\, \lambda + \nu^2 \right) \right) - 8\, X_0\, \nu\, \tau \right)] \\ &/\ [16\, X_0\, (2\, Y_0 + x\, \gamma)^2\, \lambda]. \end{aligned} \tag{29}$$

The numerator of the expression (29) is a polynomial of degree four with respect to x, whereas the denominator is a polynomial of degree two with respect to x. In more detail, the expanded numerator num$(V(x))$ and denominator den$(V(x))$ are given by

$$\begin{aligned}
&\mathrm{num}(V(x)) = x^4\,\left(-2\,\gamma^2\,\lambda^2 - \gamma^2\,\lambda\,\nu^2\right) + x^3\,\left(-16\,Y_0\,\gamma\,\lambda^2 - 8\,Y_0\,\gamma\,\lambda\,\nu^2\right)\\
&+x^2\,\left(-16\,Q_0\,X_0\,\gamma\,\lambda - 32\,{Y_0}^2\,\lambda^2 - 8\,Q_0\,X_0\,\gamma\,\nu^2 - 16\,{Y_0}^2\,\lambda\,\nu^2 - 8\,X_0\,\gamma\,\nu\,\tau\right)\\
&+x\,\left(-64\,Q_0\,X_0\,Y_0\,\lambda - 32\,Q_0\,X_0\,Y_0\,\nu^2 - 32\,X_0\,Y_0\,\nu\,\tau\right) + 16\,V_0\,X_0\,\lambda,
\end{aligned} \tag{30}$$

$$\mathrm{den}(V(x)) = 16\,X_0\,\gamma^2\,\lambda\,x^2 + 64\,X_0\,Y_0\,\gamma\,\lambda\,x + 64\,X_0\,{Y_0}^2\,\lambda. \tag{31}$$

We now look at some special cases. Since the coefficients of the different powers of x are interrelated, it is not clear that we can get any combination of powers in the numerator (30) and denominator (31). Let us first assume that we want the term $\sim x^4$ in the numerator to vanish. We need to impose that

$$\begin{aligned}
-2\,\gamma^2\,\lambda^2 - \gamma^2\,\lambda\,\nu^2 &= 0,\\
-\lambda &= -\frac{\nu^2}{2}.
\end{aligned}$$

It is now easy to see that this makes the term $\sim x^3$ in (30) vanish too. Altogether (29) becomes

$$V(x) = \frac{V_0\nu + 4Y_0\tau\;x + \gamma\tau\;x^2}{4Y_0^2\nu + 4Y_0\gamma\nu\;x + \gamma^2\nu\;x^2}.$$

Now, setting $\tau = 0$ in the last expression leads to an inverse power potential

$$V(x) = \frac{V_0}{(2Y_0 + \gamma x)^2}.$$

The examples of potentials considered in [13] can of course be reobtained from the general potential (29). If we set $\gamma = 0$ (case 1 in [13]), we get the potential of the harmonic oscillator:

$$V(x) = \frac{V_0}{4\,{Y_0}^2} - \frac{x^2\,\left(2\,\lambda + \nu^2\right)}{4\,X_0} + \frac{x\,\left(-2\,Q_0\,X_0\,Y_0\,\left(2\,\lambda + \nu^2\right) - 2\,X_0\,Y_0\,\nu\,\tau\right)}{4\,X_0\,{Y_0}^2\,\lambda}.$$

This coincides with the result in [13].

3.2. Case 2: $\beta = 0,\ X'(x) \neq 0$, and $\gamma \neq 0$

The setting $\beta = 0$ and $X'(x) \neq 0$ yields for example the Coulomb potential for the hydrogen atom or a Morse-type potential [13]. The constraints read

$$
\begin{aligned}
-\gamma X(x) - Y(x)X'(x) + 2X(x)Y'(x) &= 0, &(32)\\
-\lambda Y(x) + X(x)Q'(x) &= 0, &(33)\\
-\alpha Y(x) + 2X(x)Z'(x) + X(x)Y''(x) &= 0, &(34)\\
-\tau - \nu Q(x) - 2\lambda Z(x) + X(x)Q''(x) &= 0, &(35)\\
Q(x) + \gamma V(x) + \alpha Z(x) + Y(x)V'(x) - X(x)Z''(x) &= 0. &(36)
\end{aligned}
$$

The first of these equations is an interrelation between X and Y, so these two functions cannot be chosen independently from each other. We point this out, since in every example in [13] both X and Y were chosen simultaneously without mentioning the interrelation between them. Solving for X yields

$$X(x) = X_0 Y^2(x) \exp\left(-\gamma \int \frac{1}{Y(x)}\, dx\right). \tag{37}$$

The second equation determines Q:

$$
\begin{aligned}
Q(x) &= \lambda \int \frac{Y(x)}{X(x)}\, dx\\
&= \frac{\lambda}{X_0\gamma} \exp\left(\gamma \int \frac{1}{Y(x)}\, dx\right), \qquad (38)
\end{aligned}
$$

where in the last step we inserted (37) and did one integration. The third equation (34) yields an expression for Z (we shall use (37) again):

$$
\begin{aligned}
Z'(x) &= \frac{1}{2X(x)}\big(\alpha Y(x) - X(x)Y''(x)\big)\\
&= \frac{\alpha}{2X_0 Y(x)} \exp\left(\gamma \int \frac{1}{Y(x)}\, dx\right) - \frac{Y''(x)}{2}\\
\Rightarrow Z(x) &= \frac{\alpha}{2X_0\gamma} \exp\left(\gamma \int \frac{1}{Y(x)}\, dx\right) - \frac{Y'(x)}{2} + Z_0. \qquad (39)
\end{aligned}
$$

We now solve the fourth equation (35) for Z. At first we need an expression for Q'' depending on X and Y. We have from (33)

$$Q''(x) = \lambda \left(\frac{Y'(x)}{X(x)} - \frac{Y(x)X'(x)}{X^2(x)}\right).$$

This can be simplified using Eq. (32). Solving it for $X'(x)/X(x)$, we find

$$\begin{aligned} \frac{X'(x)}{X(x)} &= \frac{2Y'(x)-\gamma}{Y(x)} \\ \Rightarrow \frac{Y(x)X'(x)}{X^2(x)} &= \frac{2Y'(x)-\gamma}{X(x)}. \end{aligned}$$

Thus we finally have for Q'':

$$\begin{aligned} Q''(x) &= \lambda\left(\frac{Y'(x)}{X(x)} - \frac{2Y'(x)-\gamma}{X(x)}\right) \\ &= \lambda\left(\frac{\gamma - Y'(x)}{X(x)}\right). \end{aligned} \tag{40}$$

Let us now solve Eq. (35) for Z. Using (40), we get

$$Z(x) = -\frac{\tau}{2\lambda} - \frac{\nu}{2}\int \frac{Y(x)}{X(x)}\,dx - \frac{Y'(x)}{2} + \frac{\gamma}{2}. \tag{41}$$

Inserting finally (37), we obtain

$$Z(x) = -\frac{\tau}{2\lambda} - \frac{\nu\lambda}{2X_0\gamma}\exp\left(\gamma\int\frac{1}{Y(x)}\,dx\right) - Q_0 - \frac{Y'(x)}{2} + \frac{\gamma}{2}. \tag{42}$$

For consistency, (39) and (42) must be the same, which is true if we set

$$\alpha = -\nu\lambda, \tag{43}$$

$$Z_0 = -\frac{\tau}{2\lambda} - Q_0 + \frac{\gamma}{2}. \tag{44}$$

Now we can determine the potential via (36). Solving (36) yields almost the same expression as in (28); we just have to replace X_0 by $X(x)$:

$$\begin{aligned} V(x) &= \exp\left(-\gamma\int\frac{1}{Y(x)}\,dx\right) \\ &\times\left(V_0 - \int\frac{\exp\left(\gamma\int\frac{1}{Y(x)}\,dx\right)\,(Q(x)+\alpha\,Z(x)-X(x)\,Z''(x))}{Y(x)}\,dx\right). \end{aligned}$$

Inserting the functions Q and Z, we obtain after some elementary manipulations

$$\begin{aligned} V(x) &= -\exp\left(\gamma\int\frac{1}{Y(x)}\,dx\right)\left(\frac{2\lambda+\lambda^2\nu^2}{4X_0\gamma^2}\right) + \frac{1}{4}\exp\left(-\gamma\int\frac{1}{Y(x)}\,dx\right) \\ &\times\left(4V_0 + X_0(Y'(x))^2 - 2X_0Y(x)Y''(x)\right) - \frac{\nu(2Q_0\lambda+\tau)}{2\gamma}. \end{aligned} \tag{45}$$

This is the most explicit form of the potential V we are able to get. Though it still contains an integral and derivatives of the function Y, its structure is much more obvious than it is only from looking at the system of constraints (32)–(36). Consequently, expression (45) makes it much easier to choose Y in order to generate a particular potential.

As an application we reconsider case 5 from [13], that is, we set $Y(x) = x$. It follows from (37) that

$$X(x) = X_0\, x^{2-\gamma}.$$

Choosing $\gamma = 1$ and $X_0 = -1$, as in the above reference, leads to $X(x) = -x$. The potential (45) generated reads (after using (43)–(44)):

$$V(x) = \frac{V_0}{x} - \frac{\nu(2Q_0\lambda - \tau)}{2} + x\left(\frac{2\lambda + \lambda^2\nu^2}{4}\right),$$

which coincides with the result in [13] (Coulomb potential for the hydrogen atom).

In the same way we reconstruct the Morse potential (case 6 in [13]), let $Y(x) = 1$. Then we have from (37) that

$$X(x) = X_0 \exp(-\gamma x).$$

Setting $\gamma = -c$ and $X_0 = -1$, we obtain from (45) the Morse-potential:

$$V(x) = \exp(cx)V_0 + \exp(-cx)\left(\frac{2\lambda + \lambda^2\nu^2}{4c^2}\right) + \frac{2Q_0\lambda\nu + \nu\tau}{2c}.$$

3.3. Case 3: $\beta = 0,\ X'(x) \neq 0$, and $\gamma = 0$

In case $\gamma = 0$, Eqs. (32) and (36) simplify. The whole set of constraints reads

$$\begin{aligned}
-Y(x)X'(x) + 2X(x)Y'(x) &= 0, && (46)\\
-\lambda Y(x) + X(x)Q'(x) &= 0, && (47)\\
-\alpha Y(x) + 2X(x)Z'(x) + X(x)Y''(x) &= 0, && (48)\\
-\tau - \nu Q(x) - 2\lambda Z(x) + X(x)Q''(x) &= 0, && (49)\\
Q(x) + \alpha Z(x) + Y(x)V'(x) - X(x)Z''(x) &= 0. && (50)
\end{aligned}$$

Solving the first of these equations for X yields

$$X(x) = X_0 Y^2(x). \tag{51}$$

Equation (47) gives

$$\begin{aligned} Q(x) &= \lambda \int \frac{Y(x)}{X(x)}\, dx \\ &= \frac{\lambda}{X_0} \int \frac{1}{Y(x)}\, dx. \end{aligned} \tag{52}$$

The constraint (48) can be solved for Z:

$$\begin{aligned} Z'(x) &= \frac{1}{2X(x)} \big(\alpha Y(x) - X(x) Y''(x)\big) \\ &= \frac{\alpha}{2X_0 Y(x)} - \frac{1}{2} Y''(x), \end{aligned}$$

where we made use of (51) in the last step. After integration we get

$$Z(x) = \frac{\alpha}{2X_0} \int \frac{1}{Y(x)}\, dx - \frac{1}{2} Y'(x) + Z_0. \tag{53}$$

To solve the fourth constraint (49), we need Q'' in terms of X and Y. Using the result (40) for $\gamma = 0$, we arrive at

$$Q''(x) = -\frac{\lambda Y'(x)}{X(x)}.$$

Solving (49) for Z yields (41) for $\gamma = 0$, that is

$$Z(x) = -\frac{\tau}{2\lambda} - \frac{\nu}{2} \int \frac{Y(x)}{X(x)}\, dx - \frac{1}{2} Y'(x). \tag{54}$$

Equations (54) and (53) must coincide, therefore we set

$$\alpha = -\nu X_0, \tag{55}$$

$$Z_0 = -\frac{\tau}{2\lambda}. \tag{56}$$

Finally it remains to solve (50). We get

$$V(x) = V_0 - \int \frac{Q(x) + \alpha\, Z(x) - X(x)\, Z''(x)}{Y(x)}\, dx.$$

Inserting (51), (52), (54), (55), and (56) into the potential, we come to

$$\begin{aligned} V(x) = V_0 &- \left(\int \frac{1}{Y(x)}\, dx\right) \left(\frac{X_0 \nu \tau}{2\lambda}\right) - \left(\int \frac{1}{Y(x)}\, dx\right)^2 \left(\frac{1}{4} X_0 \nu^2 + \frac{\lambda}{2X_0}\right) \\ &+ \frac{1}{4} X_0 (Y'(x))^2 - \frac{1}{2} X_0 Y(x) Y''(x). \end{aligned} \tag{57}$$

There are no examples in [13] for the above potential, i.e. for the setting $\beta = \gamma = 0$ and $X'(x) \neq 0$. Choosing $Y(x) = x^k$ (k a constant and $k \neq 1$) yields for example the following power law potential:

$$V(x) = A_0 + A_1\, x^{2k-2} + A_2\, x^{-2k+2} + A_3\, x^{-k+1},$$

where A_i denote constants. Choosing $Y(x) = x$, we obtain a logarithmic potential:

$$V(x) = B_0 + B_1 \log(x) + B_2 \log^2(x),$$

where B_i denote constants. Depending on these constants, this potential can represent finite depth and a repulsive singularity at zero, set for example $B_0 = 0$ and $B_2 > 0$. Let us give another example: The choice $Y(x) = \exp(kx)$ (k a constant) leads to the following generalized Morse potential with an additional term $\sim \exp(-kx)$:

$$V(x) = C_0 + C_1 \exp(2kx) + C_2 \exp(-2kx) + C_3 \exp(-kx),$$

where C_i denote constants.

3.4. Case 4: $\beta \neq 0$ and $X'(x) \neq 0$

The constraints are now given by (12)–(16). We solve the first and second of these equations for Q and obtain the following results:

$$Q(x) = -\frac{\gamma}{\beta} - \frac{Y(x)X'(x)}{\beta X(x)} + \frac{2Y'(x)}{\beta}, \tag{58}$$

$$Q(x) = \lambda \int \frac{Y(x)}{X(x)}\, dx. \tag{59}$$

Since (58) and (59) must coincide, the following interrelation between X and Y arises:

$$\lambda \int \frac{Y(x)}{X(x)}\, dx = -\frac{\gamma}{\beta} - \frac{Y(x)X'(x)}{\beta X(x)} + \frac{2Y'(x)}{\beta}.$$

We can solve this equation for X:

$$\begin{aligned} X(x) &= \exp\left(-\gamma \int \frac{1}{Y(x)}\, dx\right) Y^2(x) \\ &\quad \times \left(X_0 - \beta\lambda \int \exp\left(\gamma \int \frac{1}{Y(x)}\, dx\right) \frac{\int Y(x)\, dx}{Y^3(x)}\, dx\right). \end{aligned} \tag{60}$$

Let us now determine Z from Eqs. (14) and (15). Equation (14) yields after integration

$$Z(x) = \frac{\alpha}{2}\int \frac{Y(x)}{X(x)}\,dx - \frac{1}{2}Y'(x) + Z_0. \tag{61}$$

To solve (15), we need Q'' in terms of X and Y. We have from (59)

$$\begin{aligned} Q''(x) &= \frac{\partial}{\partial x}\left(\lambda \int \frac{Y(x)}{X(x)}\,dx\right) \\ &= \frac{\lambda Y'(x)}{X(x)} - \frac{\lambda Y(x)X'(x)}{X^2(x)}. \end{aligned}$$

Using this, we get from Eq. (15) the following result for Z:

$$Z(x) = -\frac{\tau}{2\lambda} - \frac{\nu}{2}\int \frac{Y(x)}{X(x)}\,dx + \frac{1}{2}Y'(x) - \frac{Y(x)X'(x)}{2X^2(x)}. \tag{62}$$

Since (61) and (62) must be the same, we get the following interrelation between X and Y:

$$\frac{\alpha}{2}\int \frac{Y(x)}{X(x)}\,dx - \frac{1}{2}Y'(x) + Z_0 = -\frac{\tau}{2\lambda} - \frac{\nu}{2}\int \frac{Y(x)}{X(x)}\,dx + \frac{1}{2}Y'(x) - \frac{Y(x)X'(x)}{2X^2(x)}.$$

Solving for X, we obtain

$$\begin{aligned} X(x) &= \exp\left(\left(-2Z_0 - \frac{\tau}{\lambda}\right)\int \frac{1}{Y(x)}\,dx\right) Y^2(x)\Bigg(X_0 - (\alpha+\nu) \\ &\quad \times \int \exp\left(\left(2Z_0 + \frac{\tau}{\lambda}\right)\int \frac{1}{Y(x)}\,dx\right)\frac{\int Y(x)\,dx}{Y^3(x)}\,dx\Bigg). \end{aligned} \tag{63}$$

Equations (60) and (63) must be the same, which holds if we set

$$\gamma = 2Z_0 + \frac{\tau}{\lambda}, \tag{64}$$

$$\beta\lambda = \alpha + \nu. \tag{65}$$

Now we can obtain the potential from Eq. (16). Solving yields

$$\begin{aligned} V(x) &= \exp\left(-\int \frac{\gamma + \beta Q(x)}{Y(x)}\,dx\right)\Bigg(V_0 - \int \exp\left(\left(-\int \frac{\gamma+\beta Q(x)}{Y(x)}\,dx\right)\right) \\ &\quad \times \frac{Q(x) + \alpha Z(x) - X(x)Z''(x)}{Y(x)}\,dx\Bigg). \end{aligned} \tag{66}$$

The exponential function in the last expression becomes much simplified if we insert Q as given in (58):

$$\exp\left(-\int \frac{\gamma+\beta Q(x)}{Y(x)}\,dx\right) \;=\; \frac{X(x)}{Y^2(x)}.$$

Thus we get for the potential (66)

$$V(x) \;=\; \frac{X(x)}{Y^2(x)}\left(V_0 - \int \frac{Y(x)(Q(x)+\alpha Z(x) - X(x)Z''(x))}{X(x)}\,dx\right).$$

Inserting Q and Z as given in (59) and (61), we finally obtain the potential in terms of X and Y:

$$V(x) = \frac{\alpha}{2} + \frac{X(x)}{Y^2(x)}\left(V_0 - Z_0\,\alpha\int \frac{Y(x)}{X(x)}\,dx - \left(\int \frac{Y(x)}{X(x)}\,dx\right)^2\left(\frac{\alpha^2}{4}+\frac{\lambda}{2}\right)\right.$$
$$\left. + \frac{1}{4}(Y'(x))^2 - \frac{1}{2}Y(x)\,Y''(x)\right).$$

We omit to insert the function X as given in (63), because the above formula for V is not simplified. As in the previous section, the structure of the potential becomes much more obvious now than from the five constraints. However, if Y is a complicated function, the integrals in (63) and (67) are not solvable analytically. We omit to give examples here, because even for relatively simple functions Y the potentials (67) generated become in general very long and involved expressions.

4. CONCLUSIONS

In this note we gave the complete, closed-form solution of a system of constraints describing a class of potentials for which shift operators can be computed by a method introduced in [13]. Our main result was the explicit solutions for these potentials given in (29), (45), (57), and (67). The explicit representation of the potentials we obtained here can be used for a systematic search for new exactly-solvable special cases.

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Täpselt lahenduvate potentsiaalide klassi modelleeriva diferentsiaalvõrrandisüsteemi üldlahend. II osa: laiendatud tulemused

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On uuritud artiklites [13,14] üldise füüsikalise süsteemi nihke operaatorite leidmiseks pakutud uut meetodit. Tähtsat osa selles meetodis mängivad seosed kommutaatorite $[H, Q]$ ja $[H, P]$ vahel, kus H on süsteemi hamiltoniaan, Q

ja P on koordinaadi ja impulsi operaatorid. Seosed on määratud diferentsiaal-võrrandisüsteemiga, mille täpne lahend on leitud. Lähtudes sellest lahendist on tuletatud potentsiaalide klasside avaldised, mille puhul on võimalik konstrueerida vastavad nihke operaatorid. Töö on käsitluse [15] jätk, kusjuures praeguses uurimuses on esitatud seoste põhjalikum analüüs ja eristatud rohkem juhtusid; leitud lahenditel ja avaldistel on lihtsam kuju.