Proc. Estonian Acad. Sci. Phys. Math., 2002, **51**, 3, 160–169

# Upgrading the maximin principle for nonzero-sum games

Ants Tauts

Estonian Energy Research Institute, Paldiski mnt. 1, 10137 Tallinn, Estonia; chselts@solo.delfi.ee

Received 29 September 2001, in revised form 18 April 2002

**Abstract.** A modification of the maximin principle in the nonzero-sum game where players do not wish to harm one another but only benefit themselves is considered. First a player excludes such strategies of other players that are less favourable for them than any other choice. Assuming that other players have carried out the same procedure, in the new situation the line of reasoning similar to the first step will be repeated. So a gradual detailing process will start that with stabilization will give the final result.

**Key words:** characteristic function, nonzero-sum games, strategies, coalitions.

## 1. INTRODUCTION

The maximin principle used in market games is based on the assumption that for any player or coalition of players the behaviour of the rest of the players is unpredictable. Therefore for any step taken by a player such behaviour of the opposite side is expected, which is the most unfavourable for this player. Taking this for the basis, the player will choose the best possible step for him. Although several authors indicate (e.g. [1]) that players do not wish to harm one another, but only gain profit themselves, which is not the same for a nonzero-sum game, no practical conclusions have been drawn from these circumstances.

In this article the first step is taken in this direction in order to improve the usage of the maximin principle with taking the above considerations into account. While developing this methodology, its possible application to solving the problems arising in [2,3] was kept in mind.

## 2. FUNDAMENTAL CONCEPTS

Let $N = \{1, ..., n\}$ be the number of players. If any player $i \in N$ chooses some strategy $\tau_i$, the whole finite sequence of strategies $\tau_i$ will define the result of the game and the result for each player $k \in N$ is $\alpha_k = \alpha_k(\tau_i : i \in N)$. For the mixed strategy, the player $i \in N$ establishes the probability distribution of his possible pure strategies $\tau_i$. As a natural result, the probability distribution will also appear between the finite sequences $\langle \tau_i : i \in N \rangle$. In this case the result $\alpha_k$ is a mathematical expectation of the magnitude $\alpha_k(\tau_i : i \in N)$.

In the classical game theory the characteristic function $v$ is an essential tool by treating coalitions since it makes $v(S)$ to conform to any possible coalition $S \subseteq N$ in the following way [4].

When choosing an arbitrary finite sequence of the strategies of coalition members $\langle \tau_i : i \in S \rangle$ (it could also be a mixed strategy, i.e., a linear combination of similar strategies where multipliers are probabilities),

$$\min_{\langle \tau_j : j \in N \setminus S \rangle} \sum_{k \in S} \alpha_k \langle \tau_i : i \in N \rangle$$

can be calculated. Thus for the given $\langle \tau_i : i \in S \rangle$, the case is considered where the rest of the players together develop such a contrastrategy where the total result of the coalition $S$ is possibly small. Using the maximin principle,

$$v(S) = \max_{\langle \tau_i : i \in S \rangle} \min_{\langle \tau_j : j \in N \setminus S \rangle} \sum_{k \in S} \alpha_k \langle \tau_i : i \in N \rangle$$

is found.

It is justified for the zero-sum game since the total result

$$\sum_{k \in S} \alpha_k(\tau_i : i \in N) + \sum_{k \in N \setminus S} \alpha_k(\tau_i : i \in N)$$

is constant and the players who do not belong to the coalition $S$ are interested in reducing the second sum of the above total, because this is the only option for increasing the first sum.

Based on the definition of $v(S)$, the following main axioms must be satisfied:

1. $v(\varnothing) = 0$;
2. if $T \cap S = \varnothing$ then $v(T \cup S) \geq v(T) + v(S)$.

In case of a nonzero-sum game, the gain of some players is not related to the inevitable loss of the others. Therefore it is hardly probable that any player standing outside the coalition or the coalition consisting of such players would select a strategy that would harm the coalition $S$, while any other strategy could be more beneficial to himself.

## 3. RELATIONSHIP BETWEEN PURE STRATEGIES AND MIXED STRATEGIES

First, let us consider the relationship between pure strategies and mixed strategies. Let $S \subseteq N$. The players from the set $N \setminus S$ do not know whether $S$ is a complete coalition or is somehow divided into independently acting coalitions $S = S_1 \cup \ldots \cup S_m$, where $S_i \cap S_j = \varnothing$ for each $i \neq j$. In case of pure strategies, the finite strategy sequences $\left\langle \tau_j : j \in S_i \right\rangle$ exist for each $1 \le i \le m$. When choosing such a finite sequence for each $i$, we shall get a pure strategy for $S$. At the same time it can be well seen that every pure strategy for $S$ can be obtained in this way. Thus it makes no difference to the set of pure strategies whether $S$ is a complete coalition or disintegrated into independently acting partial coalitions.

In case of mixed strategies, the strategies of each coalition $S_i$ are linear combinations of finite sequences $\left\langle \tau_j : j \in S_i \right\rangle$ with a probability coefficient. When taking one such a linear combination for each $i$, we can make a linear combination of the finite sequences $\left\langle \tau_j : j \in S \right\rangle$ where the multiplier of each $\left\langle \tau_j : j \in S \right\rangle$ is the product of multipliers of finite sequences $\left\langle \tau_j : j \in S_i \right\rangle$ in that combination. Hence the selection of mixed strategies for the coalitions $S_1, \ldots, S_m$ gives a mixed strategy for the coalition $S$. Different from pure strategies, it turns out here that $S$ cannot be obtained for any coalition strategy in this way. Indeed, let $S = S_1 \cup S_2$. Let us take the strategies $\left\langle \tau'_j : j \in S_1 \right\rangle$ and $\left\langle \tau''_j : j \in S_1 \right\rangle$ for $S_1$ with the probabilities $\lambda_1$ and $\lambda_2$, respectively. For $S_2$, let us take the strategies $\left\langle \tau'_j : j \in S_2 \right\rangle$ and $\left\langle \tau''_j : j \in S_2 \right\rangle$ with the probabilities $\mu_1$ and $\mu_2$. Then for $S$ the finite sequences $\left\langle \tau'_j : j \in S_1;\ \tau'_j : j \in S_2 \right\rangle$, $\left\langle \tau'_j : j \in S_1;\ \tau''_j : j \in S_2 \right\rangle$, $\left\langle \tau''_j : j \in S_1;\ \tau'_j : j \in S_2 \right\rangle$, and $\left\langle \tau''_j : j \in S_1;\ \tau''_j : j \in S_2 \right\rangle$ appear with the probabilities $\lambda_1\mu_1$, $\lambda_1\mu_2$, $\lambda_2\mu_1$, and $\lambda_2\mu_2$, respectively. In this case the product of the probabilities of the first and the last finite sequence $\lambda_1\mu_1\lambda_2\mu_2$ will coincide with that of the two middle finite sequences. However, when $S$ is a complete coalition, such a probability distribution for these four finite sequences can be chosen in the way where there is no this coincidence. Thus the set of the complete mixed strategies for $S$ is wider than that obtained from combining the independently acting mixed strategies $S_1, \ldots, S_m$. A similar line of reasoning shows that if smaller partitions replace the partition $S_1, \ldots, S_m$, the set obtained by combining these mixed strategies will be even narrower.

To make the above simpler, we shall put down the strategies as pure strategies assuming that in case of mixed strategies, instead of finite sequences of a strategy, their linear combinations with probability multipliers, and instead of the results their expected values should be considered.

## 4. DOMINATION

### 4.1. Domination on the null level

Let us say that $\left\langle \tau'_i : i \in S \right\rangle$ **dominates on the null level** $\left\langle \tau''_i : i \in S \right\rangle$ if for any $\left\langle \tau_j : j \in N \setminus S \right\rangle$

$$\sum_{k\in S}\alpha_k(\tau_i':i\in S;\ \tau_j:j\in N\setminus S)\geq\sum_{k\in S}\alpha_k(\tau_i'':i\in S;\ \tau_j:j\in N\setminus S),$$

while for some $\langle\tau_j:j\in N\setminus S\rangle$ the inequality is rigorous. In this case the coalition $S$ eliminates the finite sequence $\langle\tau_i'':i\in S\rangle$ from the set of sequences worth of consideration and any player who is not a member of the coalition $S$ can assume that if the coalition $S$ exists (he may not know about the existence of $S$), he will never use the finite sequence $\langle\tau_i'':i\in S\rangle$.

According to the above, we could limit us to the case where $N\setminus S$ is a unified coalition since it involves also all cases where $N\setminus S$ can be divided into coalitions in some way. Even more, each finite sequence of mixed strategies $\langle\tau_j:j\in N\setminus S\rangle$ is a linear combination of the respective finite sequences of pure strategies with probability multipliers. If the given inequality is valid for any pure finite sequence $\langle\tau_j:j\in N\setminus S\rangle$, it will be valid also for each mixed strategy, because on both sides the linear combinations have the same multipliers and the inequality for single terms is transferred to the linear combination. In order to make inequality rigorous for some linear combination, it must occur for some pure $\langle\tau_j:j\in N\rangle$, because otherwise, if there were always equality valid for pure strategies, the equality would remain valid also for mixed strategies. Hence, without changing the contents of the definition, only pure strategies can be considered as finite sequences $\langle\tau_j:j\in N\setminus S\rangle$.

All such finite strategy sequences of the coalition $S$, which are dominated by some other finite sequence on the null level, can be named the null level dominated finite sequences.

Let us prove the following statement: if any pure strategy $\langle\tau_i:i\in S\rangle$ is dominated on the null level, the same applies to any mixed strategy while $\langle\tau_i:i\in S\rangle$ exists in its equation with certain positive probability.

Indeed, let us assume that the pure strategy $\langle\tau_i:i\in S\rangle$ is dominated by (perhaps a mixed) strategy $\langle\tau_i':i\in S\rangle$ on the null level. If now in some equation of mixed strategy there exists $\langle\tau_i:i\in S\rangle$ with the positive probability $\lambda$, then by eliminating this term from the equation and adding the $\langle\tau_i':i\in S\rangle$ equation with coefficient $\lambda$ to the remaining part, we shall get a new finite strategy sequence, which will dominate over the original mixed strategy. Thus the above statement is proved.

### 4.2. Elimination

Let us assume now that for a certain $m$, the domination is defined for any level lower than $m$ and for each possible coalition. Let us assume also that for any level lower than $m$, the following statement is valid: if any pure strategy can be dominated on the level lower than $m$, any mixed strategy, which includes this pure strategy in its expression with a positive probability, can be dominated at most on the same level.

Let us consider a certain coalition $T$. The members of this coalition are not aware about partitioning of $N\setminus T$ into separate coalitions. Let us consider some partition and a strategy for each part. As we know, the combination of these strategies can also give one possible strategy for $N \setminus T$ as a whole coalition. We can say that the particular partition **eliminates** this strategy on the $m$th level if for some $S \subseteq N \setminus T$, which is a member of this partition, the strategy selected for the coalition $S$ can be dominated on the level lower than $m$. We shall say that the finite strategy sequence $\left\langle \tau'_j : j \in N \setminus T \right\rangle$ on the $m$th level **is excluded relative to** $T$ if any partition, which with its suitably chosen combination of strategies can give $\left\langle \tau'_j : j \in N \setminus T \right\rangle$, eliminates it on the $m$th level. First, it should be noted here that among these partitions the partition of $N \setminus T$ is always included in a single piece when choosing the final strategy sequence $\left\langle \tau'_j : j \in N \setminus T \right\rangle$ for it. Second, for any other partition, if it is possible altogether to get $\left\langle \tau'_j : j \in N \setminus T \right\rangle$ as a combination of suitable strategies, this combination for the given partition will be defined uniquely: namely, for each partition $S$, the probability for each pure strategy in the set $S$ must be defined by the total of probabilities of these pure strategies of the set $N \setminus T$ where the given pure strategy of the set $S$ belongs to.

Let us take some mixed strategy sequence $\left\langle \tau'_j : j \in N \setminus T \right\rangle$ in the set $N \setminus T$. Let us assume that a certain pure strategy $\left\langle \tau_j : j \in N \setminus T \right\rangle$, which is given with positive probability in the expression $\left\langle \tau'_j : j \in N \setminus T \right\rangle$, must be excluded relative to $T$ on the $m$th level. Let us take any such partition, which as a suitably chosen combination of strategies will give $\left\langle \tau'_j : j \in N \setminus T \right\rangle$. The same partition will also give $\left\langle \tau_j : j \in N \setminus T \right\rangle$ when choosing the pure strategy $\left\langle \tau_j : j \in S \right\rangle$ for each $S$ in this partition. Since $\left\langle \tau_j : j \in N \setminus T \right\rangle$ is excluded relative to $T$ on the $m$th level, for a certain $S$, $\left\langle \tau_j : j \in S \right\rangle$ must be dominated on the level lower than $m$. However, since $\left\langle \tau_j : j \in N \setminus T \right\rangle$ in the expression $\left\langle \tau'_j : j \in N \setminus T \right\rangle$ has positive probability, the total of probabilities of all such pure strategies in the expression $\left\langle \tau'_j : j \in N \setminus T \right\rangle$, which include $\left\langle \tau_j : j \in S \right\rangle$, is definitely positive. Thus the considered mixed strategy of the set $S$ comprises $\left\langle \tau_j : j \in S \right\rangle$ with positive probability and is assumably dominated. It means that this partition eliminates $\left\langle \tau'_j : j \in N \setminus T \right\rangle$. In case of each partition that could give $\left\langle \tau'_j : j \in N \setminus T \right\rangle$, $\left\langle \tau'_j : j \in N \right\rangle$ is excluded relative to $T$ on the $m$th level.

Thus the following statement has been proved: if a pure strategy has been excluded on a certain level, any mixed strategy where this pure strategy has positive probability will also be excluded on the same level.

### 4.3. Domination on a higher level

Let $\left\langle \tau'_i : i \in T \right\rangle$ and $\left\langle \tau''_i : i \in T \right\rangle$ be such two finite strategy sequences where none of them can be dominated on the level lower than $m$. We shall say that $\left\langle \tau'_i : i \in T \right\rangle$ **dominates** over $\left\langle \tau''_i : i \in T \right\rangle$ **on the $m$th level** if for any such (mixed) strategy $\left\langle \tau_j : j \in N \setminus T \right\rangle$, which is not excluded relative to $T$ on the $m$th level,

$$\sum_{k \in T} \alpha_k (\tau'_i : i \in T; \quad \tau_j : j \in N \setminus T) \geq \sum_{k \in T} \alpha_k (\tau''_i : i \in T; \; \tau_j : j \in N \setminus T)$$

is valid, while for some such finite sequences $\left\langle \tau_j : j \in N \setminus T \right\rangle$ the inequality is rigorous.

Similar to the null level, we can show that here, too, we could limit ourselves to pure strategies $\left\langle \tau_j : j \in N \setminus T \right\rangle$ only by taking advantage of the circumstances that mixed strategies, which are not excluded relative to $T$ on the $m$th level, can comprise with positive probabilities only those pure strategies which are not excluded relative to $T$ on the $m$th level. Also, similar to the null level, we can show that if any pure strategy $\left\langle \tau_i : i \in T \right\rangle$ is dominated on the $m$th level, any mixed strategy, which includes $\left\langle \tau_i : i \in T \right\rangle$ with a positive multiplier, is also dominated.

If it can be assumed in the coalition $T$ that any player from the set $N \setminus T$ is able to think on all levels lower than $m$, this player will know that $\left\langle \tau_j : j \in N \setminus T \right\rangle$, excluded on the $m$th level relative to $T$, will not occur independent of the coalitions formed in the set $N \setminus T$. In this case the player himself can give up the finite sequence $\left\langle \tau_i'' : i \in T \right\rangle$ dominated on the $m$th level when thinking on that level. Any player from the set $N \setminus T$ who thinks on the $m$th level can assume now also that if the coalition $T$ exists, $\left\langle \tau_i'' : i \in T \right\rangle$ will not be used.

So an iterative process will emerge, which excludes a number of formally possible finite strategy sequences for each possible coalition.

## 5. MODIFICATION OF A CHARACTERISTIC FUNCTION

Let us define the characteristic function $v_0$ in the same way as the classical characteristic function $v$. Let us define $v_m$ for $m > 0$ as follows.

Let a certain coalition $T$ and $\left\langle \tau_i : i \in T \right\rangle$ be such a finite strategy sequence that cannot be dominated on any level lower than $m$. Let $\Phi_m(T)$ be the set of all such pure finite strategy sequences $\left\langle \tau_j : j \in N \setminus T \right\rangle$, which is not excluded on the $m$th level relative to $T$. Then

$$\min_{\left\langle \tau_j : j \in N \setminus T \right\rangle \in \Phi_m(T)} \sum_{k \in T} \alpha_k (\tau_i : i \in T;\ \tau_j : j \in N \setminus T)$$

can be found. It can easily be seen that the value of the expression will not change if we expand $\Phi_m(T)$ also with the mixed finite sequences $\left\langle \tau_j' : j \in N \setminus T \right\rangle$, which have not been excluded on the $m$th level relative to $T$. Doing this we shall get the set $\tilde{\Phi}_m(T)$. By taking the maximum from these minimums over the above mixed finite sequences $\left\langle \tau_i : i \in T \right\rangle$, we shall get $v_m(T)$.

Let $\left\langle \tau_i' : i \in T \right\rangle$ and $\left\langle \tau_i'' : i \in T \right\rangle$ be two such finite sequences, which cannot be dominated on the level lower than $m$. Let us assume that the second sequence is dominated by the first on the $m$th level. It means that for each $\left\langle \tau_j : j \in N \setminus T \right\rangle \in \Phi_m(T)$

$$\sum_{k\in T}\alpha_k(\tau_i':i\in T;\ \tau_j:j\in N\setminus T)\geq\sum_{k\in T}\alpha_k(\tau_i'':i\in T;\ \tau_j:j\in N\setminus T)$$

is valid.

This inequality will certainly be valid also if we take the minimum from both sides over $\langle\tau_j:j\in N\setminus T\rangle\in\Phi_m(T)$. However, it means that by taking the maximum, we can exclude $\langle\tau_i'':i\in T\rangle$ without the value of maximum being changed. Consequently, by calculating $v_m(T)$, we could restrict ourselves to only these finite strategy sequences $\langle\tau_i:i\in T\rangle$ which cannot be dominated on the level lower than $m+1$.

Let us compare now $v_m(T)$ and $v_{m+1}(T)$. Since $\Phi_{m+1}(T)\subseteq\Phi_m(T)$, the minimum over $\Phi_{m+1}(T)$ is higher or equal to the minimum of the same magnitude over $\Phi_m(T)$. When calculating $v_{m+1}(T)$, we have to take the maximum from the magnitude

$$\min_{\langle\tau_j:j\in N\setminus T\rangle\in\Phi_{m+1}(T)}\sum_{k\in T}\alpha_k(\tau_i:i\in T;\ \tau_j:j\in N\setminus T)$$

over these finite sequences $\langle\tau_i:i\in T\rangle$ which cannot be dominated on the level lower than $m+1$. But since we have seen that when calculating $v_m(T)$, we can take the maximum over the same finite sequences, but from the magnitude

$$\min_{\langle\tau_j:j\in N\setminus T\rangle\in\Phi_m(T)}\sum_{k\in T}\alpha_k(\tau_i:i\in T;\ \tau_j:j\in N\setminus T)$$

which could be only smaller or equal to the previous value, we shall get $v_m(T)\leq v_{m+1}(T)$.

The validity of the first of the main axioms of the characteristic function $v(\varnothing)=0$ for each $v_m$ follows directly from the definition. Let us check the second axiom $v(T\cup S)\geq v(T)+v(S)$, where $T\cap S=\varnothing$. If $m=0$, then $v_m=v_0$ is a classical characteristic function $v$ for which the axiom is valid. Let us consider now the case $m>0$.

Let $T,S\subseteq N$, $T\cap S=\varnothing$. Let $\langle\tau_i:i\in T\rangle$ be such a finite strategy sequence, which cannot be dominated on the level lower than $m$ and which maximizes

$$\min_{\langle\tau_j:j\in N\setminus T\rangle\in\Phi_m(T)}\sum_{k\in T}\alpha_k(\tau_i:i\in T;\ \tau_j:j\in N\setminus T),$$

giving thus the value $v_m(T)$. Let $\langle\tau_i:i\in S\rangle$ be selected in a similar way.

Let us take a random finite strategy sequence $\langle\tau_j:j\in N\setminus(T\cup S)\rangle$. Let $\langle\tau_i:i\in S;\ \tau_j:j\in N\setminus(T\cup S)\rangle$ be the finite strategy sequence in the set $N\setminus T$, which has been obtained as a combination of the last two strategy combinations. If this finite sequence does not belong to $\tilde{\Phi}_m(T)$, it must be excluded relative to $T$ at most on the $m$th level, i.e., any partition in the set $N\setminus T$, which can give this finite sequence in single pieces as a combination of strategies, must eliminate this partition at most on the $m$th level. These partitions include also all those where one piece is $S$ and the rest of pieces are such parts of the set $N\setminus(T\cup S)$

that the finite sequence $\left\langle \tau_j : j \in N \setminus (T \cup S) \right\rangle$ is available on these pieces as a combination of the given finite sequences. For each partition, a piece must be found where the finite strategy sequence selected can be dominated on some level lower than $m$. As $\left\langle \tau_i : i \in S \right\rangle$ is not the case, it must take place on some piece of the set $N \setminus (T \cup S)$. As it takes place for each partition of the set $N \setminus (T \cup S)$, where $\left\langle \tau_j : j \in N \setminus (T \cup S) \right\rangle$ can be obtained as a combination, the last mentioned finite strategy sequence relative to $T \cup S$ will be excluded at most on the $m$th level and thus will not belong to $\tilde{\Phi}_m (T \cup S)$. When reverting this conclusion, we can see that if $\left\langle \tau_j : j \in N \setminus (T \cup S) \right\rangle \in \tilde{\Phi}_m (T \cup S)$, then $\left\langle \tau_i : i \in S;\ \tau_j : j \in N \setminus (T \cup S) \right\rangle \in \tilde{\Phi}_m (T)$.

In a similar way we can get that if $\left\langle \tau_j : j \in N \setminus (T \cup S) \right\rangle \in \tilde{\Phi}_m (T \cup S)$, then $\left\langle \tau_i : i \in T;\ \tau_j : j \in N \setminus (T \cup S) \right\rangle \in \tilde{\Phi}_m (S)$.

When taking the above shown $\left\langle \tau_i : i \in T \right\rangle$ and $\left\langle \tau_i : i \in S \right\rangle$ and a random $\left\langle \tau_j : j \in N \setminus (T \cup S) \right\rangle \in \tilde{\Phi}_m (T \cup S)$, we shall get

$$
\begin{aligned}
&\sum_{k \in T \cup S} \alpha_k (\tau_i : i \in S;\ \tau_i : i \in T;\ \tau_j : j \in N \setminus (T \cup S)) \\
&\quad = \sum_{k \in T} \alpha_k (\tau_i : i \in S;\ \tau_i : i \in T;\ \tau_j : j \in N \setminus (T \cup S)) \\
&\quad\quad + \sum_{k \in S} \alpha_k (\tau_i : i \in S;\ \tau_i : i \in T;\ \tau_j : j \in N \setminus (T \cup S)) \\
&\quad \geq \min_{\left\langle \tau_j : j \in N \setminus T \right\rangle \in \tilde{\Phi}_m (T)} \sum_{k \in T} \alpha_k (\tau_i : i \in T;\ \tau_j : j \in N \setminus T) \\
&\quad\quad + \min_{\left\langle \tau_j : j \in N \setminus S \right\rangle \in \tilde{\Phi}_m (S)} \sum_{k \in S} \alpha_k (\tau_i : i \in S;\ \tau_j : j \in N \setminus S) \\
&\quad = v_m (T) + v_m (S).
\end{aligned}
$$

Since the selected $\left\langle \tau_j : j \in N \setminus (T \cup S) \right\rangle \in \tilde{\Phi}_m (T \cup S)$ was random,

$$
\begin{aligned}
\min_{\left\langle \tau_j : j \in N \setminus (T \cup S) \right\rangle \in \tilde{\Phi}_m (T \cup S)} &\sum_{k \in T \cup S} \alpha_k (\tau_i : i \in S;\ \tau_i : i \in T;\ \tau_j : j \in N \setminus (T \cup S)) \\
&\geq v_m (T) + v_m (S).
\end{aligned}
$$

When the considered finite sequence $\left\langle \tau_i : i \in T;\ \tau_i : i \in S \right\rangle$ cannot be dominated on the level lower than $m$ on the set $T \cup S$, then by replacing it with a finite sequence, which maximizes the left side of the last inequality with the finite sequences not dominating on the levels lower than $m$ over the set $T \cup S$, we shall get

$$
v_m (T \cup S) \geq v_m (T) + v_m (S).
$$

However, if the considered finite sequence $\left\langle \tau_i : i \in T;\ \tau_i : i \in S \right\rangle$ can be dominated on the level $n < m$, then by replacing it first with a dominating finite sequence, we shall increase the left side of the inequality again, since the domination will be valid for all families $\langle \tau_j : j \in N \setminus (T \cup S) \rangle \in \tilde{\Phi}_n (T \cup S)$ while $\tilde{\Phi}_n (T \cup S) \supseteq \tilde{\Phi}_m (T \cup S)$. If necessary, we shall repeat it by moving on along the intermediate levels from $n$ to $m$ with taking on a certain level always the finite sequence that maximizes the left side of the inequality. When reaching the finite sequence maximizing the left side that cannot be dominated on the $m$th level, we shall get again $v_m (T \cup S) \geq v_m (T) + v_m (S)$.

Thus the statement that both axioms will be satisfied for each $v_m$ is proved.

If the set of pure strategies is finite, then for each coalition $T$, $\Phi_m (T)$ must be constant from some $m$ on. Starting from that $m$ the concept of domination for the coalition $T$ will also be stabilized. Since the set of coalitions is also finite, the concept of overall domination and thus the concept of exclusion will be stabilized at a certain value of $m$.

Thus, such strategies $\langle \tau_i' : i \in T \rangle$, which cannot be dominated on any level, and such sets $\Phi(T)$, which form a common part for all $\Phi_m (T)$ over the values $m$, can also be considered. For each such $\langle \tau_i' : i \in T \rangle$, we shall find

$$\min_{\langle \tau_j : j \in N \setminus T \rangle \in \Phi(T)} \sum_{k \in T} \alpha_k (\tau_i' : i \in T;\ \tau_j : j \in N \setminus T)$$

and take the maximum from these magnitudes over all non-dominated finite sequences $\langle \tau_i' : i \in T \rangle$. We shall denote the obtained value by $v(T)$. Since $v \equiv v_m$ for a sufficiently high value of $m$, the principal axioms will be valid for $v$ as well.

## ACKNOWLEDGEMENT

This study was financially supported by the Estonian Science Foundation (grant No. 4791).

## REFERENCES

1. Krishna, V. and Ramesh, V. C. Intelligent agents for negotiations in market games. Parts 1, 2. *IEEE Trans. Power Syst*., 1998, **13**, 1103–1108 and 1109–1114.
2. Krumm, L., Kurrel, Ü., Tauts, A., Terno, O, Zeidmanis, I., Krišans, Z., Dale, V., Putnynsh, V. and Oleinikova I. Theory and methods of complex optimisation of the interconnected power system. In *Proceedings: World Energy Council Regional Forum, Central and East European Energy Policies, Markets and Technologies for the 21. Century, September 16–18, 1999, Lithuania. Vilnius*. 1999, 198–210.
3. Krumm, L., Kurrel, Ü., Tauts, A., Terno, O., Zeidmanis, I. and Krišans, Z. Power system control complex optimisation in the new market condition. In *Preprints: FAC Symposium on Power Plants & Power System Control April 26–29, 2000 Belgium, Brussels.* 2000, 483–488.
4. Owen, G. *Theory of Games*. Moscow, 1971 (in Russian).

# Maxmin-printsiibi täiustamine mitte-nullsumma mängudes

Ants Tauts

On vaadeldud maxmin-printsiibi modifikatsiooni mitte-nullsumma mängus, kus mängijad ei püüa üksteist kahjustada, vaid ainult ise kasu saada. Esialgu kõrvaldab mängija iga vastasmängija sellised strateegiad, mis on neile igal juhul vähem soodsad kui mõned teised. Eeldusel, et ka teised mängijad on teinud sama, korratakse tekkinud uues olukorras esimese sammuga analoogset mõttekäiku. Käivitub järk-järguline konkretiseerumisprotsess, mis stabiliseerudes annab lõpptulemuse.