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Some summability methods b-equivalent to the Cesàro methods

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Abstract. The paper deals with summability methods which are equivalent for summing bounded sequences (b-equivalent). It is well known that the Cesàro methods C_α ($\alpha > 0$) and the Abel method A are b-equivalent. More generally, different authors have proved that generalized Nörlund methods (N, a, b) and Abel-type power series methods J_q are b-equivalent under certain conditions on these methods. It turns out that quite often these conditions imply the b-equivalence of the methods (N, a, b) and J_q to C_α ($\alpha > 0$) as well. The idea of this paper is to investigate the b-equivalence of the methods (N, a, b), J_q, and C_α ($\alpha > 0$).

Key words: summability methods, generalized Nörlund methods, Cesàro methods, power series methods, b-equivalence of methods.

1. INTRODUCTION AND PRELIMINARIES

We begin with the definition of generalized Nörlund summability methods and power series methods of Abel type. Let (ξ_n) denote throughout the paper a complex sequence and $q = (q_n)$ a non-negative sequence with $q_0 > 0$ ($n \in \mathbf{N} = \{0, 1, 2, \dots\}$). For the definition of the power series method J_q (see [1]) we suppose that

$$\text{the power series } \; q(x) = \sum_{n=0}^{\infty} q_n x^n \; \text{ has the radius of convergence } R = 1. \tag{1}$$

We say that (ξ_n) is summable to ξ by the power series summability method J_q and write $\xi_n \to \xi(J_q)$ if

$$q_\xi(x) = \sum_{n=0}^{\infty} \xi_n q_n x^n \text{ converges for } |x| < 1$$

and

$$\frac{q_\xi(x)}{q(x)} \to \xi \text{ as } x \to 1- \ .$$

In particular, if $q_n \equiv 1$, then J_q is the Abel method, i.e. $J_q = A$. If $q = A^\alpha = (A_n^\alpha) = \left(\binom{n+\alpha}{n}\right)$, $\alpha > -1$, then J_q is the generalized Abel method A_α. Therefore we say that the power series method J_q is an Abel-type method (in contrast to the case with $R = \infty$ where we speak of Borel-type methods).

In the sequel the following restrictions on (q_n) will be important:

$$\sum_{k=0}^{n} q_k \to \infty \quad (n \to \infty), \tag{2}$$

$$nq_n = O\left(\sum_{k=0}^{n} q_k\right) \quad (n \to \infty), \tag{3}$$

$$\sum_{k=0}^{n} q_k = O(nq_n) \quad (n \to \infty). \tag{4}$$

We note that (4) implies (2), and the conditions (2) and (3) imply (1) as $R \leq 1$ by (2) and $R \geq 1$ by (3). By Theorem 5 in [2] the method J_q is regular, i.e. $\xi_n \to \xi \quad (n \to \infty)$ implies $\xi_n \to \xi(J_q)$, if and only if (2) holds. Notice that (3) is satisfied, for example, in case of a non-increasing and (4) in case of a non-decreasing sequence (q_n). If, in particular, $q_n = A_n^\gamma$ ($\gamma > -1$), then (3) and (4) both are satisfied. The conditions (3) and (4) are satisfied also in case of $q_n = n^\gamma L(n)$ ($n > n_0$), where $\gamma > -1$ and $L(.)$ is a slowly varying function (i.e., in case of regularly varying weights q_n, see [3] for definitions) because of the relation

$$\sum_{k=0}^{n} A_{n-k}^{\alpha-1} k^\gamma L(k) \sim \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} n^{\alpha+\gamma} L(n) \quad (n \to \infty, \ \ \alpha > 0, \ \ \gamma > -1) \tag{5}$$

(see [4], Lemma A 1), where $\Gamma(.)$ is the gamma function.

The definition of a generalized Nörlund method (N, a, b) was given in [5] and is as follows:

Let $a = (a_n)$ and $b = (b_n)$ be real sequences with the convoluted sequence

$$(a * b)_n = \sum_{k=0}^{n} a_{n-k} b_k \neq 0 \qquad (n \in \mathbf{N}).$$

We say that (ξ_n) is summable by the generalized Nörlund method (N, a, b) to ξ and write $\xi_n \to \xi(N, a, b)$ if

$$\eta_n = \frac{1}{(a * b)_n} \sum_{k=0}^{n} a_{n-k} b_k \xi_k \to \xi \qquad (n \to \infty).$$

The theorem of Toeplitz (see Theorem 2 in [2]) says that the method (N, a, b) is regular if and only if the following two conditions are satisfied:

$$\begin{aligned} &\frac{a_{n-k}b_k}{(a*b)_n} \to 0 \qquad (n\to\infty,\ \ k\in\mathbf{N}),\\ &\sum_{k=0}^{n}|a_{n-k}b_k| = O((a*b)_n)\ \ (n\to\infty). \end{aligned} \tag{6}$$

In particular, if $b_n \equiv 1$, then we have the Nörlund method $(N, a) = (N, a, \mathbf{1})$, if also $a_n = A_n^{\alpha-1}$, then we have the Cesàro methods $(N, A^{\alpha-1}, \mathbf{1}) = (C, \alpha) = C_\alpha$. If $b_n = A_n^\gamma$ and $a_n = A_n^{\alpha-1}$, then we get the generalized Cesàro methods $(N, A^{\alpha-1}, A^\gamma) = (C, \alpha, \gamma)$. If $a_n \equiv 1$, then we have the Riesz methods $(N, \mathbf{1}, b) = (\overline{N}, b)$ (for more examples see [6–13]).

For any two summability methods A and B we say that B is not weaker than A and write $A \subset B$ if $\xi_n \to \xi\ (B)$ whenever $\xi_n \to \xi\ (A)$. We say that methods A and B are equivalent and write $A \sim B$ if both the relations $A \subset B$ and $B \subset A$ hold. If the relation

$$\xi_n \to \xi(A) \Leftrightarrow \xi_n \to \xi(B)$$

is true for all bounded sequences (ξ_n), then we say that A and B are *b*-equivalent (or, A is *b*-equivalent to B).

Relations between the methods (N, a, q) and J_q were investigated in [14] and [15] in general and, in more or less general cases, also in all papers listed in References to our paper. In particular, some families of methods (N, a^α, q), where α is a discrete or continuous parameter and a^α is defined as convolution of sequences, have been constructed and relations between the methods (N, a^α, q) themselves, and between these methods and related power series methods J_q have been investigated (see [7–13]). Among other results the mentioned papers present sufficient conditions for the *b*-equivalence of the methods (N, a^α, q) to each other and to J_q. It turns out that quite often these conditions are sufficient (or almost sufficient) for the *b*-equivalence of the considered methods to the Cesàro methods C_α $(\alpha > 0)$ as well.

The idea of the present paper is to extend these investigations by studying the *b*-equivalence of the methods (N, a, q), J_q, and C_α $(\alpha > 0)$. Different sets of sufficient conditions for the *b*-equivalence of these methods will be found here.

The following inclusion relations are quite well known (see Theorem 43 in [2] and Theorem 2 in [16]):

$$C_\alpha \subset C_\beta \subset A_\gamma \qquad (\beta > \alpha > -1,\ \ \gamma > -1), \tag{7}$$

$$A_\gamma \subset A_\delta \qquad (\gamma > \delta > -1). \tag{8}$$

Also (see [17]),

$$(\overline{N}, q) \subset J_q \tag{9}$$

provided that (1) holds.

Note that the inclusion relations (7), (8), and (9) are strict, i.e. the methods compared there are not equivalent.

We take for our starting-point the following three theorems (see Theorem 92 in [2] and Theorem 4.3 in [18] together with (7) and (9), respectively, and Lemma 2 in [19]).

Theorem A. *The Cesàro methods C_α $(\alpha > 0)$ and the Abel method A are b-equivalent.*

Theorem B. *If the conditions* (2) *and* (3) *are satisfied, then the methods* $(\overline{N}, q)$ *and* J_q *are b-equivalent.*

Theorem C. *Let* (q_n) *satisfy the conditions* (1) *and* (2) *and be positive for all large* n. *If* (g_n) *is a non-negative sequence with* $g_0 > 0$ *such that* $g_n/q_n \to 1$ $(n \to \infty)$, *then the method* J_g *is b-equivalent to* J_q.

2. MAIN THEOREMS

We will present here two theorems.

Let $c = (c_n)$ and $p = (p_n)$ be two non-negative sequences such that $c_0, p_0 > 0$ and $(c * p) * q = (r_n)$ is a positive sequence. Consider the generalized Nörlund method $(N, c * p, q)$ and the power series method J_q.

Theorem 1. *Let us suppose that* (c_n) *satisfies the condition*

$$n\, c_n = O\left(\sum_{k=0}^{n} c_k\right) \qquad (n \to \infty) \tag{10}$$

and either

(i) (q_n) *is non-decreasing and satisfies* (3)

or

(ii) (q_n) *is non-increasing and satisfies* (4).

Suppose also that either

(iii) (p_n) *is non-decreasing and*

$$n\, p_n = O\left(\sum_{k=0}^{n} p_k\right) \qquad (n \to \infty) \tag{11}$$

or

(iv) (p_n) *is non-increasing and*

$$\sum_{k=0}^{n} q_k = O((p * q)_n) \qquad (n \to \infty). \tag{12}$$

Then the method $(N, c * p, q)$ *is b-equivalent to* J_q *and to the Cesàro methods* C_α $(\alpha > 0)$ *as well.*

Remark 1. Notice that the method $(N, c * p, q)$ turns into the method (N, p, q) if $c_n = \delta_{0,n}$. Thus, Theorem 1 says that the method (N, p, q) is *b*-equivalent to the Cesàro methods C_α ($\alpha > 0$) if conditions (i) or (ii) and (iii) or (iv) of Theorem 1 are satisfied. In particular, the method $(\overline{N}, q)$ is *b*-equivalent to the Cesàro methods C_α ($\alpha > 0$) if (i) or (ii) is satisfied.

In particular, if $c_n = A_n^{\alpha-1}$, then the restrictions on p_n and q_n in Theorem 1 can be weakened. Thus we get another theorem.

Denote $p_n^\alpha = \left(A^{\alpha-1} * p\right)_n$ and consider the methods

$$(N, p^\alpha, q) = (N, A^{\alpha-1} * p, q) = (N, c * p, q),$$

where α is a continuous parameter with values $\alpha > \alpha_0$ and α_0 is such a number that $p^\alpha * q = (A^{\alpha-1} * p) * q = (r_n^\alpha)$ are positive sequences. Notice that the last condition is surely satisfied if $\alpha_0 = 0$, and the relation

$$p^\beta = A^{\beta-\alpha-1} * p^\alpha \quad (\beta > \alpha_0, \ \ \alpha > \alpha_0) \tag{13}$$

holds by the properties of convolutions and the Cesàro numbers A_n^α.

The structure of the family of methods (N, p^α, q) was observed in [10,12,13] in the general case and in partial cases also in [6,8,11]. In this paper we will prove the following theorem.

Theorem 2. *Let us consider the methods* $(N, p^\alpha, q) = (N, A^{\alpha-1} * p, q)$ *with* $\alpha > 0$. *Suppose that* (q_n) *and* (p_n) *satisfy the conditions* (1), (3), *and* (11), *respectively.*

(i) *Then the methods* (N, p^α, q) $(\alpha > 0)$ *are b-equivalent to* J_q.

(ii) *If, in addition,* (q_n) *is non-decreasing or* (q_n) *is non-increasing and satisfies* (4), *then the methods* (N, p^α, q) $(\alpha > 0)$ *are b-equivalent to the Cesàro methods* C_δ $(\delta > 0)$.

To prove Theorems 1 and 2 we need some auxiliary results.

3. AUXILIARY PROPOSITIONS

Proposition 1. *If* (q_n) *satisfies conditions* (i) *or* (ii) *of Theorem* 1, *then the methods* J_q *and* C_α $(\alpha > 0)$ *are b-equivalent. In particular, the generalized Abel methods* $J_q = A_\gamma$ $(\gamma > -1)$ *and* C_α $(\alpha > 0)$ *are b-equivalent.*

Proof. The methods J_q and $(\overline{N}, q)$ are *b*-equivalent by Theorem B because the conditions (2) and (3) both are satisfied. Further, $(\overline{N}, q) \sim C_1$ by Theorem 14 in [2] and C_1 is *b*-equivalent to C_α ($\alpha > 0$) by Theorem A. It remains to note that $q_n = A_n^\gamma$ satisfies condition (i) if $\gamma \geq 0$ and condition (ii) if $-1 < \gamma \leq 0$. $\square$

Proposition 2. *Suppose that (g_n) is a non-negative sequence with $g_0 > 0$ and $g_n \sim n^\gamma L(n)$ $(n \to \infty,\ \gamma > -1)$, where $L(.)$ is a slowly varying function. If $(n^\gamma L(n))$ is monotonic, then the methods J_g and C_α $(\alpha > 0)$ are b-equivalent.*

Proof. Our proposition is a direct conclusion from the previous one and Theorem C. Denote $q_n = n^\gamma L(n)$ $(n > n_0)$ and see from (5) that (q_n) satisfies (3) and (4). Thus conditions (i) or (ii) of Theorem 1 are satisfied and J_q is *b*-equivalent to C_α $(\alpha > 0)$. It follows now from Theorem C that J_g is *b*-equivalent to C_α $(\alpha > 0)$. $\square$

Remark 2. (i) Notice that if (q_n) is monotonic and satisfies (2) and (3), then the relation $C_1 \subset J_q$ holds (use (9) and Theorem 14 in [2]).

(ii) If $q_n = \frac{1}{n+1}$, then J_q is not *b*-equivalent to C_α $(\alpha > 0)$ because there exists a bounded sequence (ξ_n) summable by J_q but not by C_1 (see [2], Section 3.8 and Theorem 82).

The next proposition is proved in [12] as Lemma 1.1(h).

Proposition 3. *Let (q_n) satisfy (1) and the power series $\sum_{n=0}^{\infty} (c * p)_n x^n$ have the radius of convergence $R \geq 1$. If*

$$\sum_{k=0}^{n} ((c * p) * q)_k \to \infty \qquad (n \to \infty) \tag{14}$$

and

$$\sum_{n=0}^{\infty} (c * p)_n z^n \neq 0 \tag{15}$$

in the unit disc $|z| < 1$ on the complex plane then [1]

$$(N, c * p, q) \subset J_q.$$

Remark 3. In particular, if we consider the methods (N, p^α, q) $(\alpha > \alpha_0)$, then we have by Proposition 3

$$(N, p^\alpha, q) \subset J_q,$$

provided that (q_n) satisfies (1),

$$\text{the power series } \sum_{n=0}^{\infty} p_n z^n \text{ has } R \geq 1 \tag{16}$$

and $\sum_{n=0}^{\infty} p_n z^n \neq 0$ in the unit disc on the complex plane (cf. [12], Proposition 2.5). The last restriction is redundant if we apply our inclusion relation to bounded sequences (ξ_n) only.

[1] If we consider the following inclusion relation only for bounded sequences (ξ_n), then the condition (15) can be dropped. Note that c_n may be also negative for some n in this proposition.

Proposition 4. *If* (c_n) *satisfies* (10) *and either*

(i) (q_n) *is non-decreasing*

or

(ii) (q_n) *is non-increasing and satisfies* (4),

then the method $(N, c, p * q)$ *is regular.*

Proof. Since the matrix $(N, c, p * q)$ is non-negative, we have to verify only the first regularity condition (6). In case (i) we have:

$$\frac{c_{n-k}}{r_n} \leq \frac{c_{n-k}}{p_0 q_0 \sum_{\nu=0}^n c_\nu} \leq \frac{M \sum_{\nu=0}^n c_\nu}{(n-k) \sum_{\nu=0}^n c_\nu} = O\left(\frac{1}{n-k}\right) = o_k(1) \quad (n \to \infty).$$

In case (ii) we get analogously that

$$\frac{c_{n-k}}{r_n} \leq \frac{c_{n-k}}{p_0 q_n \sum_{k=0}^n c_k} \leq \frac{K\, n c_{n-k}}{Q_n \sum_{k=0}^n c_k} = O\left(\frac{n}{(n-k) Q_n}\right) = o_k(1) \quad (n \to \infty).$$

□

Proposition 5. *If the conditions of Proposition* 4 *are satisfied, then the relation*

$$(N, p, q) \subset (N, c * p, q)$$

holds.

Proof. Let us verify the equality

$$(N, c * p, q) = (N, c, p * q) \circ (N, p, q), \tag{17}$$

where the right side can be read as superposition of two transforms. Indeed, for a sequence (ξ_n) we have:

$$\begin{aligned} \frac{1}{r_n} \sum_{k=0}^n (c * p)_{n-k} q_k \xi_k &= \frac{1}{r_n} \sum_{k=0}^n \sum_{\nu=0}^{n-k} c_{n-k-\nu} p_\nu q_k \xi_k \\ &= \frac{1}{r_n} \sum_{\nu=0}^n c_{n-\nu} (p * q)_\nu \frac{1}{(p * q)_\nu} \sum_{k=0}^\nu p_{\nu-k} q_k \xi_k. \end{aligned}$$

As the method $(N, c, p * q)$ is regular by Proposition 4, our statement follows from (17). □

Remark 4. It follows from (17) and (13) with the help of Proposition 4 that

$$(N, p^\alpha, q) \subset (N, p^\beta, q) \qquad (\beta > \alpha > \alpha_0)$$

(cf. Proposition 2.2 in [12]). Indeed, it is sufficient to notice that the method

$$(N, A^{\beta-\alpha-1}, p^\alpha * q) = (N, A^{\beta-\alpha-1}, (p^{\alpha'} * q) * A^{\alpha-\alpha'-1})$$
$$(\beta > \alpha > \alpha_0, \;\; \alpha' = (\alpha + \alpha_0)/2)$$

satisfies the conditions of Proposition 4 if we take $c_n = A_n^{\beta-\alpha-1}$ and replace q_n by $A_n^{\alpha-\alpha'-1}$ and p_n by $(p^{\alpha'} * q)_n$ in it.

The following result was proved in [12] by Proposition 2.7.

Proposition 6. *If the methods $(N, p^\alpha, q) = (a_{nk}^\alpha)$ $(\alpha > \alpha_0)$ satisfy the conditions* (1), (16),

$$\sum_{k=0}^{n} |a_{nk}^\alpha| = O(1) \qquad (n \to \infty) \tag{18}$$

and

$$M_1 n^{\beta-\alpha} \le \frac{r_n^\beta}{r_n^\alpha} \le M_2 n^{\beta-\alpha} \qquad (n = 1, 2, ...) \tag{19}$$

for all $\beta > \alpha > \alpha_0$, then the implication

$$\xi_n = O(1), \;\; \xi_n \to \xi(J_q) \Rightarrow \xi_n \to \xi(N, p^\alpha, q) \tag{20}$$

is true for any $\alpha > \alpha_0$.

We need also the following proposition.

Proposition 7. *If p_n and q_n satisfy the conditions* (11) *and* (3), *respectively, then* $(p * q)_n$ *satisfies the condition*

$$n(p*q)_n = O\left(\sum_{k=0}^{n} (p*q)_k\right). \tag{21}$$

Proof. With the help of (11) and (3) we get:

$$\begin{aligned}
n\sum_{k=0}^{n} p_{n-k}q_k &= n\sum_{k=0}^{[n/2]} p_{n-k}q_k + n\sum_{k=[n/2]+1}^{n} p_{n-k}q_k \\
&\le n\sum_{k=0}^{[n/2]} p_{n-k}q_k + n\sum_{k=0}^{[n/2]} q_{n-k}p_k \\
&= n\sum_{k=0}^{[n/2]} p_{n-k}\frac{n-k}{n-k}q_k + n\sum_{k=0}^{[n/2]} q_{n-k}\frac{n-k}{n-k}p_k \\
&\le 2M_1\sum_{k=0}^{n} P_{n-k}q_k + 2M_2\sum_{k=0}^{n} Q_{n-k}p_k \\
&= 2M_1\sum_{\nu=0}^{n}(p*q)_\nu + 2M_2\sum_{\nu=0}^{n}(p*q)_\nu = O\left(\sum_{\nu=0}^{n}(p*q)_\nu\right).
\end{aligned}$$

Thus we have proved that (21) holds. □

4. PROOFS OF MAIN THEOREMS

Let us prove now Theorems 1 and 2.

Proof of Theorem 1. The methods J_q and C_α $(\alpha > 0)$ are *b*-equivalent by Proposition 1. It remains to prove that $(N, c * p, q)$ and J_q are *b*-equivalent. Notice that the power series $\sum_{n=0}^{\infty} (c * p)_n x^n$ has the radius of convergence $R \geq 1$, because this series can be seen as the product of the power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} p_n x^n$ which both have $R \geq 1$ due to the restrictions (10) and (11). Also, the condition (14) holds as

$$\sum_{k=0}^{n} ((c * p) * q)_k \geq c_0 p_0 \sum_{k=0}^{n} q_k \qquad (n \in \mathbf{N})$$

and (2) is satisfied. Thus the conditions of Proposition 3 are satisfied and we have by this proposition that the implication

$$\xi_n \to \xi(N, c * p, q) \Rightarrow \xi_n \to \xi(J_q)$$

is true for any bounded sequence (ξ_n). To complete the proof, we have to show that also the implication

$$\xi_n \to \xi(J_q) \Rightarrow \xi_n \to \xi(N, c * p, q)$$

is true for the bounded sequences (ξ_n). Indeed,

$$\xi_n \to \xi(J_q) \Rightarrow \xi_n \to \xi(\overline{N}, q)$$

by Theorem B. As the method (N, p, q) is regular (use Proposition 4), the implication

$$\xi_n \to \xi(\overline{N}, q) \Rightarrow \xi_n \to \xi(N, p, q)$$

is true by Theorem 3 in [15]. Finally, we have:

$$\xi_n \to \xi(N, p, q) \Rightarrow \xi_n \to \xi(N, c * p, q)$$

by Proposition 5. Our theorem is proved. □

Remark 5. (i) It can be seen from the proof of Theorem 1 that also the relations

$$C_1 \subset (N, p, q) \subset (N, c * p, q)$$

hold under the conditions of Theorem 1.

(ii) Note that we needed Theorem 3 from [15] and Theorem 14 from [2] in the proof of Theorem 1. That is why we could not weaken the restrictions on (p_n) and (q_n) in this theorem. These restictions are weakened in Theorem 2, where the special sequences (c_n) are considered.

Proof of Theorem 2. Let us show first that all the conditions of Proposition 6 are satisfied with $\alpha_0 = 0$. Notice that if $\alpha > 0$, then $A_n^{\alpha-1} > 0$ ($n \in \mathbf{N}$), and thus (18) is satisfied by the definition of methods (N, p^α, q). Also, the conditions (3) and (11) imply the inequalities (19) for all $\beta > \alpha > \alpha_0$ by Lemma 2.1 in [12], and (16) is satisfied due to (11). Thus the implication (20) is true by Proposition 6, and our statement (i) follows now from Proposition 3 (see also Remark 3). Statement (ii) is a direct conclusion from (i) and Proposition 1. □

Remark 6. Suppose that (q_n) is as in Theorem 1 or 2 and (g_n) is a non-negative sequence with $g_0 > 0$ such that $g_n/q_n \to 1$ ($n \to \infty$). It can be seen from proofs of Theorems 1 and 2 (with the help of Theorem C) that in the conditions of these theorems also the methods J_g and $(N, c * p, g)$ or (N, p^α, g) ($\alpha > 0$), respectively, are *b*-equivalent to C_δ ($\delta > 0$). For example, this case works if $g_n \sim q_n = n^\gamma L(n)$ ($\gamma > -1$) and (q_n) is monotonic.

5. SOME CONCLUSIONS

We derive now some corollaries from Theorems 1 and 2.

Denote $p^{*\alpha} = p * p^{*(\alpha-1)}$ and $p^{*1} = p$ ($\alpha = 1, 2, ...$) supposing that $(p^{*2})_n = (p * p)_n > 0$ ($n \in \mathbf{N}$). Realize that

$$p^{*\beta} = p^{*(\beta-\alpha)} * p^{*\alpha} \quad (\beta > \alpha, \ \ \beta, \alpha = 1, 2, ...).$$

Consider the methods $(N, p^{*\alpha}, q)$. The following result can be obtained as a corollary from Theorem 1.

Corollary 1. *Let us consider the methods $(N, p^{*\alpha}, q)$ $(\alpha = 1, 2, ...)$. If (q_n) and (p_n) satisfy the conditions of Theorem 1, then the methods $(N, p^{*\alpha}, q)$ $(\alpha = 1, 2, ...)$ are b-equivalent to J_q and to the Cesàro methods C_δ $(\delta > 0)$ as well.*

Proof. If $\alpha = 1$, then our statement follows directly from Theorem 1 if we take $c_n = \delta_{0,n}$ in it. If $\alpha > 1$, then our statement can be also derived immediately from Theorem 1 by taking $c_n = p_n^{*(\alpha-1)}$ in it and realizing that (11) implies here (10) by Proposition 7. □

In particular, if $q = p^{*\gamma}$, then Corollary 1 says as follows.

Corollary 2. *Let us consider the methods $(N, p^{*\alpha}, p^{*\gamma})$, where $\alpha, \gamma = 1, 2, ...$ If (p_n) is non-decreasing and satisfies (11), then the methods $(N, p^{*\alpha}, p^{*\gamma})$ and $J_{p^{*\gamma}}$ $(\alpha, \gamma = 1, 2, ...)$ are b-equivalent to the Cesàro methods C_δ $(\delta > 0)$.*

Proof. Our statement can be derived from Corollary 1 as a direct conclusion, because the sequence $(q_n) = (p_n^{*\gamma})$ ($\gamma = 1, 2, 3, ...$) satisfies (3) due to (11) (see Proposition 7) and is also non-decreasing:

$$p_{n+1}^{*\gamma} = \sum_{k=0}^{n+1} p_{n+1-k} p_k^{*(\gamma-1)} \geq \sum_{k=0}^{n} p_{n+1-k} p_k^{*(\gamma-1)} \geq \sum_{k=0}^{n} p_{n-k} p_k^{*(\gamma-1)} = p_n^{*\gamma}.$$

□

Remark 7. The methods $(N, p^{*\alpha}, p^{*\gamma})$ ($\alpha, \gamma = 1, 2, ...$) obeying the conditions of Corollary 2 were considered in [7,9,12], where certain inclusion, convexity and Tauberian theorems implying the *b*-equivalence of the methods $(N, p^{*\alpha}, p^{*\gamma})$ and $J_{p^{*\gamma}}$ were proved. The *b*-equivalence of these methods in the conditions of Corollary 2 was proved in [12] by Theorem 3.5(b) and Proposition 2.5. In papers [7] and [9] the restrictions on (p_n) are presented in the form $p_n = n^\delta L(n)$ (more precisely, $p_n \sim n^\delta L(n)$, $n \to \infty$), where $\delta \geq 0$, $L(.)$ is a regularly varying function and $(n^\delta L(n))$ is non-decreasing; in [12] also the case $-1 < \delta < 0$ is included. The *b*-equivalence of the methods $(N, p^{*\alpha}, p^{*\gamma})$ to the Cesàro methods was not noticed in these papers.

Finishing our paper we derive a corollary from Theorem 2.

Corollary 3. *Consider the methods* $(N, A^{\alpha-1}, q)$ *with* $\alpha > 0$. *Suppose that* (q_n) *satisfies* (1) *and* (3).

(i) *Then the methods* $(N, A^{\alpha-1}, q)$ $(\alpha > 0)$ *and* J_q *are b-equivalent.*

(ii) *If, in addition,* (q_n) *is non-decreasing or* (q_n) *is non-increasing and satisfies* (4), *then the methods* $(N, A^{\alpha-1}, q)$ $(\alpha > 0)$ *are b-equivalent to the Cesàro methods* C_δ ($\delta > 0$).

This corollary is the immediate conclusion from Theorem 2 for the case $p_n = \delta_{0,n}$. Note that statement (i) was proved in [8] in stronger conditions (2) and (3) (see Theorem 1 and Proposition 1 in [8]).

Remark 8. If $q_n = \frac{1}{n+1}$ and (p_n) satisfies the condition (11), then (N, p^α, q) ($\alpha > 0$) and J_q are *b*-equivalent by Theorem 2. It should be mentioned that the methods (N, p^α, q) are not *b*-equivalent to C_δ ($\delta > 0$) (see Remark 2). In particular, the method $(\overline{N}, q)$ is *b*-equivalent to J_q but not to C_δ ($\delta > 0$).

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Cesàro menetlustega *b*-ekvivalentsetest summeerimismenetlustest

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Artiklis on käsitletud summeerimismenetlusi, mis on tõkestatud jadade summeerimisel ekvivalentsed (*b*-ekvivalentsed). On hästi teada, et Cesàro menetlused C_α ($\alpha > 0$) ja Abeli menetlus A on *b*-ekvivalentsed. Üldisemalt, mitmed autorid on tõestanud, et üldistatud Nörlundi menetlus (N, a, b) ja Abeli tüüpi astmerea menetlus J_q on teatavatel tingimustel *b*-ekvivalentsed. Osutub, et küllalt sageli on saadud tingimustel menetlused (N, a, b) ja J_q *b*-ekvivalentsed ühtlasi ka Cesàro menetlustega C_α ($\alpha > 0$). Käesolevas töös on leitud erinevaid piisavaid tingimusi nimetatud menetluste *b*-ekvivalentsuseks.