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Singular and hypersingular integral equations with the Hilbert kernel, delta-function, and method of discrete vortices

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Abstract. The singular and hypersingular integral equations with the Hilbert kernel, having the delta-function in their right-hand sides, are studied. For these equations a method of discrete vortices type is constructed and justified.

Key words: singular, hypersingular, integral equation, discrete vortex method.

The present paper develops further the joint results by G. Vainikko and I. K. Lifanov (see [1–3]). In these works the singular integral equations on closed and open curves in the weighted Sobolev spaces H_ρ^λ for distributions are studied. Here these equations are considered on a segment and with the Hilbert kernel in the case when there is a delta-function in the right-hand side. For example:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\gamma(x)\mathrm{d}x}{x_0 - x} = f(x_0) + f_{\delta,q}(x_0), \quad x_0, q \in (-1,1), \tag{1}$$

where $f_{\delta,q}(x_0) = Q\delta(x_0 - x)$, Q is an arbitrary real number,

$$\delta(x_0 - x) = 0, \quad x_0 \neq q, \text{ and } \delta(x_0 - x) = \infty, \quad x_0 = q, \tag{2}$$

$$\int_{-1}^{1}\delta(x_0 - q)\,\mathrm{d}x_0 = 1, \tag{3}$$

or

$$\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{\theta_0-\theta}{2}\gamma(\theta)\mathrm{d}\theta = f(\theta_0)+f_{\delta,q}(\theta_0), \quad \theta_0, q\in[0, 2\pi]. \tag{4}$$

Problems of aerodynamics of ideal incompressible fluid flow around a profile having suction of external flow can be reduced to such equations. In aerodynamics these problems have been solved numerically by the discrete vortices method. The introduction of the delta-function in the right-hand side of the singular integral equations allows us to simplify the algorithms of numerical solution of Eqs. (1) and (4) with the help of the discrete vortices method. The justification of the convergence of numerical solutions by the discrete vortices method for Eq. (1) is given in [2].

In this paper numerical algorithms for solving Eq. (4) by the discrete vortices method are presented and the justification of the convergence of numerical solutions to exact one is given. Analogous results are presented also for the equation

$$\frac{1}{4\pi}\int_0^{2\pi}\frac{g(\theta)\mathrm{d}\theta}{\sin^2(\theta_0-\theta)/2} = f(\theta_0)+f_{\delta,q}(\theta_0), \quad \theta_0, q\in[0, 2\pi]. \tag{5}$$

So, let us consider Eq. (4), where $f(\theta)$ belongs to the Hölder class $H(\alpha)$, $0<\alpha\le 1$, on $[0, 2\pi]$ and which is periodic with the period 2π (as all functions in this paper). Equation (4) has a solution only when the equality

$$\int_0^{2\pi}(f(\theta_0)+f_{\delta,q}(\theta_0))\mathrm{d}\theta_0 = 0 \tag{6}$$

holds. Since due to (3) we have

$$\int_0^{2\pi} f_{\delta,q}(\theta_0)\mathrm{d}\theta_0 = Q, \tag{7}$$

it follows from equalities (6) and (7) that the equality

$$\int_0^{2\pi} f(\theta_0)\mathrm{d}\theta_0 = -Q \tag{8}$$

is the condition for solvability of Eq. (4).

If equality (6) holds, then the solution of Eq. (4) is given by the formula

$$\gamma(\theta) = -\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{\theta-\theta_0}{2} f(\theta_0)\mathrm{d}\theta_0 - \frac{Q}{2\pi}\cot\frac{\theta-q}{2} + C, \theta, q\in[0, 2\pi], \tag{9}$$

where

$$\frac{1}{2\pi}\int_{0}^{2\pi}\gamma(\theta)\mathrm{d}\theta = C. \tag{10}$$

Let us construct a method for numerical solution of Eq. (4) using the ideas of the discrete vortices method. Let the segment $[0, 2\pi]$ be divided by two sets of points $E = \{\theta_k, k = 1, 2, \ldots, n\}$ and $E_0 = \{\theta_{0j}, j = 1, 2, \ldots, n\}$, so that $\theta_{k+1} - \theta_k = h = 2\pi/n$, $k = 1, 2, \ldots, n$, $\theta_{n+1} = \theta_1$, $\theta_{0j} = \theta_j + h/2$, and $q \in E_0$, $q = \theta_{0j_q}$. Now we replace Eq. (4) by the following system of linear algebraic equations:

$$\begin{gathered}\gamma_{0n} + \frac{1}{2\pi}\sum_{k=1}^{n}\cot\frac{\theta_{0j}-\theta_k}{2}\gamma_n(\theta_k)\frac{2\pi}{n} = f(\theta_{0j}) + Q\delta_h(\theta_{0j}-q), \quad j = 1, 2, \ldots, n, \\ \frac{1}{2\pi}\sum_{k=1}^{n}\gamma_n(\theta_k)\frac{2\pi}{n} = C,\end{gathered} \tag{11}$$

where $\delta_h(\theta - q) = 0$, $\theta \in (\theta_{0j}, \theta_{0j+1})$, $j \neq j_q$, and $\delta_h(\theta - q) = 1/h$, $\theta \in (\theta_{0j_q}, \theta_{0j_q+1})$, $h = 2\pi/n$. Summing the first n equations in (11), we obtain

$$\gamma_{0n} = \frac{1}{2\pi}\sum_{j=1}^{n} f(\theta_{0j})\frac{2\pi}{n} + \frac{Q}{2\pi}. \tag{12}$$

From equality (8) it follows that the condition $\gamma_{0n} \to 0$ as $n \to \infty$ holds if and only if equality (8) holds.

Theorem 1. *Let function $f(\theta)$ belong to $H(\alpha)$, $0 < \alpha \le 1$, and it be periodic with the period 2π. Then for the solution of system* (11) *and function $\gamma(\theta)$ from* (9) *the following inequality holds*:

$$\left|\gamma(\theta_k) - \gamma_n(\theta_k)\right| \le O(n^{-\alpha}\ln n), \quad k = 1, 2, \ldots, n. \tag{13}$$

Proof. There is some analogy with the corresponding proof from [4]. We transform system (11) in the following way:

$$\begin{aligned}&-\frac{i}{2}\sum_{k=1}^{n}\gamma_n(\theta_k)\frac{2\pi}{n} + \frac{1}{2}\sum_{k=1}^{n}\cot\frac{\theta_{0j}-\theta_k}{2}\gamma_n(\theta_k)\frac{2\pi}{n} \\ &= \left[\pi\Big(f(\theta_{0j}) + Q\delta_h(\theta_{0j}-q)\Big) - \left(\frac{1}{2}\sum_{j=1}^{n} f(\theta_{0j})\frac{2\pi}{h} + \frac{Q}{2}\right)\right] - i\pi C, \quad j = 1, 2, \ldots, n,\end{aligned} \tag{14}$$

or

$$\sum_{k=1}^{n}\frac{\overset{\Lambda}{\gamma}_n(t_k)a_k}{t_{0j}-t_k}=\overset{\Lambda}{f}(t_{0j}),\qquad j=1,2,\ldots n,$$

$$_{k}=e^{i\theta_k},\ t_{0j}=e^{i\theta_{0j}},\ \overset{\Lambda}{\gamma}_n(t_k)=\gamma_n(\theta_k),\ a_k=\frac{2\pi i t_k}{n},\tag{15}$$

$$\overset{\Lambda}{f}(t_{0j})=\left[\pi\Big(f(\theta_{0j})+Q\delta_h(\theta_{0j}-q)\Big)-\left(\frac{1}{2}\sum_{k=1}^{n}f(\theta_{0k})\frac{2\pi}{h}+\frac{Q}{2}\right)\right]-i\pi C.$$

As it was shown in [4], system (15) implies

$$\overset{\Lambda}{\gamma}_n(t_k)=-\frac{1}{\pi^2}\sum_{j=1}^{n}\frac{f(t_{0j})b_j}{t_k-t_{0j}},\qquad k=1,2,\ldots,n,\tag{16}$$

where

$$b_j=\frac{2\pi i t_{0j}}{n}.$$

Using the notations of system (15), we now have

$$\gamma_n(\theta_k)=-\frac{1}{\pi^2}\sum_{j=1}^{n}\left(\frac{1}{2}\cot\frac{\theta_k-\theta_{0j}}{2}-\frac{i}{2}\right)$$

$$\times\left\{\left[\pi\Big(f(\theta_{0j})+Q\delta_h(\theta_{0j}-q)\Big)-\left(\frac{1}{2}\sum_{k=1}^{n}f(\theta_{0k})\frac{2\pi}{h}+\frac{Q}{2}\right)\right]-i\pi C\right\}\frac{2\pi}{n}$$

$$=-\frac{1}{2\pi}\sum_{j=1}^{n}\cot\frac{\theta_k-\theta_{0j}}{2}f(\theta_{0j})\frac{2\pi}{n}-\frac{Q}{2\pi}\cot\frac{\theta_k-\theta_{0j_q}}{2}+\frac{i}{2\pi}\sum_{j=1}^{n}f(\theta_{0j})\frac{2\pi}{h}$$

$$+\frac{i}{2\pi}Q-\frac{i}{2\pi}\sum_{k=1}^{n}f(\theta_{0k})\frac{2\pi}{h}-\frac{i}{2\pi}Q+C\tag{17}$$

$$=-\frac{1}{2\pi}\sum_{j=1}^{n}\cot\frac{\theta_k-\theta_{0j}}{2}f(\theta_{0j})\frac{2\pi}{n}-\frac{Q}{2\pi}\cot\frac{\theta_k-\theta_{0j_q}}{2}+C,$$

$$k=1,2,\ldots,n.$$

Now inequality (13) follows from the quadrature formulas for the discrete vortices method for the integral with the Hilbert kernel [4].

In [1] it is shown that, for the Hilbert operator extended by continuity onto the periodic Sobolev space H^{λ}, there holds the equality

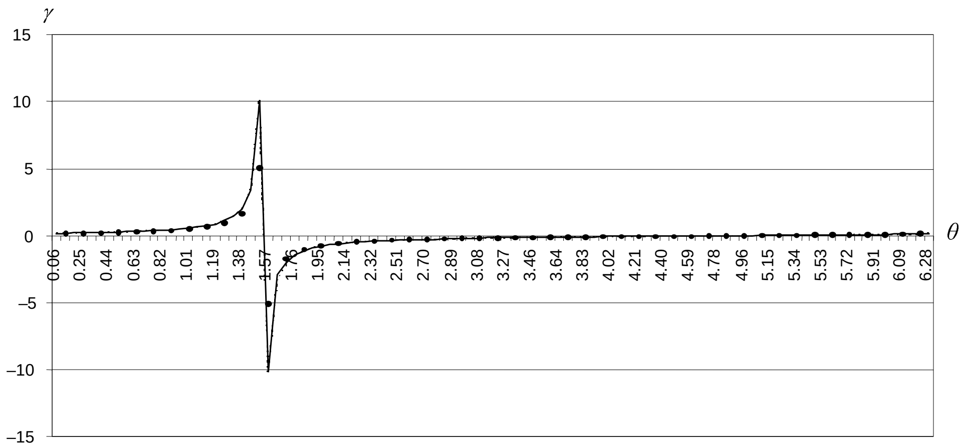

Fig. 1. Numerical solution of Eq. (4) for q = 1.57. • numerical solution n = 50; – exact solution.

$$\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{\theta_0-\theta}{2}\left(-\frac{Q}{2\pi}\cot\frac{\theta-q}{2}\right)\mathrm{d}\theta=-\frac{Q}{2\pi}+Q\delta(\theta_0),\ \ \theta_0\in[0,2\pi]. \tag{18}$$

Therefore the solution of Eq. (4) is given by the formula

$$\gamma(\theta)=-\frac{Q}{2\pi}\cot\frac{\theta-q}{2}, \tag{19}$$

when the right-hand side of this equation has the form as the right-hand side of equality (18) and $C=0$ in (10). Just in this case we have considered the numerical solution of Eq. (4) with $q=1.57.$ The result is good (see Fig. 1).

Now we consider Eq. (5) with the hypersingular periodic kernel. Equation (5) is equivalent to the equation

$$-\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{\theta_0-\theta}{2}g'(\theta)\,\mathrm{d}\theta=f(\theta_0)+f_{\delta,q}(\theta_0),\qquad \theta_0,q\in[0,2\pi]. \tag{20}$$

Since the equality

$$\int_0^{2\pi}\frac{\mathrm{d}\theta}{\sin^2(\theta_0-\theta)/2}=0,\qquad \theta_0\in[0,2\pi], \tag{21}$$

holds, equality (8) must hold for Eq. (5). As the equality

$$\int_0^{2\pi}g'(\theta)\,\mathrm{d}\theta=0, \tag{22}$$

holds, for the function $g'(\theta)$ from Eq. (20) we have

$$g'(\theta) = \frac{1}{2\pi}\int_0^{2\pi}\cot\frac{\theta-\theta_0}{2}(f(\theta_0) + f_{\delta,q}(\theta_0))\,\mathrm{d}\theta_0$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\cot\frac{\theta-\theta_0}{2} f(\theta_0)\,\mathrm{d}\theta_0 + \frac{Q}{2\pi}\cot\frac{\theta-q}{2}, \quad \theta\in[0, 2\pi]. \tag{23}$$

Integrating equality (23), we obtain

$$g(\theta) = \frac{1}{\pi}\int_0^{2\pi}\ln\left|\sin\frac{\theta-\theta_0}{2}\right| f(\theta_0)\,\mathrm{d}\theta_0 + \frac{Q}{\pi}\ln\left|\sin\frac{\theta-q}{2}\right| + C. \tag{24}$$

For the numerical solution of Eq. (5) we again consider two sets E and E_0 of the points setting $q\in E_0$, $q=\theta_{0j_q}$. Now we replace Eq. (5) by the following system of linear algebraic equations:

$$\gamma_{0n} + \frac{1}{2\pi}\sum_{k=1}^{n} g_n(\theta_{0k})\left[\cot\frac{\theta_{0j}-\theta_{k+1}}{2} - \cot\frac{\theta_{0j}-\theta_k}{2}\right] = f(\theta_{0j}) + Q\delta_h(\theta_{0j}-q),$$

$$j = 1, 2, \ldots, n, \tag{25}$$

$$\sum_{k=1}^{n} g_n(\theta_{0k})\frac{2\pi}{n} = C.$$

System (25) can be written in the form

$$\gamma_{0n} - \frac{1}{2\pi}\sum_{k=1}^{n}\cot\frac{\theta_{0j}-\theta_k}{2} g_n'(\theta_k)\frac{2\pi}{n} = f(\theta_{0j}) + Q\delta_h(\theta_{0j}-q), \quad j = 1, 2, \ldots, n,$$

$$\sum_{k=1}^{n} g_n'(\theta_k)\frac{2\pi}{n} = 0, \tag{26}$$

where

$$g_n'(\theta_k) = \frac{g_n(\theta_{0k}) - g_n(\theta_{0k-1})}{2\pi/n}, \quad g_n(\theta_{00}) = g_n(\theta_{0n}).$$

Similarly as for Eq. (4) it may be shown that

$$\left|g(\theta_{0k}) - g_n(\theta_{0k})\right| \le O(n^{-\alpha}\ln n), \quad 0 < \alpha \le 1. \tag{27}$$

Now we consider the equation

$$\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{\theta_0-\theta}{2}\gamma(\theta)\,\mathrm{d}\theta + \int_0^{2\pi}K(\theta_0,\theta)\gamma(\theta)\,\mathrm{d}\theta = f(\theta_0) + f_{\delta,q}(\theta_0), \quad \theta_0, q\in[0, 2\pi], \tag{28}$$

where the functions $f(\theta_0)$, $f_{\delta,q}(\theta_0)$ are the same as in (4) and the function $K(\theta_0,\theta)$ belongs to $H(\alpha)$ on $[0, 2\pi]\times[0, 2\pi]$ and has period 2π with respect to θ_0 and θ.

The solution $\gamma(\theta)$ of Eq. (28) we will seek in the form

$$\gamma(\theta) = \psi(\theta) - \frac{Q}{2\pi}\cot\frac{\theta - q}{2}, \quad \theta, q \in [0, 2\pi]. \tag{29}$$

From equality (18) it follows that function $\psi(\theta)$ satisfies the equation

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{\theta_0-\theta}{2}\psi(\theta)\,\mathrm{d}\theta + \int_0^{2\pi}K(\theta_0,\theta)\psi(\theta)\,\mathrm{d}\theta \\ &= f(\theta) + \frac{Q}{2\pi} + \frac{Q}{2\pi}\int_0^{2\pi}K(\theta_0,\theta)\cot\frac{\theta-q}{2}\,\mathrm{d}\theta, \\ &\theta_0 \in [0, 2\pi]. \end{aligned} \tag{30}$$

Now the condition for the solvability of Eq. (30) obtains the form

$$\int_0^{2\pi}\left[f(\theta) + \frac{Q}{2\pi} + \frac{Q}{2\pi}\int_0^{2\pi}K(\theta_0,\theta)\cot\frac{\theta-q}{2}\,\mathrm{d}\theta - \int_0^{2\pi}K(\theta_0,\theta)\psi(\theta)\,\mathrm{d}\theta \right]\mathrm{d}\theta_0 = 0. \tag{31}$$

We will assume that Eq. (28) has a unique solution under condition (10). Then for numerical solution of Eq. (28) we take a system of type (11), adding to the left-hand side for $j = 1, 2, \dots, n$ the member

$$\sum_{k=1}^{n} K(\theta_{0j},\theta_k)\gamma(\theta_k)\frac{2\pi}{n}.$$

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Hilberti tuumaga singulaarne ja hüpersingulaarne integraalvõrrand, deltafunktsioon ja diskreetsete keeriste meetod

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Käsitletakse Hilberti tuumaga singulaarset ja hüpersingulaarset integraalvõrrandit, mille vabaliige sisaldab deltafunktsiooni. Esitatakse diskreetsete keeriste meetod taoliste ülesannete lahendamiseks ning tõestatakse meetodi koonduvus.