

FAST SOLVERS OF GENERALIZED AIRFOIL EQUATION OF INDEX -1

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Received 28 May 2001

Abstract. We consider the generalized airfoil equation in the situation where the index of the problem is -1 . We periodize the problem, then discretize it by a fully discrete version of the trigonometric collocation method and apply the conjugate gradient method to solve the discretized problem. The approximate solution appears to be of optimal accuracy in a scale of Sobolev norms, and the N parameters of the approximate solution can be determined by $\mathcal{O}(N \log N)$ arithmetical operations.

Key words: airfoil equations, fast solvers.

1. THE GENERALIZED AIRFOIL EQUATION AND ITS PERIODIZATION

Consider the generalized airfoil equation

$$(Bv)(x) := \int_{-1}^1 \left(\frac{1}{\pi} \frac{1}{x-y} + b_1(x, y) \log |x-y| + b_2(x, y) \right) v(y) dy = g(x),$$
$$-1 < x < 1. \tag{1}$$

We assume that the kernel functions b_1 and b_2 are smooth. It is well known (see, e.g., [1–4]) that B represents a linear continuous Fredholm operator in different weighted spaces $L_\sigma^2(-1, 1)$; the index of B depends on the weight. Particularly, $\text{ind}(B) = 0$ if $\sigma(x) = \sqrt{(1+x)/(1-x)}$ or $\sigma(x) = \sqrt{(1-x)/(1+x)}$, and

$\text{ind}(B) = 1$ if $\sigma(x) = \sqrt{1-x^2}$. Collocation solvers of Eq. (1) in these cases have been examined in [2] and [4], respectively. In the present paper, we put

$$\sigma(x) = \frac{1}{\sqrt{1-x^2}}, \quad (u, v)_{L_\sigma^2} = \int_{-1}^1 \sigma(x)u(x)\overline{v(x)}dx;$$

then the index of $B \in \mathcal{L}(L_\sigma^2(-1, 1))$ is -1 . We assume that the homogeneous equation $Bv = 0$ has in $L_\sigma^2(-1, 1)$ only the trivial solution $v = 0$; then the range $\mathcal{R}(B) = BL_\sigma^2(-1, 1)$ is of codimension 1. Let us fix a smooth function $\psi \in L_\sigma^2(-1, 1)$ outside $\mathcal{R}(B)$. For any $g \in L_\sigma^2(-1, 1)$ there exists a unique pair $(\omega, v) \in \mathbb{C} \times L_\sigma^2(-1, 1)$ satisfying $\omega\psi + Bv = g$, and this pair can be treated as a generalized solution of (1). If $g \in \mathcal{R}(B)$, then $\omega = 0$, and the generalized solution $(0, v)$ can be identified with the usual solution $v \in L_\sigma^2(-1, 1)$ of (1). In the sequel we design a numerical method yielding approximations (ω_N, v_N) such that $|\omega_N - \omega| \rightarrow 0$, $\|v_N - v\|_{L_\sigma^2} \rightarrow 0$ with a certain velocity. Thus, the convergence $\omega_N \rightarrow 0$ as $N \rightarrow \infty$ indicates that $\omega = 0$, $g \in \mathcal{R}(B)$, and (1) is solvable in $L_\sigma^2(-1, 1)$ in the usual sense. An interpretation of the generalized solution (ω, v) with $\omega \neq 0$ can be given considering the flow ejection through a point of the airfoil (see [5]). In any case, the generalized solution (ω, v) is of interest also if $\omega \neq 0$, i.e. $g \notin \mathcal{R}(B)$. So we do not assume that $g \in \mathcal{R}(B)$.

With the cosine transformation

$$x = x(t) = -\cos(2\pi t) \left(0 \leq t \leq \frac{1}{2}\right), \quad y = x(s) = -\cos(2\pi s) \left(0 \leq s \leq \frac{1}{2}\right),$$

Eq. (1) can be reduced (see [3] for details) to the 1-periodic integral equation

$$\mathcal{A}u := A_0u + A_1u + A_2u = f, \quad (2)$$

where

$$\begin{aligned} (A_0u)(t) &= \int_{-1/2}^{1/2} \cot \pi(t-s)u(s)ds \quad (\text{the Hilbert transformation}), \\ (A_1u)(t) &= \int_{-1/2}^{1/2} a_1(t, s) \log |\sin \pi(t-s)|u(s)ds, \\ (A_2u)(t) &= \int_{-1/2}^{1/2} a_2(t, s)u(s)ds, \end{aligned}$$

$$\begin{aligned}
f(t) &= g(x(t)), \quad t \in \mathbb{R}, \\
a_1(t, s) &= b_1(x(t), x(s))x'(s), \\
a_2(t, s) &= \frac{1}{2} [b_2(x(t), x(s)) + (\log 2)b_1(x(t), x(s))]x'(s), \quad t, s \in \mathbb{R}.
\end{aligned}$$

Clearly, f is 1-periodic and even, whereas a_1 and a_2 are 1-biperiodic, even in t and odd in s . The relation between solutions of (1) and (2) is somewhat more sophisticated: for $s \in \left(-\frac{1}{2}, \frac{1}{2}\right]$

$$u(s) = \begin{cases} v(x(s)), & 0 \leq s \leq \frac{1}{2}, \\ -v(x(-s)), & -\frac{1}{2} < s < 0, \end{cases}$$

and after that u is extended from $\left(-\frac{1}{2}, \frac{1}{2}\right]$ to \mathbb{R} 1-periodically. Thus u is a 1-periodic odd function. To the generalized solution (ω, v) of (1) there corresponds the generalized solution (ω, u) of (2) satisfying $\omega\varphi + \mathcal{A}u = f$, where $\varphi(t) = \psi(x(t))$, $t \in \mathbb{R}$.

2. SOLVABILITY OF THE PROBLEM

Notice that $a_1, a_2 \in C^m(\mathbb{R})$, $f, \varphi \in C^m(\mathbb{R})$ if $b_1, b_2 \in C^m([-1, 1] \times [-1, 1])$, $g, \psi \in C^m[-1, 1]$. Introduce the Sobolev space H^λ , $\lambda \geq 0$, of 1-periodic functions u having a finite norm

$$\|u\|_\lambda = \left(\sum_{k \in \mathbb{Z}} \underline{k}^{2\lambda} |\hat{u}(k)|^2 \right)^{1/2}, \quad \underline{k} = \max\{1, |k|\}, \quad \hat{u}(k) = \int_{-1/2}^{1/2} u(s) e^{-ik2\pi s} ds.$$

We have $H^\lambda = H_{\text{ev}}^\lambda \oplus H_{\text{od}}^\lambda$, where H_{ev}^λ and H_{od}^λ are closed subspaces of H^λ consisting of even and odd functions, respectively. An orthogonal basis of H_{ev}^λ is given by $\{\cos(k2\pi t)\}_{k \geq 0}$, and an orthogonal basis of H_{od}^λ is given by $\{\sin(k2\pi t)\}_{k \geq 1}$. We also introduce the Sobolev space H^{λ_1, λ_2} , $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, of 1-biperiodic functions a having a finite norm

$$\begin{aligned}
\|a\|_{\lambda_1, \lambda_2} &= \left(\sum_{(k_1, k_2) \in \mathbb{Z}^2} \underline{k}_1^{2\lambda_1} \underline{k}_2^{2\lambda_2} |\hat{a}(k_1, k_2)|^2 \right)^{1/2}, \\
\hat{a}(k_1, k_2) &= \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} a(t, s) e^{-ik_1 2\pi t} e^{-ik_2 2\pi s} ds dt,
\end{aligned}$$

and the subspace $H_{\text{ev,od}}^{\lambda_1, \lambda_2}$ of functions which are even in the first argument and odd in the second argument.

It is well known that

$$A_0 \sin(k2\pi t) = -\cos(k2\pi t), \quad k \geq 1,$$

$$A_0 1 = 0, \quad A_0 \cos(k2\pi t) = \sin(k2\pi t), \quad k \geq 1.$$

Thus $A_0 \in \mathcal{L}(H_{\text{od}}^\lambda, H_{\text{ev}}^\lambda)$ is a Fredholm operator of index -1 for every $\lambda \geq 0$.

Lemma 2.1. *If $a_1 \in H_{\text{ev,od}}^{\mu, \nu} \cap H_{\text{ev,od}}^{\nu, \mu}$, $\frac{1}{2} < \nu \leq \mu$, then $A_1 \in \mathcal{L}(H_{\text{od}}^\lambda, H_{\text{ev}}^\lambda)$ is compact for every $\lambda \in [0, \mu]$.*

Lemma 2.2. *If $a_2 \in H_{\text{ev,od}}^{\mu, 0}$, $\mu \geq 0$, then $A_2 \in \mathcal{L}(H_{\text{od}}^\lambda, H_{\text{ev}}^\lambda)$ is compact for every $\lambda \in [0, \mu]$.*

Lemma 2.3. *Assume that $a_1 \in H_{\text{ev,od}}^{\mu, \nu} \cap H_{\text{ev,od}}^{\nu, \mu}$, $a_2 \in H_{\text{ev,od}}^{\mu, 0}$, $\frac{1}{2} < \nu \leq \mu$. Then $\mathcal{A} = A_0 + A_1 + A_2 \in \mathcal{L}(H_{\text{od}}^\lambda, H_{\text{ev}}^\lambda)$ is a Fredholm operator of index -1 for every $\lambda \in [0, \mu]$.*

The proofs of Lemmas 2.1–2.3 can be constructed following the ideas of [4]. As a consequence of Lemma 2.3 we obtain the following result.

Theorem 2.1. *Assume the conditions of Lemma 2.3. Assume also that the homogeneous equation $\mathcal{A}u = 0$ has in H_{od}^μ only the trivial solution. Then the range $\mathcal{A}H_{\text{od}}^\mu \subset H_{\text{ev}}^\mu$ is of codimension 1. Fixing a $\varphi \in H_{\text{ev}}^\mu \setminus \mathcal{A}H_{\text{od}}^\mu$, for every $f \in H_{\text{ev}}^\mu$ we get a unique pair $(\omega, u) \in \mathbb{C} \times H_{\text{od}}^\mu$ such that $\omega\varphi + \mathcal{A}u = f$, and this generalized solution of (2) is unique in $\mathbb{C} \times H_{\text{od}}^0$.*

We have $H^\mu \subset C^m(\mathbb{R})$ for $m < \mu - \frac{1}{2}$, $\mu > \frac{1}{2}$, and under conditions of Theorem 2.1, $u \in C^m(\mathbb{R})$. For (ω, v) , the generalized solution of (1), we have

$$v(x) = u\left(\frac{1}{2\pi} \arccos(-x)\right), \quad 1 \leq x \leq 1.$$

So v is continuous on $[-1, 1]$, C^m -smooth in $(-1, 1)$, satisfies $v(-1) = u(0) = 0$, $v(1) = u(1/2) = 0$, but the derivatives of v have certain singularities at the end points of the interval $(-1, 1)$, e.g. $v \in C^1(-1, 1)$ for $\mu > \frac{3}{2}$,

$$v'(x) - \frac{u'(0)}{2\pi\sqrt{1-x^2}} \rightarrow 0 \text{ as } x \rightarrow -1, \quad v'(x) - \frac{u'(1/2)}{2\pi\sqrt{1-x^2}} \rightarrow 0 \text{ as } x \rightarrow 1.$$

3. A FULLY DISCRETE COLLOCATION METHOD

For $N \in \mathbb{N}$, introduce $m, M, n \in \mathbb{N}$ such that

$$\begin{aligned} 2m \leq M \leq n \leq N, \quad m \sim N^\varrho, \quad M \sim N^\sigma, \quad n \sim N^\tau, \\ 0 < \varrho \leq \sigma \leq \tau < 1, \quad \sigma \leq \frac{1}{2}, \quad \frac{\mu}{\mu+1} \leq \tau < 1, \end{aligned} \quad (3)$$

where $n \sim N^\tau$ means that there are positive constants c_1 and c_2 such that $c_1 \leq nN^{-\tau} \leq c_2$ as $N \rightarrow \infty$. We approximate $\mathcal{A} = A_0 + A_1 + A_2 \in \mathcal{L}(H_{\text{od}}^0, H_{\text{ev}}^0)$ by $\mathcal{A}_N \in \mathcal{L}(H_{\text{od}}^0, H_{\text{ev}}^0)$ defined by

$$\mathcal{A}_N = A_0 + Q_M^{\text{ev}}(A_1^{(M)} + A_2^{(M)})P_m^{\text{od}} + Q_n^{\text{ev}}A_1^{[d]}(P_n^{\text{od}} - P_m^{\text{od}}), \quad (4)$$

where P_n^{od} is the orthogonal projection operator in H_{od}^0 to

$$\mathcal{T}_n^{\text{od}} = \text{span} \{ \sin(k2\pi t), \quad k = 1, \dots, n \};$$

Q_n^{ev} is the interpolation projection operator defined by

$$\begin{aligned} Q_n^{\text{ev}}u \in \mathcal{T}_n^{\text{ev}} = \text{span} \{ \cos(k2\pi t), \quad k = 0, 1, \dots, n \}, \\ (Q_n u) \left(\frac{j}{2n+1} \right) = u \left(\frac{j}{2n+1} \right), \quad j = 0, 1, \dots, n, \quad u \in H_{\text{ev}}^\mu, \quad \mu > \frac{1}{2}; \end{aligned}$$

the product integration approximations $A_1^{(M)}, A_2^{(M)} \in \mathcal{L}(H_{\text{od}}^\mu, H_{\text{ev}}^0)$ are defined by

$$\begin{aligned} (A_1^{(M)}u)(t) &= \int_{-1/2}^{1/2} \log |\sin \pi(t-s)| Q_{M,s}^{\text{ev}}(a_1(t,s)u(s))ds, \\ (A_2^{(M)}u)(t) &= \int_{-1/2}^{1/2} Q_{M,s}^{\text{ev}}(a_2(t,s)u(s))ds, \quad u \in H_{\text{od}}^\mu, \quad \mu > \frac{1}{2}, \end{aligned}$$

where the index s in $Q_{M,s}^{\text{ev}}$ indicates the interpolation with respect to the argument s ; the asymptotic approximation $A_1^{[d]} \in \mathcal{L}(H_{\text{od}}^0, H_{\text{ev}}^0)$ of A_1 is defined by

$$A_1^{[d]} \sin(k2\pi t) = \sum_{j=0}^{d-2} k^{-1-j} b_j(t) \begin{cases} \sin(k2\pi t), & j \text{ even} \\ \cos(k2\pi t), & j \text{ odd} \end{cases}, \quad k = 1, 2, \dots,$$

$$b_j(t) = - \begin{cases} (-1)^{j/2}, & j \text{ even} \\ (-1)^{(j-1)/2}, & j \text{ odd} \end{cases} \frac{1}{2} \frac{1}{(2\pi)^j} \left(\frac{\partial}{\partial s} \right)^j a_1(t,s) \Big|_{s=t}, \\ j = 0, \dots, d-2,$$

$\mathbb{N} \ni d \geq \frac{1-\varrho}{\varrho}\mu, \quad \mu > \frac{1}{2}; \quad d = 1, \quad A_1^{[d]} = 0$ may be set if $\frac{1-\varrho}{\varrho}\mu \leq 1$.

Lemma 3.1. *Let (3) be fulfilled with a $\mu > \frac{1}{2}$, and let $d \geq \frac{1-\varrho}{\varrho}\mu$. Further, assume that $a_i = a_i(t, s)$, $i = 1, 2$, are even in t , odd in s and with a $\nu > 1/2$,*

$$\begin{aligned} a_1 &\in H^{\nu, d+\nu} \cap H^{\mu+1, \nu} \cap H^{\nu+\mu(1-\sigma)/\sigma, \mu/\sigma} \cap H^{\mu/\sigma, \nu+\mu(1-\sigma)/\sigma}, \\ a_2 &\in H^{\nu, \mu/\sigma} \cap H^{\mu/\sigma, 0} \cap H^{0, \mu(1-\varrho)/\varrho}. \end{aligned}$$

Then

$$\begin{aligned} \|\mathcal{A} - \mathcal{A}_N\|_{\lambda, \mu} &:= \|\mathcal{A} - \mathcal{A}_N\|_{\mathcal{L}(H_{\text{ev}}^\mu, H_{\text{od}}^\lambda)} \leq cN^{\lambda-\mu} \quad (0 \leq \lambda \leq \mu), \\ \|\mathcal{A} - \mathcal{A}_N\|_{\lambda, \lambda} &\rightarrow 0 \text{ as } N \rightarrow \infty \quad (0 \leq \lambda \leq \mu). \end{aligned}$$

Theorem 3.1. *Assume the conditions of Lemma 3.1. Assume also that the homogeneous equation $\mathcal{A}u = 0$ has in H_{od}^μ only the trivial solution. Let $\varphi \in H_{\text{ev}}^\mu \setminus \mathcal{A}H_{\text{od}}^\mu$. Then there is a $N_0 \in \mathbb{N}$ such that for $N \geq N_0$, the approximate problem*

$$\omega Q_N^{\text{ev}} \varphi + \mathcal{A}_N u = Q_N^{\text{ev}} f \quad (5)$$

has for every $f \in H_{\text{ev}}^\mu$ a solution $(\omega_N, u_N) \in \mathbb{C} \times T_N^{\text{od}}$ which is unique in $\mathbb{C} \times H_{\text{od}}^0$, and

$$|\omega_N - \omega| \leq cN^{-\mu} \|f\|_\mu, \quad \|u_N - u\|_\lambda \leq cN^{\lambda-\mu} \|f\|_\mu \quad (0 \leq \lambda \leq \mu),$$

where $(\omega, u) \in \mathbb{C} \times H_{\text{ev}}^\mu$ is the (unique) solution of the problem $\omega\varphi + \mathcal{A}u = f$.

Notice that to (ω_N, u_N) , $u_N = \sum_{j=1}^N c_j \sin(j2\pi t)$, there corresponds the approximate generalized solution (ω_N, v_N) of (1) with

$$\begin{aligned} v_N(x) &= u_N \left(\frac{1}{2\pi} \arccos(-x) \right) = \sum_{j=1}^N c_j \sin(j \arccos(-x)) \\ &= \sqrt{1-x^2} \sum_{j=1}^N c_j U_{j-1}(-x), \end{aligned}$$

where $U_j(x) = \sin((j+1) \arccos x) / \sqrt{1-x^2}$, $j = 0, 1, \dots$, are the Chebyshev polynomials of the second kind. Moreover, by Theorem 3.1

$$\|v_N - v\|_{L_{\varrho}^2} = \|u_N - u\|_0 \leq cN^{-\mu} \|f\|_\mu,$$

where (ω, v) , $v(x) = u(\frac{1}{2\pi} \arccos(-x))$, is the generalized solution of problem (1). Also estimates of $v_N - v$ in weighted Sobolev norms follow from Theorem 3.1.

4. MATRIX FORM OF THE METHOD AND CONJUGATE GRADIENTS

The dimension of the problem (5) can be reduced from N to n . Namely, if (ω_N, u_N) with $u_N = \sum_{j=1}^N c_j \sin(j2\pi t)$ is the solution of (5), then $\omega_n = \omega_N$, $u_n = P_n^{\text{od}} u_N = \sum_{j=1}^n c_j \sin(j2\pi t)$ is the solution of the problem

$$\omega \varphi_n + \mathcal{A}_N u = f_n \quad (6)$$

with $\varphi_n = P_n^{\text{ev}} Q_N^{\text{ev}} \varphi$, $f_n = P_n^{\text{ev}} Q_N^{\text{ev}} f$, and u_N can be reconstructed by the formula $u_N = u_n + \sum_{j=n+1}^N (\omega_n \alpha_j - d_j) \sin(j2\pi t)$, where α_j and d_j are the Fourier coefficients of $Q_N^{\text{ev}} \varphi$ and $Q_N^{\text{ev}} f$, respectively,

$$Q_N^{\text{ev}} \varphi = \sum_{j=0}^N \alpha_j \cos(j2\pi t), \quad Q_N^{\text{ev}} f = \sum_{j=0}^N d_j \cos(j2\pi t).$$

Denoting $\underline{c}_n = (c_1, \dots, c_n)^\top$, $\underline{d}_n = (d_0, d_1, \dots, d_n)^\top$, $\underline{\alpha}_n = (\alpha_0, \alpha_1, \dots, \alpha_n)^\top$, we have problem (6) in the matrix form

$$\omega \underline{\alpha}_n + \mathbb{M}_n \underline{c}_n = \underline{d}_n \quad (7)$$

with the $(n+1) \times n$ matrix \mathbb{M}_n defined by

$$\begin{aligned} \mathbb{M}_n &= \mathbb{A}_0 + \mathbb{I}_{n,M} \tilde{\mathcal{C}}_M (\mathbb{A}_1^{(M)} + \mathbb{A}_2^{(M)}) \mathcal{S}_M \mathbb{P}_{M,m,n} \\ &\quad + \tilde{\mathcal{C}}_n \sum_{j=0}^{d-2} \mathbb{B}_n^{(j)} \left\{ \begin{array}{l} \mathcal{C}_n \mathbb{J}_n, \quad j \text{ even} \\ \mathbb{J}_n \mathcal{S}_n, \quad j \text{ odd} \end{array} \right\} \mathbb{G}_N^{(j)}, \end{aligned}$$

where

$$\mathbb{A}_0 = -\mathbb{J}_n, \quad \mathbb{J}_n = \begin{pmatrix} 0 \\ \mathbb{I}_n \end{pmatrix} \text{ are } (n+1) \times n \text{ matrices,}$$

\mathbb{I}_n is an $n \times n$ identity matrix,

$$\mathbb{I}_{n,M} = \begin{pmatrix} \mathbb{I}_{M+1} \\ 0 \end{pmatrix} \text{ is an } (n+1) \times (M+1) \text{ matrix,}$$

$$\mathbb{P}_{M,m,n} = \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & 0 \end{pmatrix} \text{ is an } M \times n \text{ matrix;}$$

$$\mathcal{C}_n = \left(\cos \left(kj \frac{2\pi}{2n+1} \right) \right)_{j,k=0}^n, \quad \tilde{\mathcal{C}}_n = \frac{4}{2n+1} \mathbb{D}_n \mathcal{C}_n \mathbb{D}_n,$$

$$\mathbb{D}_n = \text{diag} \left\{ \frac{1}{2}, 1, \dots, 1 \right\}, \quad \mathcal{S}_n = \left(\sin \left(kj \frac{2\pi}{2n+1} \right) \right)_{j,k=1}^n;$$

$$\mathbb{A}_1^{(M)} = \left(a_{kj}^{(1)} \right), \quad \mathbb{A}_2^{(M)} = \left(a_{kj}^{(2)} \right) \text{ are } (M+1) \times M \text{ matrices with the entries}$$

$$\begin{aligned}
a_{kj}^{(1)} &= -\frac{1}{2M+1}a_1\left(\frac{k}{2M+1}, \frac{j}{2M+1}\right)(\gamma_{|k-j|} + \gamma_{k+j}), \\
a_{kj}^{(2)} &= \frac{2}{2M+1}a_2\left(\frac{k}{2M+1}, \frac{j}{2M+1}\right), \quad k = 0, 1, \dots, M, \quad j = 1, \dots, M, \\
\gamma_k &= \log 2 + \sum_{l=1}^M \frac{1}{l} \cos\left(kl \frac{2\pi}{2M+1}\right), \quad k = 0, 1, \dots, M, \\
\gamma_{M+k} &= \gamma_{M+1-k}, \quad 1 \leq k \leq M;
\end{aligned}$$

$\mathbb{G}_n^{(j)} = \text{diag}\{0, \dots, 0, (m+1)^{-1-j}, \dots, n^{-1-j}\}$ is an $n \times n$ matrix,

$\mathbb{B}_n^{(j)} = \text{diag}\left\{b_j(0), b_j\left(\frac{1}{2n+1}\right), \dots, b_j\left(\frac{n}{2n+1}\right)\right\}$ is an $(n+1) \times (n+1)$ matrix.

The application of \mathbb{M}_n to an n -vector, as well as the application of \mathbb{M}'_n , the Hermite adjoint matrix of \mathbb{M}_n , to an $(n+1)$ -vector costs $\mathcal{O}(n \log n) + \mathcal{O}(M^2) = \mathcal{O}(N^\tau \log N) + \mathcal{O}(N^{2\sigma})$ arithmetical operations, provided that the fast Fourier technique is used for the cosine and sine transformations \mathcal{C}_n and \mathcal{S}_n . The computation of the entries of \mathbb{M}_n costs $\mathcal{O}(M^2) + \mathcal{O}(N) = \mathcal{O}(N)$ arithmetical operations. This enables us to design fast solvers of problem (2) on the basis of iteration methods. We specify a classical conjugate gradient iteration algorithm (see [6,7]) to solve (7).

Denote by $\underline{x}_n = (\omega, c_1, \dots, c_n)$ the $(n+1)$ -vector of unknowns and rewrite the system (7) in the form

$$\mathbb{A}_n \underline{x}_n = \underline{d}_n,$$

where

$$\mathbb{A}_n = \begin{pmatrix} \underline{\alpha}_n & \mathbb{M}_n \end{pmatrix} \text{ is an } (n+1) \times (n+1) \text{ matrix.}$$

Algorithm 1.

Step 0: $\underline{x}_n^0 = 0$, $\underline{y}_n^0 = -\underline{d}_n$, $\underline{r}_n^0 = -\mathbb{A}'_n \underline{d}_n$.

For $k = 0, 1, 2 \dots$:

- (i) if $\|\underline{y}_n^k\| \leq \|\underline{d}_n\| \delta N^{-\mu}$, then terminate;
- (ii) if $\|\underline{y}_n^k\| > \|\underline{d}_n\| \delta N^{-\mu}$, then go to step $k+1$, and compute

$$\begin{aligned}
\underline{z}_n^k &= \begin{cases} -\underline{r}_n^0, & k = 0, \\ -\underline{r}_n^k + (\|\underline{r}_n^k\| / \|\underline{r}_n^{k-1}\|)^2 \underline{z}_n^{k-1}, & k \geq 1, \end{cases} \\
\underline{x}_n^{k+1} &= \underline{x}_n^k + \gamma_k \underline{z}_n^k, \quad \gamma_k = (\|\underline{r}_n^k\| / \|\mathbb{A}_n \underline{z}_n^k\|)^2, \\
\underline{y}_n^{k+1} &= \underline{y}_n^k + \gamma_k \mathbb{A}_n \underline{z}_n^k, \\
\underline{r}_n^{k+1} &= \underline{r}_n^k + \gamma_k \mathbb{A}'_n \mathbb{A}_n \underline{z}_n^k.
\end{aligned}$$

In this algorithm the usual norm $\|\underline{d}_n\| = (\sum_{k=0}^n |d_k|^2)^{1/2}$ is used for $(n+1)$ -vectors. We have incorporated the residual termination rule into the algorithm: the iterations stop on the first k such that $\|\mathbb{A}_n \underline{x}_n^k - \underline{d}_n\| \leq \|\underline{d}_n\| \delta N^{-\mu}$. Here $\delta > 0$ is a parameter.

Theorem 4.1. *Under conditions of Theorem 3.1, for $N \geq N_0$, Algorithm 1 terminates at an iteration number k of order $o(\log N)$ as $N \rightarrow \infty$. The corresponding iteration approximation $\underline{x}_n^k = (\omega^k, c_1^k, \dots, c_n^k)$ defines an iteration solution (ω_N^k, u_N^k) to (5) with*

$$\omega_N^k = \omega^k, \quad u_N^k = \sum_{j=1}^n c_j^k \sin(j2\pi t) + \sum_{j=n+1}^N (\omega^k \alpha_j - d_j) \sin(j2\pi t)$$

for which there hold the optimal order estimates

$$|\omega_N^k - \omega| \leq cN^{-\mu} \|f\|_\mu, \quad \|u_N^k - u\|_\lambda \leq cN^{\lambda-\mu} \|f\|_\mu, \quad 0 \leq \lambda \leq \mu,$$

where $(\omega, u) \in \mathbb{C} \times H_{\text{od}}^\mu$ is the unique generalized solution of integral equation (2).

The computation of $\underline{d}_N = \tilde{\mathcal{C}}_N \underline{f}_N$ and $\underline{\alpha}_N = \tilde{\mathcal{C}}_N \underline{\varphi}_N$ from the vectors of grid values $\underline{f}_N = (f(0), f(\frac{1}{2N+1}), \dots, f(\frac{N}{2N+1}))$ and $\underline{\varphi}_N = (\varphi(0), \varphi(\frac{1}{2N+1}), \dots, \varphi(\frac{N}{2N+1}))$ by the fast algorithm costs $\mathcal{O}(N \log N)$ arithmetical operations. All other computations are cheaper, costing asymptotically $o(\log N)(\mathcal{O}(N^\tau \log N) + \mathcal{O}(N^{2\sigma}))$ arithmetical operations, which is $o(N)$ for $\sigma < \frac{1}{2}$; notice that an iteration step by Algorithm 1 contains one application of \mathbb{A}_n and one application of \mathbb{A}'_n . If the Fourier coefficients of f and φ with respect to $\cos(k2\pi t)$ ($k = 0, 1, 2, \dots$) are known, we can use $P_N^{\text{ev}} f$ and $P_N^{\text{ev}} \varphi$ instead of $Q_N^{\text{ev}} f$ and $Q_N^{\text{ev}} \varphi$. In this case the full number of arithmetical and logical operations reduces to $N + o(N)$.

If functions a_1, a_2 , and φ are real, then \mathbb{A}_n is real.

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ÜLDISTATUD TIIVAVÕRRANDI KIIRED LAHENDUSMEETODID INDEKSI -1 KORRAL

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Üldistatud tiivavõrrandit on käsitletud situatsioonis, kui vastava integraaloperaatori Fredholmi indeks on -1 . On esitatud vastava laiendatud ülesande lahendusmeetod, mis põhineb trigonomeetrilisele kollokatsioonimeetodile, on aga täielikult diskreetne ning võimaldab teatud mõttes optimaalse täpsusastmega lähilahendi N parameetrit määrata $\mathcal{O}(N \log N)$ aritmeetilise tehtega.