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On the summability of Fourier expansions in Banach spaces

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Abstract. Let X be a Banach space with an orthogonal system of projections. Let Z^r ($r > 0$) be the method of Zygmund, $M^\varphi = (\varphi(k/(n+1)))$ the triangular method of summation, generated by the differentiable function φ, and $Z^r_n x$, $M^\varphi_n x$ be Z^r- and M^φ-means of Fourier expansions of $x \in X$, respectively. The author of this paper has proved the theorem (see *Facta Univ. Niš. Ser. Math. Inform.*, 1997, **12**, 233–238) that gives sufficient conditions for $(n+1)^{\alpha+\gamma-1} \parallel M^\varphi_n x_0 - x_0 \parallel = O(1)$ ($x_0 \in X$) if it is assumed that $(n+1)^\alpha \parallel Z^r_n x_0 - x_0 \parallel = O(1)$ for the same x_0, and $g(t) = t^{1-r}\varphi'(t) \in \mathrm{Lip}\,\gamma$ ($\gamma \in]0,1]$) on $]0,1[$. In the present paper this theorem is applied in the cases, where M^φ is either the method of Riesz, Jackson–de La Vallée Poussin, Bohman–Korovkin, Zhuk or Favard.

Key words: Fourier expansions, summability methods, approximation order.

Everywhere in this paper we suppose that X is a Banach space, where there exists a total sequence of mutually orthogonal continuous projections (T_k) $(k = 0, 1, ...)$ on X. It means that T_k is a bounded linear operator of X into itself, $T_k x = 0$ for all k implies $x = 0$, and $T_j T_k = \delta_{jk} T_k$, where δ_{jk} is the Kronecker symbol. Then, with each $x \in X$ one may associate its formal Fourier expansion

$$x \sim \sum_k T_k x.$$

It is known (cf. [1], pp. 74–75, 85–86) that the sequence of projections (T_k) exists in several Banach spaces. For example, if $X = C_{2\pi}$ is the space of all 2π-periodic and continuous functions on $]-\infty, \infty[$ or $X = L^p_{2\pi}$ $(1 \le p < \infty)$ is the space of all 2π-periodic functions, Lebesgue integrable to the pth power over $]-\pi, \pi[$, then the projections are formed by the Fourier coefficients multiplied with associated trigonometric harmonics.

Let us consider now the sequence of projections (T_k) in $L^p(-\infty,\infty)$ $(1 \le p < \infty)$ – the space of all functions, Lebesgue integrable to the pth power over $]-\infty,\infty[$. For this purpose we consider the Hermite polynomials defined by

$$H_k(t) = (-1)^k e^{t^2} \frac{d^k(e^{-t^2})}{dt^k} \qquad (k \ge 0).$$

If we set

$$\varphi_k(t) = (2^k k! \sqrt{\pi})^{-1/2} e^{-t^2/2} H_k(t),$$

(φ_k) is an orthonormal sequence of functions on $]-\infty,\infty[$ (cf. [1], pp. 85–86). Thus the projections

$$T_k x(t) = \left[\int\limits_{-\infty}^{\infty} x(s)\varphi_k(s)ds \right] \varphi_k(t)$$

are mutually orthogonal. One can define the sequence of projections (T_k), for example, also with the help of Laguerre or Jacobi polynomials respectively in $L^p(0,\infty)$ $(1 \le p < \infty)$ – the space of all functions, Lebesgue integrable to the pth power over $]0,\infty[$, and in $C[-1,1]$ – the space of all measurable functions, continuous on $[-1,1]$ (cf. [1], pp. 84, 87).

The summability method of Zygmund Z^r $(r > 0)$ is defined by the equality

$$Z_n^r x = \sum_{k=0}^{n} \left[1 - \left(\frac{k}{n+1} \right)^r \right] T_k x. \tag{1}$$

Let the summability method M^φ be defined by a function φ, continuous on $[0,1]$ and differentiable on $]0,1[$, where $\varphi(0) = 1$ and $\varphi(1) = 0$, as follows:

$$M_n^\varphi x = \sum_{k=0}^{n} \varphi\left(\frac{k}{n+1} \right) T_k x. \tag{2}$$

If $X = C_{2\pi}$ or $X = L^p_{2\pi}$ $(1 \le p < \infty)$, then it is well known that for the classical trigonometric system (T_k) and for $\alpha \in]0,1[$ the relation

$$(n+1)^\alpha \parallel Z_n^1 x - x \parallel = O_x(1)$$

holds if and only if

$$x \in \operatorname{Lip} \alpha = \{ x \in X | \parallel x(t+h) - x(t) \parallel = O_x(h^\alpha) \}$$

(cf. [2], p. 106). Several results, where the order of approximation can be characterized via Lipschitz conditions, are known (cf. [2], pp. 67–88, 106–107). In [3] the order of approximation of the element $x \in X$ by M^φ-means of Fourier expansions was described via the order of approximation by Z^r-means of Fourier expansions, i.e. the following result (see [3], pp. 236–237) holds

Theorem A. *Let* Z_n^r $(r > 0)$ *and* M_n^φ *be defined by* (1) *and* (2), *respectively. Assume that for* $g(t) = t^{1-r}\varphi'(t)$ *on* $]0,1[$ *we have* $g \in \operatorname{Lip}\gamma$, *where* $\gamma \in]0,1]$. *If for some* $x_0 \in X$ *and for* $\alpha \in]1-\gamma, r[$ *the estimation*

$$(n+1)^\alpha \parallel Z_n^r x_0 - x_0 \parallel = O(1) \tag{3}$$

holds, then

$$(n+1)^{\alpha+\gamma-1} \parallel M_n^\varphi x_0 - x_0 \parallel = O(1).$$

The cases, where M^φ is the method of Zygmund or the method of Rogosinski, are studied in [4] and [3], respectively. Now we consider the functions φ_i $(i = 1, ..., 5)$, defined on $[0,1]$ as follows:

$$\varphi_1(t) = (1-t^\omega)^\sigma \ (\omega, \sigma > 0); \tag{4}$$

$$\varphi_2(t) = \begin{cases} 1 - 6t^2 + 6t^3 & (t \in \ [0, \frac{1}{2}]), \\ 2(1-t)^3 & (t \in \ [\frac{1}{2}, 1]); \end{cases} \tag{5}$$

$$\varphi_3(t) = (1-t)\cos(\pi t) + \frac{1}{\pi}\sin(\pi t); \tag{6}$$

$$\varphi_4(t) = 1 - \tan^2\left(\frac{\pi t}{4}\right); \tag{7}$$

$$\varphi_5(t) = \begin{cases} 1 & (t = 0), \\ \dfrac{\pi t}{2}\cot\left(\dfrac{\pi t}{2}\right) & (t \in]0,1]). \end{cases} \tag{8}$$

In this paper we apply Theorem A in the case, where $M^\varphi = M^{\varphi_i}$ $(i = 1, ..., 5)$. The method M^{φ_1} is called the method of Riesz (cf. [2], pp. 265, 475), M^{φ_2} the method of Jackson–de La Vallée Poussin (cf. [2], p. 205), M^{φ_3} the method of Bohman–Korovkin (cf. [5], p. 305), M^{φ_4} the method of Zhuk (cf. [6], p. 319), and M^{φ_5} the method of Favard (cf. [7], p. 161).

Theorem. *Let* M^{φ_i} $(i = 1, \ldots, 5)$ *be the summation methods defined by* (4)–(8). *Assume that for some* $x_0 \in X$ *and for* $\alpha \in]0, r[$ *the estimation* (3) *is valid.*

(I) *The estimation*

$$(n+1)^{\alpha+\gamma-1} \left\| \sum_{k=0}^{n} \left[1 - \left(\frac{k}{n+1} \right)^\omega \right]^\sigma T_k x_0 - x_0 \right\| = O(1)$$

holds if at least one of the following conditions 1–6 *is fulfilled*:

1. $\gamma = 1$, $\sigma \geq 2$ *and* $\omega \geq r+1$ *or* $\omega = r \geq 1$,
2. $\max\{0, 1-\alpha\} < \gamma = \omega - r < 1$ *and* $\sigma \geq 2$,
3. $\max\{0, 1-\alpha\} < \gamma = \sigma - 1 < 1$ *and* $\omega \geq r+1$ *or* $\omega = r \geq 1$,
4. $\max\{0, 1-\alpha\} < \gamma = \min\{\omega - r, \sigma - 1\}$ *and* $\max\{\omega - r, \sigma - 1\} < 1$,
5. $\max\{0, 1-\alpha\} < \gamma = \omega = r < 1$ *and* $\sigma \geq 2$,
6. $\max\{0, 1-\alpha\} < \gamma = \min\{\omega, \sigma - 1\}$, $\max\{\omega, \sigma - 1\} < 1$ *and* $\omega = r$.

(II) *The estimation*

$$(n+1)^{\alpha+\gamma-1} \parallel M_n^{\varphi_i} x_0 - x_0 \parallel = O_i(1) \tag{9}$$

holds for $i = 2, 3, 4$ *if at least one of the following conditions* 7–9 *is fulfilled*:
7. $\gamma = 1$ *and* $r \leq 1$,
8. $\max\{0, 1-\alpha\} < \gamma = 2 - r < 1$,
9. $\gamma = 1$ *and* $r = 2$.

(III) *The estimation* (9) *holds for* $i = 5$ *if condition* 7 *or condition* 8 *is fulfilled.*

Proof. Let the estimation (3) be fulfilled. It is sufficient to show that the validity of at least one of conditions 1–9 implies the validity of the conditions of Theorem A for suitable $\varphi = \varphi_i$. As the method of proof for all conditions 1–9 is quite similar, we give the proof of this theorem only partly, for example, for conditions 1, 3, 6, and for condition 8 if $i = 2, 3$.

First assume condition 1 is fulfilled and denote

$$g_i(t) = t^{1-r} \varphi_i'(t) \ (t \in]0, 1[, \ i = 1, \ldots, 5).$$

Then

$$g_1(t) = -\sigma\omega(1 - t^\omega)^{\sigma-1} t^{\omega - r} \ (t \in]0, 1[).$$

As now

$$g_1'(t) = -\sigma\omega t^{\omega-r-1}(1 - t^\omega)^{\sigma-2}[(\omega - r)(1 - t\omega) - (\sigma - 1)\omega t^\omega] \quad (t \in]0, 1[),$$

g_1' for $\omega \geq r+1$ is bounded on $]0, 1[$. Also, g_1' is bounded on $]0, 1[$ for $\omega = r$, because in this case

$$g_1'(t) = \sigma(\sigma - 1)\omega^2 t^{\omega-1}(1 - t^\omega)^{\sigma-2} \quad (t \in]0, 1[).$$

Therefore, $g_1 \in \mathrm{Lip}\, 1$ on $]0, 1[$. Thus the conditions of Theorem A are fulfilled.

Suppose condition 3 is fulfilled and let $a \in]0, 1[$. Then due to $\omega \geq 1$, the derivative g_1' is bounded on $]0, a]$. Hence, $g_1 \in \mathrm{Lip}\, 1$ on $]0, a]$. Moreover, $g_1 \in \mathrm{Lip}\,(\sigma - 1)$ on $]a, 1[$, because g_1 is equivalent to $-\sigma\omega(1 - t^\omega)^{\sigma-1}$ in the limit process $t \to 1-$ if $\omega > r+1$, and

$$g_1(t) = -\sigma\omega(1 - t^\omega)^{\sigma-1} \quad (t \in]0, 1[) \tag{10}$$

if $\omega = r$. Therefore, $g_1 \in \mathrm{Lip}\,(\sigma - 1)$ on $]0,1[$. Consequently, the conditions of Theorem A are fulfilled.

Let condition 6 be fulfilled. Then by (10) we have $g_1 \in \mathrm{Lip}\,(\sigma - 1)$ on $]a,1[$ for $a \in]0,1[$. Moreover, $g_1 \in \mathrm{Lip}\,\omega$ on $]0,a]$, because $t^\omega \in \mathrm{Lip}\,\omega$ on $]0,a]$ for $\omega \in]0,1[$. Therefore $g_1 \in \mathrm{Lip}\,(\min\{\omega, \sigma - 1\})$ on $]0,1[$. Thus the conditions of Theorem A are fulfilled.

Suppose condition 8 is fulfilled for $i = 2$ and $i = 3$. Then

$$g_2(t) = \begin{cases} -6t^{2-r}(2-3t) & (t \in]0, \frac{1}{2}]), \\ -6t^{1-r}(1-t)^2 & (t \in]\frac{1}{2}, 1[) \end{cases}$$

and

$$g_3(t) = \pi(t-1)t^{1-r}\sin(\pi t) \ (t \in]0,1[).$$

As we have now

$$g_2'(t) = \begin{cases} -6t^{1-r}[(2-r)(2-3t) - 3t] & (t \in]0, \frac{1}{2}]), \\ -6t^{-r}[(1-r)(1-t)^2 - 2(1-t)t] & (t \in]\frac{1}{2}, 1[) \end{cases}$$

and

$$g_3'(t) = \pi t^{1-r}\left[\pi(t + (1-r)(t-1))\frac{\sin(\pi t)}{\pi t} + \pi(t-1)\cos(\pi t)\right] \ (t \in]0,1[),$$

the derivatives g_2' and g_3' are bounded on $[a,1[$ for each $a \in]0,1[$. Hence, $g_2, g_3 \in \mathrm{Lip}\,1$ on $[a,1[$ for $a \in]0,1[$. In addition, in the limit process $t \to 0+$ the function g_2 is equivalent to $-12t^{2-r}$ and g_3 to $-\pi^2 t^{2-r}$, because we can rewrite g_3 as

$$g_3(t) = \pi^2(t-1)t^{2-r}\frac{\sin(\pi t)}{\pi t} \ (t \in]0,1[).$$

Therefore, $g_2, g_3 \in \mathrm{Lip}\,(2-r)$ on $]0,1[$. Thus the conditions of Theorem A are fulfilled.

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Fourier' arenduste summeeruvusest Banachi ruumides

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Olgu X Banachi ruum, milles eksisteerib projektorite ortogonaalne süsteem. Olgu Z^r ($r > 0$) Zygmundi meetod, $M^\varphi = (\varphi(k/(n+1)))$ kolmnurkne maatriksmeetod, mis on defineeritud mingi diferentseeruva funktsiooni φ abil, kus $\varphi(0) = 1$ ja $\varphi(1) = 0$, ning $Z_n^r x$, $M_n^\varphi x$ olgu vastavalt elemendi $x \in X$ Zygmundi ja M^φ keskmised. Autori varasemas töös [3] on tõestatud teoreem, mis annab piisavad tingimused selleks, et hinnangust $(n+1)^\alpha \parallel Z_n^r x_0 - x_0 \parallel = O(1)$ ($x_0 \in X$) järelduks sama x_0 jaoks hinnang $(n+1)^{\alpha+\gamma-1} \parallel M_n^\varphi x_0 - x_0 \parallel = O(1)$ eeldusel, et $g(t) = t^{1-r}\varphi'(t) \in \mathrm{Lip}\,\gamma$ ($0 < \gamma \leq 1$) vahemikus $]0,1[$. Siinses artiklis rakendatakse seda teoreemi juhtudel, kui M^φ on kas Rieszi, Jacksoni–de La Vallée Poussini, Bohmani–Korovkini, Zhuki või Favardi menetlus.