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Analysis of large deformations of curved surfaces in holographic interferometry with remarks concerning nonspherical gravitational fields and rotating bodies

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Abstract. The principles of analysis of large deformations in holographic interferometry are briefly outlined. Modifications at the reconstruction are necessary to recover the previously invisible fringes. The spacing and contrast are characterized by the fringe and visibility vectors. The relevant first derivative of the path difference involves the polar decomposition of the deformation gradient and affine connections. By the modification the image aberration must be considered together with changes in geodesic curvatures and surface curvatures. This leads to similar aspects for hypersurfaces, and in particular to an interpretation of the Schwarzschild-solution by virtual deformations. Remarks concerning nonspherical gravitational fields and a tentative approach to the Kerr-solution for rotating bodies are added.

Key words: deformation analysis, holographic interferometry, curved surfaces.

1. INTRODUCTION: OPTICAL PATH DIFFERENCE

The linear basic equation of holographic interferometry for investigating the deformation of a small surface by the double exposure technique reads for the optical path difference

$$D = \boldsymbol{u} \cdot (\boldsymbol{k} - \boldsymbol{h}). \tag{1}$$

Here $\boldsymbol{u}$ denotes the displacement vector, $\boldsymbol{h}$ and $\boldsymbol{k}$ are unit vectors on the incident ray and on the reflected ray. Note that the path difference $D = \lambda \nu$ is also related to the wavelength λ and to the fringe order ν. An extended reference is given in [[1]]. Sometimes modification of the optical arrangement at the reconstruction [2] is necessary in order to change the fringe order. The simplest procedure for this purpose can be achieved by a phase shift $\Delta\psi$ within the real time technique. Here the actual wave field of the deformed surface interferes at the reconstruction with the diffracted wave field from the single exposed hologram of the undeformed surface. We have then $D = \boldsymbol{u}\cdot(\boldsymbol{k}-\boldsymbol{h}) - \boldsymbol{t}\cdot(\boldsymbol{k}-\boldsymbol{c}) - \lambda\Delta\psi/2\pi$. The vector $\boldsymbol{t}$ denotes a shift of the hologram, either of fringe control or of a repositioning error, and $\boldsymbol{c}$ is the unit vector on the reference ray. In case of a large deformation, when using two holograms and an appropriate modification at the reconstruction, we must use the exact expression $D = (\lambda/2\pi)(\tilde{\varphi} - \tilde{\varphi}') - (\tilde{L} - \tilde{L}')$ for the optical path difference, where $\tilde{L}$, $\tilde{L}'$ are the distances from the image points $\tilde{\mathrm{P}}$, $\tilde{\mathrm{P}}'$ to a point $\tilde{\mathrm{K}}$ of fringe localization (Fig. 1). The phases $\tilde{\varphi}$, $\tilde{\varphi}'$ at $\tilde{\mathrm{P}}$, $\tilde{\mathrm{P}}'$ are determined by the interference-diffraction identities between recording and reconstruction: $\tilde{\varphi} = 2\pi(L_\mathrm{T} + L_\mathrm{S} + p - q - q_\mathrm{T} - \tilde{p} + \tilde{q} + \tilde{q}_\mathrm{T})/\lambda + \pi + \tilde{\psi}$, $\tilde{\varphi}' = \ldots + \Delta\psi$. They contain the phase $\tilde{\psi}$ at $\tilde{\mathrm{T}}$ and the distances $L_\mathrm{T}, L_\mathrm{S}, p, q, q_\mathrm{T}, \tilde{p}, \tilde{q}, \tilde{q}_\mathrm{T}, \ldots$ If we substitute the phases in the exact expression of D, we obtain the general basic equation (see, e.g., [3], p. 153, Eq. (5.3))

$$D = L_\mathrm{S} - L'_\mathrm{S} - (\tilde{L} + \tilde{p}) + (\tilde{L}' + \tilde{p}') + p - q - p' + q' \\ + \tilde{q} - \tilde{q}' + L_\mathrm{T} - L'_\mathrm{T} + \tilde{q}_\mathrm{T} - \tilde{q}'_\mathrm{T} - \frac{\lambda}{2\pi}\Delta\psi. \tag{2}$$

Many authors (see, e.g., [4–10]) have studied the recovering of fringes, the elimination of the rotation, and the analysis of the modified fringes. As these fringes appear generally only in a small vicinity of a singularity, one should render the optical path difference quasi-stationary by the modification. The spacing and the contrast of the fringes depend, in fact, on the smallness of the derivative of D. The fringe spacing and direction lead also to the strains. Therefore this derivative will be primary in the following.

2. DERIVATIVE OF THE PATH DIFFERENCE. STRAIN AND ROTATION. CURVATURE

The differential of the path difference Eq. (2) reads $\mathrm{d}D = \mathrm{d}L_\mathrm{S} - \mathrm{d}L'_\mathrm{S} - \mathrm{d}(\tilde{L} + \tilde{p}) + \ldots + \mathrm{d}\tilde{q} - \mathrm{d}\tilde{q}'$, where we have, for example, $\mathrm{d}L_\mathrm{S} = \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{N}\nabla L_\mathrm{S} = \mathrm{d}\boldsymbol{r}\cdot\nabla_\mathrm{n} L_\mathrm{S} = \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{N}\boldsymbol{h}$ with the normal projector $\boldsymbol{N} = \boldsymbol{I} - \boldsymbol{n}\otimes\boldsymbol{n}$. We use here the rules $\boldsymbol{v}\,(\boldsymbol{a}\otimes\boldsymbol{b}) = (\boldsymbol{v}\cdot\boldsymbol{a})\boldsymbol{b}$, $(\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{w} = \boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{w})$ for a dyadic $\boldsymbol{a}\otimes\boldsymbol{b}$ or any tensor $\boldsymbol{T}$, which assimilate a vector either from left or from right by a scalar product in a linear transformation. The 2D-operator $\nabla_\mathrm{n} = \boldsymbol{N}\nabla = \boldsymbol{a}^\alpha\,\partial/\partial\theta^\alpha$ (α from 1 to 2) appears as a formal projection of the 3D-operator $\nabla = \boldsymbol{e}_x\,\partial/\partial x + \boldsymbol{e}_y\,\partial/\partial y + \boldsymbol{e}_z\,\partial/\partial z$. The index n recalls the surface normal $\boldsymbol{n}$ (or $\boldsymbol{n}', \ldots$). It is also expressed by curvilinear coordinates θ^1, θ^2 and a base $\boldsymbol{a}^1, \boldsymbol{a}^2$ on the surface, where $\boldsymbol{a}^\alpha\cdot\boldsymbol{a}_\beta = \delta^\alpha_\beta$, $\boldsymbol{a}_\beta = \partial\boldsymbol{r}/\partial\theta^\beta$. With $\boldsymbol{a}_\alpha\cdot\boldsymbol{a}_\beta = a_{\alpha\beta}$ we can also write $\boldsymbol{N} = a_{\alpha\beta}\boldsymbol{a}^\alpha\otimes\boldsymbol{a}^\beta$. Using the projectors $\boldsymbol{N}, \boldsymbol{N}', \hat{\boldsymbol{N}}, \hat{\boldsymbol{N}}', \overset{\frown}{\boldsymbol{N}}, \overset{\frown}{\boldsymbol{N}}', \tilde{\boldsymbol{K}}_{\tilde{p}}$, we get thus

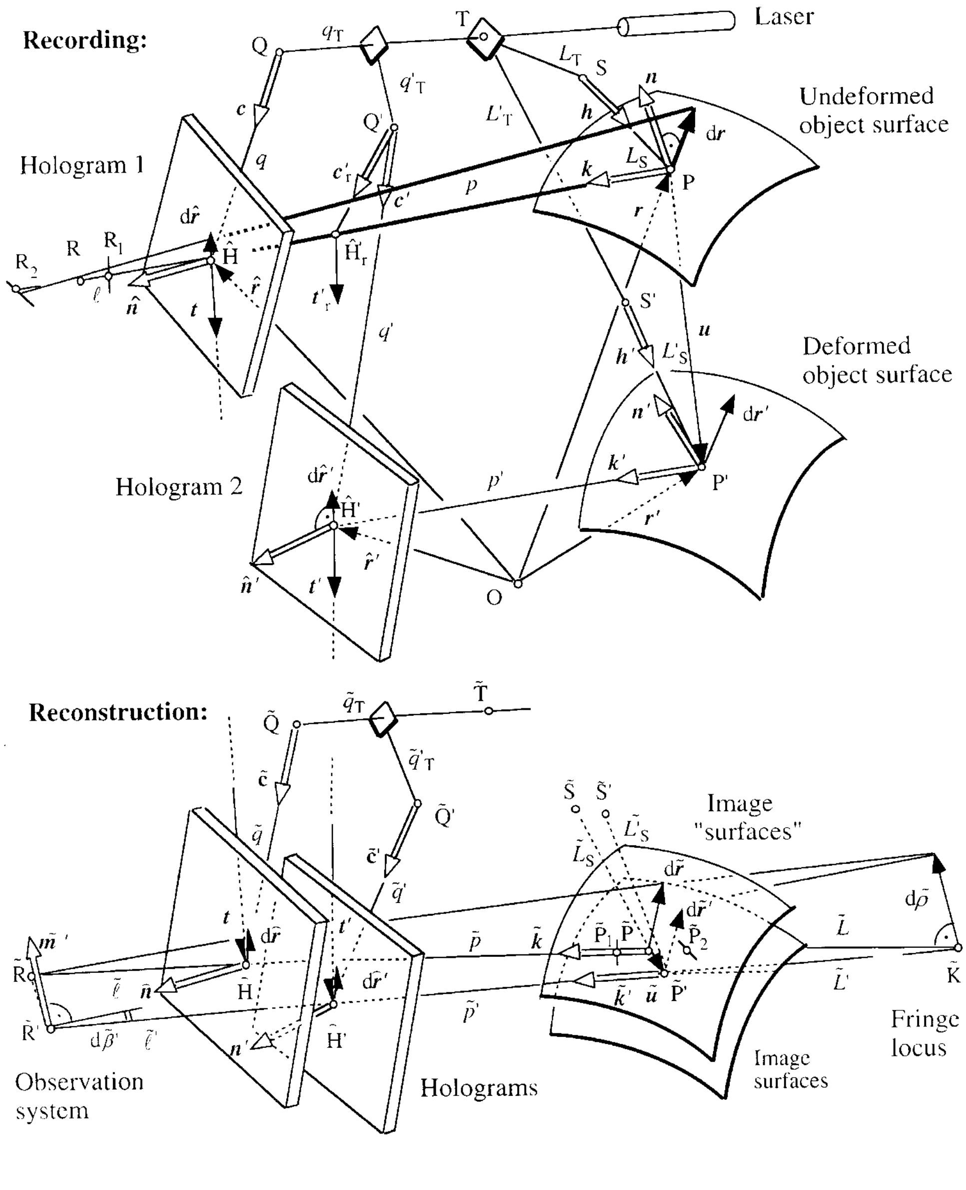

Fig. 1. Recording of a large object deformation. Reconstruction with a modification.

$$\mathrm{d}D = \mathrm{d}\boldsymbol{r}\boldsymbol{N}\cdot\boldsymbol{h} - \mathrm{d}\boldsymbol{r}'\boldsymbol{N}'\cdot\boldsymbol{h}' - (\mathrm{d}\widehat{\boldsymbol{r}}\widehat{\boldsymbol{N}} - \mathrm{d}\widetilde{\rho}\widetilde{\boldsymbol{K}}_{\widetilde{\rho}})\cdot\widetilde{\boldsymbol{k}} + (\mathrm{d}\widehat{\boldsymbol{r}}'\widehat{\boldsymbol{N}}' - \mathrm{d}\widetilde{\rho}\widetilde{\boldsymbol{K}}_{\widetilde{\rho}})\cdot\widetilde{\boldsymbol{k}}'$$
$$+ (\mathrm{d}\hat{\boldsymbol{r}}\hat{\boldsymbol{N}} - \mathrm{d}\boldsymbol{r}\boldsymbol{N})\cdot\boldsymbol{k} - \mathrm{d}\hat{\boldsymbol{r}}\hat{\boldsymbol{N}}\cdot\boldsymbol{c} - (\mathrm{d}\hat{\boldsymbol{r}}'\hat{\boldsymbol{N}}' - \mathrm{d}\boldsymbol{r}'\boldsymbol{N}')\cdot\boldsymbol{k}' + \mathrm{d}\hat{\boldsymbol{r}}'\hat{\boldsymbol{N}}'\cdot\boldsymbol{c}'$$
$$+ \mathrm{d}\widehat{\boldsymbol{r}}\widehat{\boldsymbol{N}}\cdot\widetilde{\boldsymbol{c}} - \mathrm{d}\widehat{\boldsymbol{r}}'\widehat{\boldsymbol{N}}'\cdot\widetilde{\boldsymbol{c}}. \tag{3}$$

The deformations of the surfaces are $\boldsymbol{N}'\mathrm{d}\boldsymbol{r}' = \boldsymbol{F}\boldsymbol{N}\mathrm{d}\boldsymbol{r}$, $\widehat{\boldsymbol{N}}\mathrm{d}\widehat{\boldsymbol{r}} = \hat{\boldsymbol{F}}\hat{\boldsymbol{N}}\mathrm{d}\hat{\boldsymbol{r}},\ldots,$ so that only the semiprojections $\boldsymbol{F}\boldsymbol{N},\ldots$ of the 3D-deformation gradients $\boldsymbol{F},\ldots$ intervene. The polar decomposition would be $\boldsymbol{F} = \boldsymbol{Q}\boldsymbol{U}$ with the symmetric dilatation $\boldsymbol{U}$, defined by the Cauchy–Green tensor $\boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{U}\boldsymbol{U}$, and the orthogonal rotation tensor $\boldsymbol{Q}$ of the volume element. At the surface element the polar decomposition is with the in-plane dilatation $\boldsymbol{V}$, defined by the full projection $\boldsymbol{N}\boldsymbol{F}^T\boldsymbol{F}\boldsymbol{N} = \boldsymbol{V}\boldsymbol{V}$, and the surface rotation tensor $\boldsymbol{Q}_\mathrm{n}$, implying the orthogonality $\boldsymbol{Q}_\mathrm{n}^T\boldsymbol{Q}_\mathrm{n} = \boldsymbol{I}$:

$$\boldsymbol{F}\boldsymbol{N} = \boldsymbol{Q}_\mathrm{n}\boldsymbol{V}. \tag{4}$$

The surface strain tensor would be $\gamma = (\boldsymbol{V}\boldsymbol{V} - \boldsymbol{N})/2$ and the rotation of the unit normal $\boldsymbol{n}$ is given by $\boldsymbol{Q}_\mathrm{n}\boldsymbol{n} = \boldsymbol{n}'$. In case of a small strain tensor γ, a small inclination vector ω, and a pivot rotation scalar Ω we write the additive decomposition $\boldsymbol{F}\boldsymbol{N} = \boldsymbol{N} + \gamma - \Omega\boldsymbol{E} + \boldsymbol{n}\otimes\omega$, where $\boldsymbol{Q}_\mathrm{n} \cong \boldsymbol{I} - \omega\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\omega - \Omega\boldsymbol{E}$ and $\boldsymbol{V} \cong \boldsymbol{N} + \gamma$. The 2D-permutation tensor $\boldsymbol{E} = E_{\alpha\beta}\boldsymbol{a}^\alpha\otimes\boldsymbol{a}^\beta = \mathbf{E}\boldsymbol{n}$ has the components

$$E_{12} = -E_{21} = \sqrt{a_{11}a_{22} - (a_{12})^2}, \qquad E_{11} = E_{22} = 0;$$

it appears also as "normal part" of the 3D-permutation tensor $\mathbf{E}$. We use further the derivatives

$$\nabla_\mathrm{n}\otimes\boldsymbol{n} = -\boldsymbol{B}, \quad \nabla_\mathrm{n}\otimes\boldsymbol{N} = \boldsymbol{B}\otimes\boldsymbol{n} + \boldsymbol{B}\otimes\boldsymbol{n}]^T. \tag{5}$$

The tensor $\boldsymbol{B} = B_{\alpha\beta}\boldsymbol{a}^\alpha\otimes\boldsymbol{a}^\beta = (\boldsymbol{e}_1\otimes\boldsymbol{e}_1)/r_1 + (\boldsymbol{e}_2\otimes\boldsymbol{e}_2)/r_2$ describes the (exterior) curvature of the surface with principal values $1/r_1, 1/r_2$. Equations (5) correspond to the equations of Frenet $\mathrm{d}\boldsymbol{n}/\mathrm{d}s = -\boldsymbol{e}/r$, $\mathrm{d}\boldsymbol{e}/\mathrm{d}s = \boldsymbol{n}/r$ for a plane curve with the intrinsic unit base $\boldsymbol{e}, \boldsymbol{n}$ and the radius r. The open square bracket $]^T$ in Eq. (5_2) indicates a transposition of the last two factors in the triadic, so that $\boldsymbol{B}\otimes\boldsymbol{n}]^T = B_{\alpha\beta}\boldsymbol{a}^\alpha\otimes\boldsymbol{n}\otimes\boldsymbol{a}^\beta$. Equations (5) help also for any decomposition. If the displacement $\boldsymbol{u} = \boldsymbol{v} + w\boldsymbol{n}$ has the tangential (interior) part $\boldsymbol{v} = \boldsymbol{N}\boldsymbol{u} = \boldsymbol{N}\boldsymbol{v}$ and the normal (exterior) part $w\boldsymbol{n}$, we get with the product rule the derivative $\nabla_\mathrm{n}\otimes\boldsymbol{u} = (\nabla_\mathrm{n}\otimes\boldsymbol{v})\boldsymbol{N} - \boldsymbol{B}w + (\nabla_\mathrm{n}w + \boldsymbol{B}\boldsymbol{v})\otimes\boldsymbol{n}$, and in case of a small deformation the strain tensor $\gamma = [(\nabla_\mathrm{n}\otimes\boldsymbol{v})\boldsymbol{N} + \boldsymbol{N}(\nabla_\mathrm{n}\otimes\boldsymbol{v})^T]/2 - \boldsymbol{B}w$. At the free surface of an elastic isotropic body the stress–strain relations read $\gamma = (\tau + \nu\boldsymbol{E}\tau\boldsymbol{E})/E$ with the coefficients of elasticity E, ν. The involution $\boldsymbol{E}(\ldots)\boldsymbol{E}$ will intervene several times later on. But we return now again to the large deformation. Equation (3) becomes with Eq. (4) in a Lagrange form

$$\mathrm{d}D = \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{N}[\boldsymbol{V}\boldsymbol{Q}_\mathrm{n}^T(\boldsymbol{k}' - \boldsymbol{h}') - (\boldsymbol{k} - \boldsymbol{h})] + \mathrm{d}\widetilde{\rho}\cdot\widetilde{\boldsymbol{K}}_{\widetilde{\rho}}(\widetilde{\boldsymbol{k}} - \widetilde{\boldsymbol{k}}')$$
$$- \mathrm{d}\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{N}}[\hat{\boldsymbol{V}}\hat{\boldsymbol{Q}}_{\hat{\mathrm{n}}}^T(\widetilde{\boldsymbol{k}} - \widetilde{\boldsymbol{c}}) - (\boldsymbol{k} - \boldsymbol{c})] + \mathrm{d}\hat{\boldsymbol{r}}'\cdot\hat{\boldsymbol{N}}'[\hat{\boldsymbol{V}}'\hat{\boldsymbol{Q}}'^T_{\hat{\mathrm{n}}}(\widetilde{\boldsymbol{k}}' - \widetilde{\boldsymbol{c}}') - (\boldsymbol{k}' - \boldsymbol{c}')]. \tag{6}$$

However, the directions to the images $\langle\tilde{\mathrm{P}}\rangle, \langle\tilde{\mathrm{P}}'\rangle$ of P, P' are defined by $\mathrm{d}\theta_\mathrm{P} = 0, \ldots$ of the phase difference $\theta_\mathrm{P} = (2\pi/\lambda)[(\tilde{p} - \tilde{q}) - (p - q)], \ldots$ for the rays through the aperture, so that the second line in Eq. (6) vanishes. The remaining first line can also be written in Euler's form

$$\mathrm{d}D = \mathrm{d}\boldsymbol{r}' \cdot \boldsymbol{N}'[(\boldsymbol{k}' - \boldsymbol{h}') - \boldsymbol{V}'\boldsymbol{Q}'^T_{\mathrm{n}'}(\boldsymbol{k} - \boldsymbol{h})] + \mathrm{d}\tilde{\boldsymbol{\rho}} \cdot \tilde{\boldsymbol{K}}_{\tilde{\rho}}(\tilde{\boldsymbol{k}} - \tilde{\boldsymbol{k}}'), \tag{7}$$

where $\boldsymbol{V}'\boldsymbol{Q}'^T_{\mathrm{n}'} = \boldsymbol{Q}_\mathrm{n}\boldsymbol{V}^{(-1)}$. The tensor $\boldsymbol{V}^{(-1)}$ represents a sort of 2D-"inverse" dilatation. It is defined by the relation $\boldsymbol{V}^{(-1)}\boldsymbol{V} = \boldsymbol{N}$ and will play an important role later on. Further, the increment $\mathrm{d}\boldsymbol{r}'$ in Eq. (7) must be transformed to the reconstruction. That implies the aberration of neighbouring rays, so that the second differential of the phase difference will intervene. Therefore we outline here some relations about changes of curvature. First, the equation of a geodesic curve with the arc s on the undeformed surface reads $\boldsymbol{N}\mathrm{d}^2\boldsymbol{r}/\mathrm{d}s^2 = 0,$ because the osculating plane contains the normal $\boldsymbol{n}$. For any curve and its image on the deformed surface we have

$$\boldsymbol{N}'\mathrm{d}^2\boldsymbol{r}' = \boldsymbol{N}'\mathrm{d}(\mathrm{d}\boldsymbol{r}') = \boldsymbol{N}'[\mathrm{d}(\boldsymbol{F}\boldsymbol{N})\mathrm{d}\boldsymbol{r} + \boldsymbol{F}\boldsymbol{N}\mathrm{d}^2\boldsymbol{r}] = \boldsymbol{N}'[(\mathrm{d}\boldsymbol{Q}_\mathrm{n}\boldsymbol{V} + \boldsymbol{Q}_\mathrm{n}\mathrm{d}\boldsymbol{V})\mathrm{d}\boldsymbol{r} + \boldsymbol{Q}_\mathrm{n}\boldsymbol{V}\mathrm{d}^2\boldsymbol{r}]. \tag{8}$$

Multiplying from left by $\boldsymbol{V}^{(-1)}\boldsymbol{Q}^T_\mathrm{n}$, we obtain for the change of geodesic curvature and the change of arc length the reversed equation of (8) in a short form

$$\begin{aligned} \boldsymbol{N}\mathrm{d}^2\boldsymbol{r} &= \boldsymbol{V}^{(-1)}[\boldsymbol{Q}^T_\mathrm{n}\boldsymbol{N}'\mathrm{d}^2\boldsymbol{r}' - (\mathrm{d}\boldsymbol{r}\boldsymbol{D}_\mathrm{V}\mathrm{d}\boldsymbol{r})], \\ \boldsymbol{D}_\mathrm{V} &= [(\nabla_\mathrm{n} \otimes \boldsymbol{V})\boldsymbol{N}] | \boldsymbol{N} + [(\nabla_\mathrm{n} \otimes \boldsymbol{Q}_\mathrm{n})\boldsymbol{V}] | \boldsymbol{N}'\boldsymbol{Q}_\mathrm{n}. \end{aligned} \tag{9}$$

If we use the integrability $\nabla_\mathrm{n}(\boldsymbol{E}\boldsymbol{F}^T) = 0,$ we can eliminate the derivative of the rotation and will get $\boldsymbol{D}_\mathrm{V} = [(\nabla_\mathrm{n} \otimes \boldsymbol{V})\boldsymbol{N}] | \boldsymbol{N} + \nabla_\mathrm{n}(\boldsymbol{E}\boldsymbol{V}\boldsymbol{E})\boldsymbol{V}^{(-1)}\boldsymbol{E} \otimes \boldsymbol{E}\boldsymbol{V}$ for Eq. (9_2). The involution $\boldsymbol{E}(\ldots)\boldsymbol{E}$ appears here once again. The triadic is thus interior and depends mainly on the dilatation $\boldsymbol{V}$, recalled by the index V. The sign $|\boldsymbol{N}$ marks a projection of the middle factor in $\nabla_\mathrm{n} \otimes \boldsymbol{V}$. Second, by means of the relation $\mathrm{d}\boldsymbol{r}' \cdot \nabla_{\mathrm{n}'} = \mathrm{d}\boldsymbol{r} \cdot \nabla_\mathrm{n} = \mathrm{d}\boldsymbol{r}' \cdot \boldsymbol{Q}_\mathrm{n}\boldsymbol{V}^{(-1)}\nabla_\mathrm{n}$, we get a useful equation for the curvature of the deformed surface:

$$\begin{aligned} \boldsymbol{B}' &= -\nabla_{\mathrm{n}'} \otimes \boldsymbol{n}' = -\boldsymbol{Q}_\mathrm{n}\boldsymbol{V}^{(-1)}(\nabla_\mathrm{n} \otimes \boldsymbol{n}') = \boldsymbol{Q}_\mathrm{n}\boldsymbol{V}^{(-1)}[\boldsymbol{B}\boldsymbol{Q}^T_\mathrm{n} - (\nabla_\mathrm{n} \otimes \boldsymbol{Q}_\mathrm{n})\boldsymbol{n}] \\ &= \boldsymbol{Q}_\mathrm{n}\boldsymbol{V}^{(-1)}(\boldsymbol{B} - \kappa)\boldsymbol{V}^{(-1)}\boldsymbol{Q}^T_\mathrm{n}, \end{aligned} \tag{10}$$

where κ describes the (symmetric) change of curvature; in case of a small isometric deformation it turns into the reduced curvature of Koiter–Sanders from the shell theory.

3. DERIVATIVES OF PHASES. IMAGE ABERRATION. FRINGE AND VISIBILITY VECTORS

In the following we consider $\theta_{\tilde{\mathrm{R}}} = (2\pi/\lambda)[(\ell + q) - (\tilde{\ell} + \tilde{q})]$ instead of $\theta_{\mathrm{P}} = (2\pi/\lambda)[(\tilde{p} - \tilde{q}) - (p - q)]$. As a matter of fact, $\mathrm{d}\theta_{\tilde{\mathrm{R}}} = 0,\ \mathrm{d}\theta_{\mathrm{P}} = 0$ lead both to the same equation

$$\hat{\boldsymbol{N}}[\hat{\boldsymbol{V}}\hat{\boldsymbol{Q}}_{\hat{\mathrm{n}}}^{T}(\tilde{\boldsymbol{k}} - \tilde{\boldsymbol{c}}) - (\boldsymbol{k} - \boldsymbol{c})] = 0. \tag{11}$$

For neighbouring rays we have $\mathrm{d}\overline{\theta}_{\tilde{\mathrm{R}}} = \mathrm{d}\theta_{\tilde{\mathrm{R}}} + \mathrm{d}^2\theta_{\tilde{\mathrm{R}}}/2 + \ldots$ and thus $\mathrm{d}^2(\ell + q - \tilde{\ell} - \tilde{q}) = 0.$ As $\mathrm{d}^2 q = \mathrm{d}(\mathrm{d}\hat{\boldsymbol{r}}\hat{\boldsymbol{N}} \cdot \boldsymbol{c}) = \mathrm{d}^2\hat{\boldsymbol{r}}\mathrm{d}\hat{\boldsymbol{N}} \cdot \boldsymbol{c} + \mathrm{d}\hat{\boldsymbol{r}}\mathrm{d}\hat{\boldsymbol{N}} \cdot \boldsymbol{c} + \mathrm{d}\hat{\boldsymbol{r}}\hat{\boldsymbol{N}} \cdot \mathrm{d}\boldsymbol{c}$ and $\mathrm{d}\hat{\boldsymbol{r}}\mathrm{d}\hat{\boldsymbol{N}} \cdot \boldsymbol{c} = (\mathrm{d}\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{B}}\mathrm{d}\hat{\boldsymbol{r}})(\hat{\boldsymbol{n}} \cdot \boldsymbol{c})$ according to Eq. (5), the curvature tensor $\hat{\boldsymbol{B}} = -\nabla_{\hat{\mathrm{n}}} \otimes \hat{\boldsymbol{n}}$ of the hologram intervenes if it is curved. The last increment gives $\mathrm{d}\boldsymbol{c} = \boldsymbol{C}\hat{\boldsymbol{N}}\mathrm{d}\hat{\boldsymbol{r}}/q$ with the projector $C = \boldsymbol{I} - \boldsymbol{c} \otimes \boldsymbol{c}.$ In summary, we obtain

$$\begin{aligned}
&\mathrm{d}^2\tilde{q} = (\widehat{\boldsymbol{N}}\,\mathrm{d}^2\,\widehat{\boldsymbol{r}}) \cdot \tilde{\boldsymbol{c}} + \mathrm{d}\,\widehat{\boldsymbol{r}} \cdot \widehat{\boldsymbol{B}}_{\tilde{\mathrm{C}}}\,\mathrm{d}\,\widehat{\boldsymbol{r}}\,,\ \mathrm{d}^2 q = (\hat{\boldsymbol{N}}\,\mathrm{d}^2\,\hat{\boldsymbol{r}}) \cdot \boldsymbol{c} + \mathrm{d}\,\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{B}}_{\mathrm{C}}\,\mathrm{d}\,\hat{\boldsymbol{r}}\,,\\
&\mathrm{d}^2\tilde{\ell} = -(\widehat{\boldsymbol{N}}\,\mathrm{d}^2\,\widehat{\boldsymbol{r}}) \cdot \tilde{\boldsymbol{k}} - \mathrm{d}\,\widehat{\boldsymbol{r}} \cdot \widehat{\boldsymbol{B}}_{\tilde{\mathrm{K}}}\,\mathrm{d}\,\widehat{\boldsymbol{r}} \quad \hat{\boldsymbol{B}}_{\mathrm{C}} = \hat{\boldsymbol{B}}(\hat{\boldsymbol{n}} \cdot \boldsymbol{c}) + \hat{\boldsymbol{N}}\boldsymbol{C}\hat{\boldsymbol{N}}/q,\\
&\widehat{\boldsymbol{B}}_{\tilde{\mathrm{C}}} = \widehat{\boldsymbol{B}}(\widehat{\boldsymbol{n}} \cdot \tilde{\boldsymbol{c}}) + \widehat{\boldsymbol{N}}\tilde{\boldsymbol{C}}\widehat{\boldsymbol{N}}/\tilde{q}, \quad \widehat{\boldsymbol{B}}_{\tilde{\mathrm{K}}} = \widehat{\boldsymbol{B}}(\widehat{\boldsymbol{n}} \cdot \tilde{\boldsymbol{k}}) - \widehat{\boldsymbol{N}}\tilde{\boldsymbol{K}}\widehat{\boldsymbol{N}}/\tilde{\ell}.
\end{aligned} \tag{12}$$

On the other hand, we may write for the second differentials of any curve according to Eqs. (9):

$$\begin{aligned}
&\widehat{\boldsymbol{N}}\,\mathrm{d}^2\,\widehat{\boldsymbol{r}} = \hat{\boldsymbol{Q}}_{\hat{\mathrm{n}}}\hat{\boldsymbol{V}}\hat{\boldsymbol{N}}\,\mathrm{d}^2\,\hat{\boldsymbol{r}} - \widehat{\boldsymbol{V}}^{(-1)}(\mathrm{d}\,\widehat{\boldsymbol{r}}\widehat{\boldsymbol{D}}_{\widehat{\mathrm{V}}}\,\mathrm{d}\,\widehat{\boldsymbol{r}}),\\
&\widehat{\boldsymbol{D}}_{\widehat{\mathrm{V}}} = [(\nabla_{\widehat{\mathrm{n}}} \otimes \widehat{\boldsymbol{V}})\,\widehat{\boldsymbol{N}}]\,|\,\widehat{\boldsymbol{N}} + [(\nabla_{\widehat{\mathrm{n}}} \otimes \widehat{\boldsymbol{Q}}_{\widehat{\mathrm{n}}})\widehat{\boldsymbol{V}}]\,|\,\hat{\boldsymbol{N}}\widehat{\boldsymbol{Q}}_{\widehat{\mathrm{n}}}\,.
\end{aligned} \tag{13}$$

Further, the relations $\mathrm{d}\widehat{\boldsymbol{r}} = -\tilde{\ell}\widehat{\boldsymbol{M}}^{T}\mathrm{d}\tilde{\boldsymbol{k}}$, $\mathrm{d}\widehat{\boldsymbol{r}} = -\ell\hat{\boldsymbol{M}}^{T}\mathrm{d}\boldsymbol{k}$ describe two affine connections, where $\widehat{\boldsymbol{M}} = \boldsymbol{I} - (\widehat{\boldsymbol{n}} \otimes \tilde{\boldsymbol{k}})/(\widehat{\boldsymbol{n}} \cdot \tilde{\boldsymbol{k}})$, $\widehat{\boldsymbol{M}} = \boldsymbol{I} - (\hat{\boldsymbol{n}} \otimes \boldsymbol{k})/(\hat{\boldsymbol{n}} \cdot \boldsymbol{k})$ are oblique projectors. We note also that $\mathrm{d}^2\underline{\ell} = -(\hat{N}\mathrm{d}^2\,\hat{\boldsymbol{r}}) \cdot \boldsymbol{k} - (\mathrm{d}\,\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{B}}\,\mathrm{d}\,\hat{\boldsymbol{r}})(\hat{\boldsymbol{n}} \cdot \boldsymbol{k})$ $-\mathrm{d}\,\hat{\boldsymbol{r}}\,\hat{N} \cdot \mathrm{d}\,\overline{\boldsymbol{k}}$ does not give a projector for $\mathrm{d}\overline{\boldsymbol{k}}$, because the origins of two intersecting rays at $\tilde{\mathrm{R}}$ are two skewed rays with an astigmatic interval $\langle\mathrm{R}\rangle$. If we resolve $\mathrm{d}^2(\ell + q - \tilde{\ell} - \tilde{q}) = 0$ with respect to $\mathrm{d}\hat{\boldsymbol{r}}\hat{\boldsymbol{N}} \cdot \mathrm{d}\overline{\boldsymbol{k}}$, the terms with the factor $\hat{\boldsymbol{N}}\mathrm{d}^2\hat{\boldsymbol{r}}$ are cancelled because of Eq. (11). The isolation of a factor $\mathrm{d}\boldsymbol{k}$ gives then $\mathrm{d}\overline{\boldsymbol{k}} = \ell\boldsymbol{T}\mathrm{d}\boldsymbol{k}$, where

$$\begin{aligned}
\boldsymbol{T} = &-\hat{\boldsymbol{M}}\{\hat{\boldsymbol{B}}_{\mathrm{C}} - \hat{\boldsymbol{B}}(\hat{\boldsymbol{n}} \cdot \boldsymbol{k})\\
&-\widehat{\boldsymbol{Q}}_{\widehat{\mathrm{n}}}\widehat{\boldsymbol{V}}^{(-1)}[\widehat{\boldsymbol{B}}_{\tilde{\mathrm{C}}} - \widehat{\boldsymbol{B}}(\widehat{\boldsymbol{n}} \cdot \tilde{\boldsymbol{k}}) + \widehat{\boldsymbol{D}}_{\widehat{\mathrm{V}}}\,|\,\widehat{\boldsymbol{V}}^{(-1)}(\tilde{\boldsymbol{k}} - \tilde{\boldsymbol{c}}) + \tilde{\boldsymbol{K}}/\tilde{\ell}]\widehat{\boldsymbol{V}}^{(-1)}\widehat{\boldsymbol{Q}}_{\widehat{\mathrm{n}}}^{T}\}\,\hat{\boldsymbol{M}}^{T}
\end{aligned} \tag{14}$$

describes the curvature of the converging nonspherical wavefront in $\hat{\mathrm{H}}$. The inverse values of the distances ℓ_1, ℓ_2 to the focal lines are the eigenvalues of the symmetric (!) tensor $\boldsymbol{T}$. The quadrangle between the hologram and the object shows the equation of transverse ray aberration $\boldsymbol{K}\mathrm{d}\boldsymbol{r} = \boldsymbol{K}\mathrm{d}\hat{\boldsymbol{r}} - p\mathrm{d}\overline{\boldsymbol{k}} = -\ell(\mathrm{d}\boldsymbol{k} + p\boldsymbol{T}\mathrm{d}\boldsymbol{k})$. We combine now both affine connections with the deformation gradient and the polar decomposition (4) to find the bridge

$\ell \mathrm{d}\boldsymbol{k} = \tilde{\ell}\boldsymbol{K}\hat{\boldsymbol{V}}^{(-1)}\hat{\boldsymbol{Q}}_{\hat{\mathrm{n}}}^{T}\hat{\boldsymbol{M}}^{T}\mathrm{d}\tilde{\boldsymbol{k}}$ between reconstruction and recording. It leads to the inverse projected virtual image deformation

$$\boldsymbol{K}\mathrm{d}\boldsymbol{r} = -(\tilde{\ell} + \tilde{p})\mathrm{d}\tilde{\boldsymbol{k}}\tilde{\boldsymbol{G}}_{\tilde{\mathrm{R}}}^{T} = \tilde{\boldsymbol{G}}_{\tilde{\mathrm{R}}}(\tilde{\boldsymbol{K}}\mathrm{d}\tilde{\boldsymbol{r}}), \quad \tilde{\boldsymbol{G}}_{\tilde{\mathrm{R}}} = \frac{\tilde{\ell}p}{\tilde{\ell} + \tilde{p}}\left(\boldsymbol{T} + \frac{1}{p}\boldsymbol{K}\right)\hat{\boldsymbol{V}}^{(-1)}\hat{\boldsymbol{Q}}_{\hat{\mathrm{n}}}^{T}\hat{\boldsymbol{M}}^{T}. \quad (15)$$

If there is a sufficient apparent overlapping of the corresponding surface areas $\tilde{A}, \tilde{A}'$, projected by the aperture A_0, we may write $\tilde{\boldsymbol{k}} - \tilde{\boldsymbol{k}}' \cong -\tilde{\boldsymbol{K}}\tilde{\boldsymbol{u}}/\tilde{L}'$, where the superposition vector $\tilde{\boldsymbol{f}}_S = \tilde{\boldsymbol{K}}\tilde{\boldsymbol{u}}$ should be very small compared to the diameter of these areas, otherwise the correlation is not possible. The strain should not be too large, so that the sizes of the areas do not differ much. We use further the collineation $\tilde{\boldsymbol{K}}'\mathrm{d}\tilde{\boldsymbol{\rho}} = -(\tilde{\ell}' + \tilde{p}' + \tilde{L}')\mathrm{d}\tilde{\boldsymbol{k}}'$ and the relation $\mathrm{d}\boldsymbol{r}' = \boldsymbol{M}'^{T}(\boldsymbol{K}'\mathrm{d}\boldsymbol{r}')$. The projector $\boldsymbol{M}'^{T} = \boldsymbol{I} - (\boldsymbol{k}' \otimes \boldsymbol{n}')/(\boldsymbol{k}' \cdot \boldsymbol{n}')$ is called shadow. We introduce now Eq. (15_1) (with primes) into Eq. (7) and express $\mathrm{d}\tilde{\boldsymbol{k}}' = -\tilde{\boldsymbol{m}}'\mathrm{d}\tilde{\boldsymbol{\beta}}'$ by a lateral unit vector $\tilde{\boldsymbol{m}}'$ and an increment $\mathrm{d}\tilde{\boldsymbol{\beta}}'$. So we can write $\mathrm{d}D_{\tilde{\mathrm{R}}'}/\mathrm{d}\tilde{\boldsymbol{\beta}}' = \tilde{\boldsymbol{m}}' \cdot \tilde{\boldsymbol{f}}'_{\tilde{\mathrm{R}}'}$, which implies the fringe vector [11]

$$\tilde{\boldsymbol{f}}'_{\tilde{\mathrm{R}}'} = (\tilde{\ell}' + \tilde{p}')\tilde{\boldsymbol{G}}'^{T}_{\tilde{\mathrm{R}}'}\boldsymbol{M}'[(\boldsymbol{k}' - \boldsymbol{h}') - \boldsymbol{Q}_{\mathrm{n}}\boldsymbol{V}^{(-1)}(\boldsymbol{k} - \boldsymbol{h})] - \tilde{\boldsymbol{K}}'\tilde{\boldsymbol{f}}_S(\tilde{\ell}' + \tilde{p}' + \tilde{L}')/\tilde{L}'. \quad (16)$$

It is normal to the fringe in the plane of sight and its inverse value is a measure of the fringe spacing. At this point it is perhaps judicious to specialize in the real-time technique, applied to a small object deformation and combined with a rigid body motion of the hologram. Here we perform the following approximations: $\boldsymbol{Q}_{\mathrm{n}}\boldsymbol{V}^{(-1)} \cong \boldsymbol{N} - \gamma - \Omega\boldsymbol{E} + \boldsymbol{n} \otimes \omega$, $\boldsymbol{h}' \cong \boldsymbol{h} + \boldsymbol{H}\boldsymbol{u}/L_{\mathrm{S}}$, $\tilde{\boldsymbol{f}}_S \cong \boldsymbol{K}\tilde{\boldsymbol{u}}$, $\boldsymbol{k}' \cong \tilde{\boldsymbol{k}} + \boldsymbol{K}\tilde{\boldsymbol{u}}/L$, $\hat{\boldsymbol{V}}\hat{\boldsymbol{Q}}_{\hat{\mathrm{n}}}^{T} \cong \hat{\boldsymbol{N}} + \hat{\omega} \otimes \hat{\boldsymbol{n}} + \hat{\Omega}\hat{\boldsymbol{E}}$, $\tilde{\boldsymbol{k}} \cong \boldsymbol{k} + \boldsymbol{K}(\boldsymbol{t} - \boldsymbol{u} + \tilde{\boldsymbol{u}})/p$, $\tilde{\boldsymbol{c}} \cong c + \boldsymbol{C}\boldsymbol{t}/q$. If we introduce the last three relations into Eq. (11), we obtain the linearized image condition

$$\boldsymbol{K}(\boldsymbol{t} - \boldsymbol{u} + \tilde{\boldsymbol{u}})/p = -\hat{\boldsymbol{M}}[(\hat{\omega} \otimes \hat{\boldsymbol{n}} + \hat{\Omega}\hat{\boldsymbol{E}})(\boldsymbol{k} - \boldsymbol{c}) - \boldsymbol{C}\boldsymbol{t}/q]. \quad (17)$$

On the other hand, if we insert the first four approximations into Eq. (16), we get with $\boldsymbol{M}' = \boldsymbol{M}'\boldsymbol{N}'$, $\boldsymbol{N}' \cong \boldsymbol{N} + \omega \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \omega$, $\tilde{\boldsymbol{G}}'^{T}_{\tilde{\mathrm{R}}'} \cong \boldsymbol{K}$ and Eq. (17) the linearized fringe vector

$$\begin{aligned}\tilde{\boldsymbol{f}}'_{\tilde{\mathrm{R}}'} \cong\ & (\ell + p)\boldsymbol{M}[(\gamma + \Omega\boldsymbol{E} + \omega \otimes \boldsymbol{n})(\boldsymbol{k} - \boldsymbol{h}) - \boldsymbol{H}\boldsymbol{u}/L_{\mathrm{S}}] \\ & - \ell\hat{\boldsymbol{M}}[(\hat{\omega} \otimes \hat{\boldsymbol{n}} + \hat{\Omega}\boldsymbol{E})(\boldsymbol{k} - \boldsymbol{c}) - \boldsymbol{C}\boldsymbol{t}/q] + \boldsymbol{K}(\boldsymbol{t} - \boldsymbol{u}). \quad (18)\end{aligned}$$

As we expect, the translation and the rotation of the hologram are related to the strain and the rotation of the object surface by means of the fringe interspace and direction. The visibility is determined like in the standard case [12]. The complex amplitudes over the areas $\tilde{A}, \tilde{A}'$ are

$$U = \iint_{\tilde{A}}(\tilde{S}\overline{\tilde{K}}\overline{\tilde{G}}/\overline{\tilde{L}}_{\mathrm{S}} \mid \overline{\tilde{L}} \mid)\exp[\mathrm{i}\overline{\tilde{\varphi}} - 2\pi\mathrm{i}\overline{\tilde{L}}/\lambda]\mathrm{d}\tilde{A}$$

and

$$U' = \iint_{\tilde{A}'}(\tilde{S}'...)\exp[\mathrm{i}\overline{\tilde{\varphi}} - 2\pi\mathrm{i}(\overline{\tilde{L}} - \overline{D})/\lambda]\mathrm{d}\tilde{A}'.$$

Here S and K denote source- and inclination-factors, whereas $\overline{D} = D + \Delta D_{\tilde{\mathrm{K}}} \cong D + \mathrm{d}\tilde{\boldsymbol{k}}' \cdot \boldsymbol{f}'_{\tilde{\mathrm{K}}}$. If $G'(\boldsymbol{r}') \cong G(\boldsymbol{r})$ is the statistical function of the roughness with

$$\left\langle G(\bar{\boldsymbol{r}})G^*(\bar{\bar{\boldsymbol{r}}})\right\rangle = C\delta_0(\bar{\boldsymbol{r}} - \bar{\bar{\boldsymbol{r}}})$$

(Dirac-function) and if

$$\tilde{S}\tilde{\overline{K}}\mathrm{d}\tilde{A}\big/\tilde{\overline{L}}_{\mathrm{S}}\tilde{\overline{L}} \cong \tilde{S}'\tilde{\overline{K}}'\mathrm{d}\tilde{A}'\big/\tilde{\overline{L}}'_{\mathrm{S}}\tilde{\overline{L}}',$$

the cross-correlation reads

$$\Gamma = \frac{1}{2}\left\langle U'U^*\right\rangle = |\Gamma| \exp\left[-\frac{2\pi \mathrm{i}}{\lambda}(D+\delta)\right]$$
$$\cong \exp\left[-\frac{2\pi \mathrm{i}}{\lambda}D\right]\iint_{\tilde{A}} \frac{\tilde{S}^2\tilde{\overline{K}}^2\tilde{C}}{\tilde{\overline{L}}_{\mathrm{S}}{}^2\tilde{\overline{L}}^2} \exp\left[-\frac{2\pi \mathrm{i}}{\lambda}\Delta D_{\tilde{\mathrm{K}}}\right]\mathrm{d}\tilde{A}, \tag{19}$$

where δ summarizes the influence of $\Delta D_{\tilde{\mathrm{K}}}$. The visibility becomes then

$$V = \frac{|\Gamma|}{I} = \frac{1}{A_0}\left|\iint_{A_0} \exp\left(-\frac{2\pi \mathrm{i}}{\lambda}\mathrm{d}\tilde{\boldsymbol{k}}' \cdot \tilde{\boldsymbol{f}}'_{\tilde{\mathrm{K}}}\right)\mathrm{d}A_0\right|. \tag{20}$$

$I = \langle UU^*\rangle/2 \cong I'$ denotes the intensity of one field. Since $\tilde{\mathrm{K}}$ is now the fixed point, we must exchange $\tilde{\ell}' + \tilde{p}'$ by $-\tilde{L}'$ in Eq. (16) to obtain the explicit expression of the visibility vector

$$\tilde{\boldsymbol{f}}'_{\tilde{\mathrm{K}}} = \tilde{L}'\tilde{\boldsymbol{G}}'^{T}_{\tilde{\mathrm{K}}}\boldsymbol{M}'[(\boldsymbol{k}' - \boldsymbol{h}') - \boldsymbol{Q}_{\mathrm{n}}\boldsymbol{V}^{(-1)}(\boldsymbol{k} - \boldsymbol{h})]. \tag{21}$$

It gives the shortest distance of the skewed homologous rays. The fringes are contrasted if the value of this vector, divided by the distance to $\tilde{\mathrm{K}}$, is small compared to the ratio of wavelength and aperture diameter. The visibility depends thus only of the derivative of the path difference. The intensity of the fringe field is $J = I + I' + \Gamma + \Gamma^* \cong 2I\{1 + V\cos[2\pi(D+\delta)/\lambda]\}$. Again, let us specialize in the real-time technique. According to Eq. (1) in the introduction, we define $2\pi[\boldsymbol{u}\cdot(\boldsymbol{k}-\boldsymbol{h}) - \boldsymbol{t}\cdot(\boldsymbol{k}-\boldsymbol{c}) + \delta]/\lambda = \Delta\phi$, where δ is usually neglected. We get then, for example, with three frames $\Delta\psi_1 = 2\pi/3$, $\Delta\psi_2 = 0$, $\Delta\psi_3 = -2\pi/3$ three intensities $J_1 = 2I[1 + V\cos(\Delta\phi - 2\pi/3)]$, $J_2 = 2I[1 + V\cos(\Delta\phi)]$, $J_3 = 2I[1 + V\cos(\Delta\phi + 2\pi/3)]$, so that the "detected visibility" becomes

$$V = \sqrt{3(J_1 - J_3)^2 + (J_1 + J_3 - 2J_2)^2}\Big/6I.$$

Finally, if we have a good contrast of the fringes ($\tilde{\boldsymbol{f}}'_{\tilde{\mathrm{K}}} \cong 0$), the second derivative of D (see also [13]) becomes after some calculations

$$\frac{\mathrm{d}^2 D_{\tilde{\mathrm{R}}'}}{\mathrm{d}\tilde{\beta}'^2} = \frac{\tilde{\boldsymbol{f}}'_{\tilde{\mathrm{R}}'} \cdot \tilde{\boldsymbol{K}}' \mathrm{d}^2 \tilde{\rho}}{(\tilde{L}' + \tilde{p}' + \tilde{\ell}')\mathrm{d}\tilde{\beta}'^2} - (\tilde{\ell}' + \tilde{p}')^2 \tilde{\boldsymbol{m}}' \cdot \tilde{\boldsymbol{G}}'^T_{\tilde{\mathrm{R}}'} \left(\boldsymbol{T}'_F + \frac{1}{p'} \boldsymbol{K}' \right) \tilde{\boldsymbol{G}}'_{\tilde{\mathrm{R}}'} \tilde{\boldsymbol{m}}'$$
$$- \tilde{\ell}^2 \tilde{\boldsymbol{m}} \cdot \left(\tilde{\boldsymbol{T}} + \frac{1}{\tilde{\ell}} \tilde{\boldsymbol{K}} \right) \tilde{\boldsymbol{m}} - \tilde{\ell}'^2 \tilde{\boldsymbol{m}}' \ldots \tag{22}$$

The symmetric tensor

$$\tilde{\boldsymbol{T}} = \hat{\boldsymbol{M}} \{ \hat{\boldsymbol{B}}_{\tilde{\mathrm{C}}} - \hat{\boldsymbol{B}}(\hat{\boldsymbol{n}} \cdot \tilde{\boldsymbol{k}}) - \hat{\boldsymbol{Q}}_{\hat{\mathrm{n}}} \hat{\boldsymbol{V}}^{(-1)} [\hat{\boldsymbol{B}}_{\mathrm{C}} - \hat{\boldsymbol{B}}(\hat{\boldsymbol{n}} \cdot \boldsymbol{k})$$
$$+ \hat{\boldsymbol{D}}_{\hat{\mathrm{V}}} \, | \hat{\boldsymbol{V}}^{(-1)}(\boldsymbol{k} - \boldsymbol{c}) - \boldsymbol{K}/p] \hat{\boldsymbol{V}}^{(-1)} \hat{\boldsymbol{Q}}^T_{\hat{\mathrm{n}}} \} \hat{\boldsymbol{M}}^T \tag{23}$$

is dual to the tensor $\boldsymbol{T}$ of Eq. (14) and describes the curvature of the diverging nonspherical wavefront in $\hat{\mathrm{H}}$. It determines also the astigmatic interval $\langle \tilde{\mathrm{P}} \rangle$, whereas the symmetric tensor

$$\boldsymbol{T}'_{\mathrm{F}} = \boldsymbol{M}' \{ \boldsymbol{B}'_{\mathrm{H}'} - \boldsymbol{B}'(\boldsymbol{n}' \cdot \boldsymbol{k}') - \boldsymbol{Q}_{\mathrm{n}} \boldsymbol{V}^{(-1)} [\boldsymbol{B}_{\mathrm{H}} - \boldsymbol{B}(\boldsymbol{n} \cdot \boldsymbol{k})$$
$$+ \boldsymbol{D}_{\mathrm{V}} \, | \boldsymbol{V}^{(-1)}(\boldsymbol{k} - \boldsymbol{h}) + \boldsymbol{K}/p] \boldsymbol{V}^{(-1)} \boldsymbol{Q}^T_{\mathrm{n}} \} \boldsymbol{M}'^T \tag{24}$$

expresses the influence of the changes of geodesic and surface curvatures on the deformation.

4. ASPECTS IN GENERAL. AN INTERPRETATION OF SCHWARZSCHILD'S SOLUTION

The remaining sections are only indirectly related to the previous subject. An extension should first illustrate Eqs. (9) and (10), which contain projected deformations, the polar decomposition Eq. (4), and the curvature Eqs. (5). But second, this procedure should also focus on the problem of general gravitational fields. For $\boldsymbol{B} = 0$, Eq. (10) gives the curvature $\boldsymbol{B}'$ of a surface $A^2 \subset R^3$ as a deformed part of R^2. This can be generalized to a hypersurface $A^k \subset R^n$, where it leads to the Ricci tensor $\boldsymbol{R}$. We recall the components $R_{\alpha\beta} = \Gamma^{\lambda}_{\alpha\lambda,\beta} - \Gamma^{\lambda}_{\alpha\beta,\lambda} + \Gamma^{\mu}_{\alpha\lambda}\Gamma^{\lambda}_{\mu\beta} - \Gamma^{\mu}_{\alpha\beta}\Gamma^{\lambda}_{\mu\lambda}$ with Christoffel symbols $\Gamma^{\lambda}_{\alpha\beta} = a^{\lambda\mu}(a_{\mu\alpha,\beta} + a_{\mu\beta,\alpha} - a_{\alpha\beta,\mu})/2$ and the "metric tensor" $a_{\alpha\beta}$. The latter shows the components of the projector $\boldsymbol{N}' = \boldsymbol{I} - \boldsymbol{n}'_i \otimes \boldsymbol{n}'_i = a_{\alpha\beta}\boldsymbol{a}^{\alpha} \otimes \boldsymbol{a}^{\beta} = \boldsymbol{a}^{\alpha} \otimes \boldsymbol{a}_{\alpha}$, with unit normals $\boldsymbol{n}'_i$ (i sum 1 to $n-k$; α, β sum 1 to k or 0 to $k-1$). The Ricci tensor is the contraction of the 4th-order Riemann–Christoffel tensor $\mathbf{R}$. It can be seen that we may write (see also, e.g. [14], pp. 34–36)

$$\mathbf{R}^T = \boldsymbol{N}' | \{ \boldsymbol{N}' [\nabla_{\mathrm{n}'} \otimes (\nabla_{\mathrm{n}'} \otimes \boldsymbol{N}') - \nabla_{\mathrm{n}'} \otimes (\nabla_{\mathrm{n}'} \otimes \boldsymbol{N}')^T] \boldsymbol{N}' \} | \boldsymbol{N}'$$
$$= \boldsymbol{B}'_i \otimes \boldsymbol{B}'_i - \boldsymbol{B}'_i \otimes \boldsymbol{B}'_i]]^T , \tag{25}$$

where $\boldsymbol{B}'_i = -\boldsymbol{Q}_{\mathrm{n}} \boldsymbol{V}^{(-1)} (\nabla_{\mathrm{n}} \otimes \boldsymbol{n}'_i) \boldsymbol{N}' = -\boldsymbol{Q}_{\mathrm{n}} \boldsymbol{V}^{(-1)} (\nabla_{\mathrm{n}} \otimes \boldsymbol{Q}_{\mathrm{n}} \boldsymbol{n}_i) \boldsymbol{N}'$ according to Eqs. (4), (5), (10). The double square bracket indicates a transposition of the factors 2 and 4 in the last quadratic. The Ricci tensor becomes then by contraction alternatively

$$\boldsymbol{R} = \boldsymbol{B}'_i \boldsymbol{B}'_i - \boldsymbol{B}'_i (\boldsymbol{B}'_i \cdot \boldsymbol{N}'), \tag{26}$$

with the trace $\boldsymbol{B}'_i \cdot \boldsymbol{N}'$. We outline now an interpretation by two virtual deformations of Schwarzschild's solution for the central gravitation (round a spherical star) as a simple example. The well-known fundamental form [15] reads

$$\mathrm{d}\sigma'^2 = -\left(1 - \frac{2M}{r}\right) c^2 \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - 2M/r} + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \, \mathrm{d}\varphi^2, \tag{27}$$

with coordinates $r, \theta, \varphi,$ time $t,$ velocity of light $c,$ and Schwarzschild radius $2M,$ so that $a_{00} = -(1 - 2M/r),$ $a_{11} = (1 - 2M/r)^{-1},$ $a_{22} = r^2,$ $a_{33} = r^2 \sin^2\theta.$ For the following it is convenient to introduce an angle $\psi,$ defined by $\sin\psi = \sqrt{2M/r}.$ We write further the spherical part in the form $r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \, \mathrm{d}\varphi^2 = \mathrm{d}\boldsymbol{r} \cdot \overline{\boldsymbol{K}} \mathrm{d}\boldsymbol{r},$ where $\overline{\boldsymbol{K}} = \boldsymbol{N}(\boldsymbol{I} - \boldsymbol{k} \otimes \boldsymbol{k})\boldsymbol{N} = \boldsymbol{N} - \boldsymbol{k} \otimes \boldsymbol{k}$ denotes a projector referring to the radial unit vector $\boldsymbol{k}(\theta, \varphi).$ First we consider only the partial fundamental form of the space part $\mathrm{d}s'^2 = \mathrm{d}\boldsymbol{r} \cdot [(\boldsymbol{k} \otimes \boldsymbol{k})/\cos^2\psi + \overline{\boldsymbol{K}}] \mathrm{d}\boldsymbol{r} = \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{V}\boldsymbol{V} \mathrm{d}\boldsymbol{r},$ which leads immediately to the inverse dilatation $\boldsymbol{V}^{(-1)} = (\boldsymbol{k} \otimes \boldsymbol{k}) \cos\psi + \overline{\boldsymbol{K}},$ and also to the components $R_{11} = \tan^2\psi / r^2,$ $R_{22} = -\sin^2\psi/2,$ $R_{33} = R_{22} \sin^2\theta$ of the 3D-Ricci tensor $\boldsymbol{R}_{3\mathrm{D}}.$ Alternatively, this tensor and a hypersurface $A^3 \subset R^4$ may be got from $\boldsymbol{r}' = r\boldsymbol{k} + w(r)\boldsymbol{n},$ where the vector $\boldsymbol{n}$ denotes the unit normal of the subspace $R^3 \subset R^4.$ We have then $\mathrm{d}\boldsymbol{r}' = \mathrm{d}\boldsymbol{r}[\boldsymbol{k} \otimes (\boldsymbol{k} + w_{,r}\boldsymbol{n}) + \overline{\boldsymbol{K}}] = \mathrm{d}\boldsymbol{r}\, \boldsymbol{N}\boldsymbol{F}^T,$ because $\nabla_{\mathrm{n}} \otimes \boldsymbol{k} = (\boldsymbol{N} - \boldsymbol{k} \otimes \boldsymbol{k})/r = \overline{\boldsymbol{K}}/r.$ As we have $\mathrm{d}s'^2 = \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{N}\boldsymbol{F}^T \boldsymbol{F}\boldsymbol{N} \, \mathrm{d}\boldsymbol{r} = \mathrm{d}\boldsymbol{r} \cdot [(1 + w_{,r}^2)(\boldsymbol{k} \otimes \boldsymbol{k}) + \overline{\boldsymbol{K}}] \, \mathrm{d}\boldsymbol{r},$ a comparison gives $\cos\psi = 1 \big/ \sqrt{1 + w_{,r}^2};$ thus ψ turns out to be the inclination. The polar Eq. (4) is $\boldsymbol{Q}_{\mathrm{n}} \boldsymbol{N} = \boldsymbol{F}\boldsymbol{N}\boldsymbol{V}^{(-1)} = \boldsymbol{k}' \otimes \boldsymbol{k} + \overline{\boldsymbol{K}},$ where $\boldsymbol{k}' = \boldsymbol{k} \cos\psi + \boldsymbol{n} \sin\psi,$ and further $\boldsymbol{Q}_{\mathrm{n}} = \boldsymbol{k}' \otimes \boldsymbol{k} + \boldsymbol{n}' \otimes \boldsymbol{n} + \overline{\boldsymbol{K}},$ $\boldsymbol{Q}_{\mathrm{n}} \boldsymbol{V}^{(-1)} = \boldsymbol{k}' \otimes \boldsymbol{k} \cos\psi + \overline{\boldsymbol{K}}$ with the normal $\boldsymbol{n}' = -\boldsymbol{k} \sin\psi + \boldsymbol{n} \cos\psi.$ By the use of a sort of key relation $(\sin\psi)_{,r} = -\sin\psi/2r,$ the derivative is $\nabla_{\mathrm{n}} \otimes \boldsymbol{n}' = \boldsymbol{k} \otimes \boldsymbol{k}' \tan\psi/2r - \overline{\boldsymbol{K}} \sin\psi/r.$ We obtain then with $\overline{\boldsymbol{K}} \cdot \boldsymbol{N}' = 2 (r_2 = r_3)$ the curvature $\boldsymbol{B}' = -\boldsymbol{Q}_{\mathrm{n}} \boldsymbol{V}^{(-1)} (\nabla_{\mathrm{n}} \otimes \boldsymbol{n}')\boldsymbol{N}' = (\sin\psi/r)[-(\boldsymbol{k}' \otimes \boldsymbol{k}')/2 + \overline{\boldsymbol{K}}] = (\boldsymbol{k}' \otimes \boldsymbol{k}')/r_1 + \overline{\boldsymbol{K}}/r_2,$ as well as

$$\begin{aligned} \boldsymbol{R}_{3\mathrm{D}} &= \boldsymbol{B}'\boldsymbol{B}' - \boldsymbol{B}'(\boldsymbol{B}' \cdot \boldsymbol{N}') \\ &= -\frac{2}{r_1 r_2}(\boldsymbol{k}' \otimes \boldsymbol{k}') - \frac{1}{r_2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\overline{\boldsymbol{K}} = \frac{\sin^2\psi}{r^2}\left(\boldsymbol{k}' \otimes \boldsymbol{k}' - \frac{1}{2}\overline{\boldsymbol{K}}\right). \end{aligned} \tag{28}$$

In fact, we have here $R_1^1 = \sin^2\psi / r^2 = R_{11} a^{11},$ $R_2^2 = -\sin^2\psi / 2r^2 \ldots$ conformal to the previous components. In 2D, where $\overline{\boldsymbol{K}} \cdot \boldsymbol{N}' = 1,$ we would obtain $\boldsymbol{R}'_{2\mathrm{D}} = (\boldsymbol{k}' \otimes \boldsymbol{k}' + \overline{\boldsymbol{K}}) \sin^2 \psi / 2r^2$ and $w_{,r} = (r/2M - 1)^{-1/2}$ determines the well-known vase-like surface $r = 2M + w^2/8M$ ([16], p. 837). Second, the fundamental form $\mathrm{d}\bar{s}'^2 = -(\cos^2 \psi) c^2 \mathrm{d}t^2 + (\cos^2 \psi)^{-1} \mathrm{d}r^2$ of the first two terms of $\mathrm{d}\sigma'^2$ gives $\overline{R}_0^0 = \overline{R}_1^1 = -\sin^2 \psi / r^2$. Alternatively to [17] we introduce with another normal $\overline{\boldsymbol{n}}$ a vector $\bar{\boldsymbol{r}}' = 2M \cos\psi \boldsymbol{k} + \overline{w}(r) \overline{\boldsymbol{n}}$ for $\boldsymbol{k} = \boldsymbol{k}_0 \cos \omega t + \boldsymbol{h}_0 \sin \omega t$, $\boldsymbol{h} = -\boldsymbol{k}_0 \sin \omega t + \boldsymbol{h}_0 \cos \omega t$. The virtual deformation reads $\mathrm{d}\bar{\boldsymbol{r}}' = 2M[\mathrm{d}\boldsymbol{k} \cos\psi + \mathrm{d}r\boldsymbol{k}(\cos\psi)_{,r}] + \mathrm{d}r\overline{w}_{,r}\overline{\boldsymbol{n}} = \mathrm{d}\bar{\boldsymbol{r}}\overline{\boldsymbol{N}\boldsymbol{F}}^T$, where we have $\mathrm{d}\boldsymbol{k} = \mathrm{d}\bar{\boldsymbol{r}}(\boldsymbol{h} \otimes \boldsymbol{h})/r$, $(\cos\psi)_{,r} = \sin^2 \psi / 2r \cos\psi$ and $\bar{\boldsymbol{r}} = r\boldsymbol{k}$. The deformation gradient is $\overline{\boldsymbol{F}\boldsymbol{N}} = (\boldsymbol{h} \otimes \boldsymbol{h}) \sin^2 \psi \cos\psi + (\boldsymbol{k} \sin^4 \psi / 2 \cos\psi + \overline{w}_{,r} \overline{\boldsymbol{n}}) \otimes \boldsymbol{k}$ and the inverse dilatation becomes $\overline{\boldsymbol{V}}^{(-1)} = \boldsymbol{h} \otimes \boldsymbol{h}(\sin^2 \psi \cos\psi)^{-1} + \boldsymbol{k} \otimes \boldsymbol{k} \cos\psi$. Therefore the reverse polar decomposition gives $\overline{\boldsymbol{Q}}_{\bar{\mathrm{n}}} \overline{\boldsymbol{N}} = \overline{\boldsymbol{F}\boldsymbol{N}}\overline{\boldsymbol{V}}^{(-1)} = \boldsymbol{h} \otimes \boldsymbol{h} + (\bar{\boldsymbol{k}} \sin^4 \psi / 2 + \overline{\boldsymbol{n}}\, \overline{w}_{,r} \cos\psi) \otimes \boldsymbol{k} = \boldsymbol{h} \otimes \boldsymbol{h} + \bar{\boldsymbol{k}}' \otimes \boldsymbol{k}$, so that we get for the angle of rotation $\pi/2 - \chi$ the relations $\sin\chi = \sin^4 \psi / 2$, $\overline{w}_{,r} = \cos\chi / \cos\psi$. In addition, we have $\bar{\boldsymbol{k}}' = \boldsymbol{k} \sin\chi + \overline{\boldsymbol{n}} \cos\chi$, $\overline{\boldsymbol{n}}' = -\boldsymbol{k} \cos\chi + \overline{\boldsymbol{n}} \sin\chi$, $(2M/r)(\mathrm{d}\bar{\boldsymbol{r}} \cdot \boldsymbol{h}) = (2M/r)\mathrm{i}r\omega\mathrm{d}t = \mathrm{i}c\mathrm{d}t$, $\omega = c/2M$. The curvature of this goblet-like surface becomes finally, with $\chi_{,r} = -2\tan\chi / r$, $\overline{\mathbf{B}}' = (\sin\psi / r)[\eta(\boldsymbol{h} \otimes \boldsymbol{h}) + (\bar{\boldsymbol{k}}' \otimes \bar{\boldsymbol{k}}')/\eta] = (\boldsymbol{h} \otimes \boldsymbol{h})/\bar{r}_0 + (\bar{\boldsymbol{k}}' \otimes \bar{\boldsymbol{k}}')/\bar{r}_1$, and the 2D-Ricci tensor is

$$\boldsymbol{R}_{2\mathrm{D}} = -(1/\bar{r}_0\bar{r}_1)(\boldsymbol{h} \otimes \boldsymbol{h} + \bar{\boldsymbol{k}}' \otimes \bar{\boldsymbol{k}}') = -(\sin^2 \psi / r^2)(\boldsymbol{h} \otimes \boldsymbol{h} + \bar{\boldsymbol{k}}' \otimes \bar{\boldsymbol{k}}'). \tag{29}$$

Note here that $\eta = \tan\psi / 2 \tan\chi$ is not relevant in Eq. (29). Third, we replace, now on a stripe $S^4 \subset R^6$ along the first meridian, $\boldsymbol{B}'$ by $\boldsymbol{B}_1' = (\boldsymbol{h} \otimes \boldsymbol{h})/r_0 + (\boldsymbol{k}' \otimes \boldsymbol{k}')/r_1 + \overline{\boldsymbol{K}}/r_2$ and $\overline{\boldsymbol{B}}'$ by $\boldsymbol{B}_2' = (\boldsymbol{h} \otimes \boldsymbol{h})/\hat{r}_0 + (\boldsymbol{k}' \otimes \boldsymbol{k}')/\hat{r}_1$, where the relation $\cos\beta / \hat{r}_1 = \sin\beta / r_1$ must hold, and $\beta, \pi/2 - \beta$ are the angles of $\boldsymbol{n}'_1, \boldsymbol{n}'_2$ with respect to $\boldsymbol{n}'$. The 4D-Ricci tensor is then by composition

$$\begin{aligned} \boldsymbol{R}_{4\mathrm{D}} &= \boldsymbol{B}_1' \boldsymbol{B}_1' - \mathbf{B}_1'(\boldsymbol{B}_1' \cdot \boldsymbol{N}') + \boldsymbol{B}_2' \boldsymbol{B}_2' - \boldsymbol{B}_2'(\boldsymbol{B}_2' \cdot \boldsymbol{N}') \\ &= -(1/r_0r_2 + 1/r_1r_2 + 1/r_2r_2)\,\overline{\boldsymbol{K}} - (1/\hat{r}_0\hat{r}_1 + 1/r_0r_1 + 2/r_0r_2)(\boldsymbol{h} \otimes \boldsymbol{h}) \\ &\quad - (1/\hat{r}_0\hat{r}_1 + 1/r_0r_1 + 2/r_1r_2)(\boldsymbol{k}' \otimes \boldsymbol{k}') = 0 \end{aligned} \tag{30}$$

valid if $1/r_0 = 1/r_1 = -1/2r_2 = -\sin\psi / 2r$, $1/\hat{r}_0 = 1/\hat{r}_1 = \tan\beta / r_1$, $\tan\beta = \sqrt{3}$, $\beta = \pi/3$. Equation (30) is compatible with Eqs. (28), (29) and therefore with the equations $R_{\alpha\beta} = 0$ in components, because $1/\hat{r}_0\hat{r}_1 + 1/r_0r_1 = 1/\bar{r}_0\bar{r}_1 = -2/r_1r_2$. Thus, Eq. (30) is an "intrinsic" form of the field equations in vacuum. The Riemann–Christoffel tensor $\mathbf{R}$ (expressed by $\mathbf{B}'_i$) has the components as, e.g., those in [18], p. 284, Eq. (10.11).

5. REMARKS CONCERNING NONSPHERICAL FIELDS

An equation similar to (30) holds in the general case $r_2 \neq r_3$ for $S^4 \subset R^8$ if $\boldsymbol{k}$ is normal to the surfaces of constant "potential" U. In this case we have $U_{,ss} + 2U_{,s}/\bar{r} = 0$, $U_{,s} = \boldsymbol{k} \cdot \nabla_{\mathrm{n}} U$ with the normal arc s. We define then $\psi \to \sin\psi = \sqrt{2}(M\mu U_{,s})^{1/4}$, so that

$$\frac{\mathrm{d}(\sin\psi)}{\mathrm{d}s} = -\rho\frac{\sin\psi}{2\bar{r}}, \quad \rho = 1 - \frac{\bar{r}}{2}\frac{\mathrm{d}(\log\mu)}{\mathrm{d}s}. \tag{31}$$

Equation (31_1) is a similar key relation as before (where $\rho = 1$). On an axis of symmetry we shall find $\rho = \bar{r}^2\bar{K}$, where $1/\bar{r} = (1/\bar{r}_2 + 1/\bar{r}_3)/2$ is the mean curvature and $\bar{K} = 1/\bar{r}_2\bar{r}_3$ is the Gauss' curvature of the surface $U = \text{const}$. At an arbitrary point, however, two "pivot"-rotations α, γ turn first $\boldsymbol{k}$ into a unit vector $\boldsymbol{k}^*$. The angles α, γ must be determined by the condition of vanishing mixed terms in the fundamental form $\mathrm{d}s'^2 = \mathrm{d}\boldsymbol{r}\cdot[(\boldsymbol{k}^*\otimes\boldsymbol{k}^*)/\cos^2\psi + \boldsymbol{K}^*]\mathrm{d}\boldsymbol{r}$. The inclination ψ rotates then from $\boldsymbol{k}^*$ (instead of $\boldsymbol{k}$) out of the flat space into the vector $\boldsymbol{k}'$. The principal result will be $\rho^* = \bar{r}r^*K^*$, where $1/r^* = (1/r_2^* + 1/r_3^*)/2$ is the mean curvature and $K^* = 1/r_2^*r_3^*$ is the Gauss' curvature of the surface with normal $\boldsymbol{k}^*$. The field equation

$$\boldsymbol{R}_{4\mathrm{D}} = -\left(\frac{1}{\hat{r}_0\hat{r}_1} + \frac{1}{r_0r_1} + \frac{2}{r_0\tilde{r}}\right)(\boldsymbol{h}\otimes\boldsymbol{h}) - \left(\frac{1}{\hat{r}_0\hat{r}_1} + \frac{1}{r_0r_1} + \frac{2}{r_1\tilde{r}}\right)(\boldsymbol{k}'\otimes\boldsymbol{k}')$$
$$-\left(\frac{1}{r_0\tilde{r}} + \frac{1}{r_1\tilde{r}} + \frac{1}{r_2r_3}\right)\boldsymbol{K}^* = 0 \tag{32}$$

looks like Eq. (30), but the projector $\boldsymbol{K}^* = \boldsymbol{N} - \boldsymbol{k}^*\otimes\boldsymbol{k}^*$ appears now instead of $\bar{\boldsymbol{K}} = \boldsymbol{N} - \boldsymbol{k}\otimes\boldsymbol{k}$. It holds apparently for $\underline{1/\tilde{r} = \sin\psi/r^*}$, $1/r_0 = 1/r_1 = -\rho^*\sin\psi/2\bar{r}$, $1/\hat{r}_0 = 1/\hat{r}_1 = \tan\beta/r_1$, $\tan\beta = \sqrt{4\bar{r}/\rho^*r^* - 1}$, and with $1/r_2 = \sin\psi/r_2^*$, $1/r_3 = \sin\psi/r_3^*$. For the proof we begin with the unit normal $\boldsymbol{n}' = -[\boldsymbol{k}\cos\alpha + (\boldsymbol{t}\cos\gamma + \boldsymbol{u}\sin\gamma)\sin\alpha]\sin\psi + \boldsymbol{n}\cos\psi$, but to simplify, we consider only rotational symmetry $\gamma \equiv 0$, since the general case leads in a similar manner to the same result. Therefore, the derivative of the unit normal becomes simply $\nabla_\mathrm{n}\otimes\boldsymbol{n}' = -\nabla_\mathrm{n}\psi\otimes\boldsymbol{k}' - [\nabla_\mathrm{n}\alpha\otimes\boldsymbol{t}^* + (\nabla_\mathrm{n}\otimes\boldsymbol{k})\cos\alpha + (\nabla_\mathrm{n}\otimes\boldsymbol{t})\sin\alpha]\sin\psi$. In addition, we have $\nabla_\mathrm{n}\otimes\boldsymbol{k} = \nabla_\mathrm{k}\otimes\boldsymbol{k} + \boldsymbol{k}\otimes\boldsymbol{k}_{,s} = \bar{\boldsymbol{B}}_\mathrm{k} + \boldsymbol{k}\otimes\boldsymbol{t}/\bar{\rho}_2$, where $1/\bar{\rho}_2$ is the curvature of the line normal to the surfaces $U = \text{const}$. The other derivative is $\nabla_\mathrm{n}\otimes\boldsymbol{t} = (\nabla_\mathrm{k}\otimes\boldsymbol{t})\boldsymbol{u}\otimes\boldsymbol{u} - \bar{\boldsymbol{B}}_\mathrm{k}\,\boldsymbol{t}\otimes\boldsymbol{k} - \boldsymbol{k}\otimes\boldsymbol{k}/\bar{\rho}_2$, with $(\nabla_\mathrm{k}\otimes\boldsymbol{t})\boldsymbol{u} = \boldsymbol{u}/\bar{r}_3\tan\phi$. The factor $1/\bar{r}_3\tan\phi$ marks the geodesic curvature of the parallel circle and ϕ denotes the angle of declination. In summary, we get with $\nabla_\mathrm{n}\psi = -\boldsymbol{k}^*\rho^*\tan\psi/2\bar{r}$ as follows: $\nabla_\mathrm{n}\otimes\boldsymbol{n}' = \boldsymbol{k}^*\otimes\boldsymbol{k}'\rho^*\tan\psi/2\bar{r} - [(\nabla_\mathrm{n}\alpha + \boldsymbol{k}/\bar{\rho}_2 + \boldsymbol{t}/\bar{r}_2)\otimes\boldsymbol{t}^*] + \boldsymbol{u}\otimes\boldsymbol{u}(\cos\alpha + \sin\alpha\cot\phi)/\bar{r}_3]\sin\psi$. The condition of a vanishing mixed term in the square bracket is $\boldsymbol{k}^*\cdot\nabla_\mathrm{n}\alpha = -(\sin\alpha/\bar{r}_2 + \cos\alpha/\bar{\rho}_2)$, so that $1/r_2^* = \cos\alpha/\bar{r}_2 - \sin\alpha/\bar{\rho}_2 + \boldsymbol{t}^*\cdot\nabla_\mathrm{n}\alpha$, $1/r_3^* = (\cos\alpha + \sin\alpha\cot\phi)/\bar{r}_3$ are the explicit expressions of principal curvature. To determine the angle α, we deduce from $\boldsymbol{t}^*\cdot\nabla_\mathrm{n}\psi = 0$ the relation $\boldsymbol{t}^*\cdot\nabla_\mathrm{n}(\mu U_{,s}) = 0$, which gives with $\boldsymbol{t}^* = \boldsymbol{t}\cos\alpha - \boldsymbol{k}\sin\alpha$

$$\tan\alpha = -\frac{\bar{r}}{2\rho}\boldsymbol{t}\cdot\nabla_{\mathrm{n}}[\log(\mu U_{,s})]. \tag{33}$$

The values ρ and μ depend on α; thus an iteration process should be used for the solution of Eq. (33).

We obtain afterwards the 3D-curvature $\boldsymbol{B}' = -\boldsymbol{Q}_\psi \boldsymbol{V}^{*(-1)}(\nabla_{\mathrm{n}}\otimes\boldsymbol{n}')\boldsymbol{N}' = (\boldsymbol{k}'\otimes\boldsymbol{k}')/r_1 + \boldsymbol{B}^*_{\mathrm{k}}\sin\psi$, where $\boldsymbol{B}^*_{\mathrm{k}} = \nabla_{\mathrm{k}^*}\otimes\boldsymbol{k}^* = (\boldsymbol{t}^*\otimes\boldsymbol{t}^*)/r_2^* + (\boldsymbol{u}\otimes\boldsymbol{u})/r_3^*$. Further, the corresponding 4D-curvature is $\boldsymbol{B}'_1 = [(\boldsymbol{h}\otimes\boldsymbol{h})/r_0 + (\boldsymbol{k}'\otimes\boldsymbol{k}')/r_1 + \boldsymbol{B}^*_{\mathrm{k}}\sin\psi]/\sqrt{2}$, and similarly we have the 2D-curvature $\boldsymbol{B}'_2 = [(\boldsymbol{h}\otimes\boldsymbol{h})/\hat{r}_0 + (\boldsymbol{k}'\otimes\boldsymbol{k}')/\hat{r}_1]/\sqrt{2}$. We write then the following combinations

$$\begin{aligned}
&\boldsymbol{B}'_1\boldsymbol{B}'_1 - \boldsymbol{B}'_1(\boldsymbol{B}'_1\cdot\boldsymbol{N}')\\
&\qquad = -\left(\frac{1}{2r_0r_1}+\frac{1}{r_0\tilde{r}}\right)(\boldsymbol{h}\otimes\boldsymbol{h}) - \left(\frac{1}{2r_0r_1}+\frac{1}{r_0\tilde{r}}\right)(\boldsymbol{k}'\otimes\boldsymbol{k}') + \frac{\sin^2\psi}{2}\left(\frac{\rho^*}{\bar{r}}\boldsymbol{B}^*_{\mathrm{k}} - K^*\boldsymbol{K}^*\right),\\
&\boldsymbol{B}'_3\boldsymbol{B}'_3 - \boldsymbol{B}'_3(\boldsymbol{B}'_3\cdot\boldsymbol{N}')\\
&\qquad = -\left(\frac{1}{2r_0r_1}+\frac{1}{r_0\tilde{r}}\right)\boldsymbol{h}\otimes\boldsymbol{h} - (\ldots)\boldsymbol{k}'\otimes\boldsymbol{k}' + \frac{\sin^2\psi}{2}\left(-\frac{\rho^*}{\bar{r}}\boldsymbol{E}^*\boldsymbol{B}^*_{\mathrm{k}}\boldsymbol{E}^* - K^*\boldsymbol{K}^*\right), \qquad (34)\\
&\boldsymbol{B}'_2\,\boldsymbol{B}'_2 - \boldsymbol{B}'_2(\boldsymbol{B}'_2\cdot\boldsymbol{N}')\\
&\qquad = -\frac{1}{2\hat{r}_0\hat{r}_1}(\boldsymbol{h}\otimes\boldsymbol{h}) - \frac{1}{2\hat{r}_0\hat{r}_1}(\boldsymbol{k}'\otimes\boldsymbol{k}') = \boldsymbol{B}'_4\,\boldsymbol{B}'_4 - \boldsymbol{B}'_4(\boldsymbol{B}'_4\cdot\boldsymbol{N}').
\end{aligned}$$

The second line shows again a 2D-permutation tensor $\boldsymbol{E}^*$, respectively an involution $\boldsymbol{E}^*(\ldots)\boldsymbol{E}^*$. Adding up the three expressions (34), we get in fact with $\boldsymbol{B}^*_{\mathrm{k}} - \boldsymbol{E}^*\boldsymbol{B}^*_{\mathrm{k}}\boldsymbol{E}^* = 2\boldsymbol{K}^*/r^*$ and with $1/\bar{r}_0 = \eta\sin\psi/r^*$, $1/\bar{r}_1 = \rho^*\sin\psi/\eta\bar{r}$, $1/\hat{r}_0\hat{r}_1 + 1/r_0r_1 = 1/\bar{r}_0\bar{r}_1 = -2/r_0\tilde{r}$ (similar to the spherical case) the vanishing 4D-Ricci tensor (i sum from 1 to 4)

$$\boldsymbol{R}_{4\mathrm{D}} = \boldsymbol{B}'_i\boldsymbol{B}'_i - \boldsymbol{B}'_i(\boldsymbol{B}'_i\cdot\boldsymbol{N}') = \sin^2\psi\left(\frac{\rho^*}{\bar{r}r^*} - K^*\right)\boldsymbol{K}^* = 0. \tag{35}$$

In some way this procedure represents a semi-inverse method to find the nonspherical field.

Besides, as to the general gravitational lens, we recall the common differential equations of any geodesic curve in components and with Christoffel symbols: $\mathrm{d}^2\theta^\lambda(s') + \Gamma^\lambda_{\alpha\beta}\mathrm{d}\theta^\alpha\mathrm{d}\theta^\beta = 0$.

Alternatively, one can use the variational principle

$$\delta\int_{\mathrm{A}}^{\mathrm{B}}\mathrm{d}s' = \int_{\vartheta_{\mathrm{A}}}^{\vartheta_{\mathrm{B}}}|\boldsymbol{r}'_{,\vartheta}|\,\mathrm{d}\vartheta = \int_{\vartheta_{\mathrm{A}}}^{\vartheta_{\mathrm{B}}}F(\boldsymbol{r}'_{,\vartheta})\,\mathrm{d}\vartheta = 0$$

and the auxiliary conditions $G_j(\boldsymbol{r}') = 0$ $(j = 1,\ldots,n-k)$ for a “hypersurface” A^k. We have then first the tangential unit vector $\partial F/\partial\boldsymbol{r}'_{,\vartheta} = \boldsymbol{a}$ and afterwards a vector in the hyperplane normal to the curve $\mathrm{d}(\partial F/\partial \boldsymbol{r}'_{,\vartheta})/\mathrm{d}\vartheta = \boldsymbol{A}\boldsymbol{r}'_{,\vartheta\vartheta}/F$, where $\boldsymbol{A} = \boldsymbol{I} - \boldsymbol{a} \otimes \boldsymbol{a}$ is the corresponding projector. As $\mu_j \partial G_j / \partial \boldsymbol{r}' = \mu_j c_{ij} \boldsymbol{n}'_i = \nu_i \boldsymbol{n}'_i$, we obtain with this notation and the application of the projector $\boldsymbol{N}' = \boldsymbol{I} - \boldsymbol{n}'_i \otimes \boldsymbol{n}'_i$ Euler's equation in the form $\boldsymbol{N}'\boldsymbol{A}\boldsymbol{r}'_{,\vartheta\vartheta} = 0$, or $\boldsymbol{N}'\mathrm{d}^2\boldsymbol{r}'(s') = 0$. To obtain the image equation by the backwards deformation into the flat space, we may then apply Eqs. (9), which are also valid in higher dimensions. A combination with four parts, similar to what we performed with Eqs. (34), but which we do not outline here in detail, leads to the result

$$\boldsymbol{N}\mathrm{d}^2 r_{4\mathrm{D}}(\sigma') = \boldsymbol{k}^* \frac{\sin^2\psi}{2} \sqrt{K^*} (\mathrm{d}\sigma'^2 - 3\mathrm{d}\boldsymbol{r} \cdot \boldsymbol{K}^* \mathrm{d}\boldsymbol{r}). \tag{36}$$

In particular, we have for a 4D-null-geodesic or *light ray* $\mathrm{d}\sigma'^2 = -k\mathrm{d}\vartheta^2$, $k \to 0$ and thus $\boldsymbol{N}\mathrm{d}^2 r_{4\mathrm{D}}(\vartheta) = \boldsymbol{k}^* \sin^2\psi \sqrt{K^*}\,(-3\mathrm{d}\boldsymbol{r} \cdot \boldsymbol{K}^* \mathrm{d}\boldsymbol{r})/2$. This is the generalization of the known result $\boldsymbol{N}\mathrm{d}^2 \boldsymbol{r}_{4\mathrm{D}}(\vartheta) = \boldsymbol{k} \sin^2\psi\,(-3\mathrm{d}\boldsymbol{r} \cdot \overline{\boldsymbol{K}}\mathrm{d}\boldsymbol{r})/2r$ from the spherical-symmetric case.

6. ROTATING BODIES. A TENTATIVE APPROACH TOWARDS THE KERR-SOLUTION

Using the previous frame for nonspherical fields and a certain *tentative composition*, we can outline a result, which is similar to the Kerr-solution for a rotating star. To simplify, we assume the body to have a spherical form, but the surrounding field is nevertheless nonspherical because of the rotation. We recall first of all some additional elements about classical gravity and inertial forces in the flat space. The Schwarzschild radius $2M = 2G\overline{M}/c^2$ contains the constant $G = 6.668 \times 10^{-11}\ \mathrm{kg}^{-1}\ \mathrm{m}^3\ \mathrm{s}^{-2}$ and the mass $\overline{M}$. If m denotes provisionally the mass of another small body lying in the system which rotates together with the central body, we can write for the gravitational potential $\overline{U} = -G\overline{M}m/r$, $U = \overline{U}/mc^2 = -M/r$ and similar for the scalar of the centrifugal force $\overline{V} = -m\Omega^2 r_0^2/2$, $V = \overline{V}/mc^2 = -(\Omega^2/2c^2)(\boldsymbol{r} \cdot \boldsymbol{K}_0 \boldsymbol{r})$. Here Ω is the angular velocity, r_0 is the distance from the axis of rotation, and $\boldsymbol{K}_0 = \boldsymbol{N} - \boldsymbol{k}_0 \otimes \boldsymbol{k}_0$ is the normal projector onto the equatorial plane. We pass now to the gradient of the *sum* $U + V$:

$$\nabla_{\mathrm{n}}(U + V) = M\left(\frac{1}{r^3}\boldsymbol{r} - \frac{\Omega^2}{Mc^2}\boldsymbol{K}_0\boldsymbol{r}\right) = \frac{M}{r^3}(\boldsymbol{N} - p^{-3}r^3\boldsymbol{K}_0)\boldsymbol{r} = \frac{M}{r^3}(\boldsymbol{N} - \chi\boldsymbol{K}_0)\boldsymbol{r}. \tag{37}$$

Here $p^{-3} = \Omega^2/Mc^2$ defines a parameter p of dimension "length" and $\chi = p^{-3}r^3 = \Omega^2 r^3/G\overline{M}$ is another dimensionless parameter for the third power of r. In some way χ describes in the equatorial plane the ratio between centrifugal and gravitational force at the distance $r = r_0$ from the centre. The unit normal of any surface $U+V=\text{const}$ is $\boldsymbol{k}=(\boldsymbol{N}-\chi\boldsymbol{K}_0)\boldsymbol{r}/W$ so that the 2D-curvature tensor reads

$$\overline{\boldsymbol{B}}_{\mathrm{k}}=\nabla_{\mathrm{k}}\otimes\boldsymbol{k}=\frac{1}{W}[\overline{\boldsymbol{K}}-\chi\overline{\boldsymbol{K}}\boldsymbol{K}_0-\nabla_{\mathrm{k}}\chi\otimes\boldsymbol{K}_0\boldsymbol{r}-\nabla_{\mathrm{k}}(\log W)\otimes(\boldsymbol{r}-\chi\boldsymbol{K}_0\boldsymbol{r})], \quad (38)$$

with the abbreviations

$$W=\sqrt{\boldsymbol{r}\cdot(\boldsymbol{k}_0\otimes\boldsymbol{k}_0)\boldsymbol{r}+(1-\chi)^2(\boldsymbol{r}\cdot\boldsymbol{K}_0\boldsymbol{r})},\ \nabla_{\mathrm{k}}\chi=\ 3p^{-3}r^2\nabla_{\mathrm{k}}r=3\chi r^{-2}\overline{\boldsymbol{K}}\boldsymbol{r}$$

and

$$\nabla_{\mathrm{k}}(\log W)=[\overline{\boldsymbol{K}}(\boldsymbol{k}_0\otimes\boldsymbol{k}_0)\boldsymbol{r}-3(1-\chi)(\chi/r^2)(\boldsymbol{r}\cdot\boldsymbol{K}_0\boldsymbol{r})\overline{\boldsymbol{K}}\boldsymbol{r}+(1-\chi)^2\,\overline{\boldsymbol{K}}\boldsymbol{K}_0\boldsymbol{r}\,]/W^2\,.$$

Note that at a general point $\boldsymbol{k}\neq\boldsymbol{r}/r$, so that one must apply the iteration with an equation of type (33) to find α. However, at points of symmetry we have $\boldsymbol{k}=\boldsymbol{r}/r$, $\nabla_{\mathrm{k}}\chi=0$, which simplifies the problem. On the axis of rotation, e.g., we get $\boldsymbol{k}=\boldsymbol{k}_0$, $W=r$, $\overline{\boldsymbol{B}}_{\mathrm{k}}=(1-\chi)\boldsymbol{K}_0/r$, $\rho=1$, but $\cos^2\psi<1-2M/r$, as can be verified. On the other hand, we obtain in the equatorial plane $\boldsymbol{k}\cdot\boldsymbol{k}_0=0$, $W=(1-\chi)r$, $\overline{\boldsymbol{B}}_{\mathrm{k}}=(\overline{\boldsymbol{K}}-\chi\overline{\boldsymbol{K}}\boldsymbol{K}_0)/(1-\chi)r$. The different curvatures and invariants are there: $1/\bar{r}_2=1/(1-\chi)r$, $1/\bar{r}_3=1/r$, $1/\bar{r}=(2-\chi)/2(1-\chi)r$, $\overline{K}=1/(1-\chi)r^2$, so that $\rho=\bar{r}^2\overline{K}=4(1-\chi)/(2-\chi)^2$. Further, as $(U+V)_{,s}=\boldsymbol{k}\cdot\nabla_{\mathrm{n}}(U+V)=WM/r^3$, we get for the inclination ψ: $\sin\psi=\sqrt{2}[M\mu(U_{,r}+V_{,r})]^{1/4}=\sqrt{2M}[\mu(r^{-2}-p^{-3}r)]^{1/4}=\sqrt{2M/r}[\mu(1-\chi)]^{1/4}$. The key relation Eq. (31$_1$) is $(\sin\psi)_{,r}=-\rho\sin\psi/2\bar{r}$, but $\rho=(2+\chi)/(2-\chi)-(\log\mu)_{,r}\bar{r}/2$ differs from Eq. (31$_2$). Thus, we find for the factor μ with $\chi_{,r}=3p^{-3}r^2=3\chi/r$ the solution

$$\frac{\mathrm{d}(\log\mu)}{\mathrm{d}r}=\frac{(4-\chi)\chi}{(2-\chi)(1-\chi)r},\qquad \mu(r)=\frac{C(1-\chi/2)^{2/3}}{1-\chi}. \quad (39)$$

If we take into account the limit $\chi=0$, $\sin^2\psi=2M/r$, $\mu=1$, we have by necessity $C=1$. As $\sin^4\psi=(2M/r)^2\,\mu(1-\chi)=(2M/r)^2\,(1-\chi/2)^{2/3}$, we get further for the other limit $\chi\to\infty$, $\sin^3\psi\to 2M\Omega/c$. This means that very far away from the centre the scalar V of the centrifugal force alone does of course not give an interior curvature of space ($\rho=0$, i.e., the constant inclination determines a developable conical surface). In summary, we obtain for the fundamental form in the rotating system and in the equatorial plane

$$\mathrm{d}\sigma'^2=-(\cos^2\psi)c^2\mathrm{d}t^2+r^2\mathrm{d}\varphi^2+(\cos^2\psi)^{-1}\mathrm{d}r^2+r^2\mathrm{d}\theta^2, \quad (40)$$

where the coefficient of “inclination” and the parameter χ are

$$\cos^2\psi=1-\frac{2M}{r}|1-\chi/2|^{1/3},\quad \chi=\frac{\Omega^2r^3}{Mc^2}=\frac{\Omega^2r^3}{G\overline{M}}. \quad (41)$$

In case of a small angular velocity Ω the approximation $\cos^2\psi \cong 1-2M/r+\Omega^2 r^2/3c^2$ holds.

If we pass now from the rotating system to the “fixed” system, we must apply the special Lorentz-transformation for the circumferential motion

$$c\mathrm{d}t=\frac{c\mathrm{d}\bar{t}-(\Omega r/c)\,r\mathrm{d}\bar{\varphi}}{\sqrt{1-(\Omega r/c)^2}},\quad r\mathrm{d}\varphi=\frac{r\mathrm{d}\bar{\varphi}-\Omega r\mathrm{d}\bar{t}}{\sqrt{1-(\Omega r/c)^2}}, \tag{42}$$

so that the fundamental form in the “fixed” system (variables with a bar on top) becomes finally

$$\mathrm{d}\sigma'^2=-(\cos^2\psi)\frac{[c\mathrm{d}\bar{t}-(\Omega r/c)\,r\mathrm{d}\bar{\varphi}]^2}{1-(\Omega r/c)^2}+\frac{(r\mathrm{d}\bar{\varphi}-\Omega r\mathrm{d}\bar{t})^2}{1-(\Omega r/c)^2}+\frac{\mathrm{d}r^2}{\cos^2\psi}+r^2\mathrm{d}\theta^2. \tag{43}$$

This last form may now be compared to the Kerr-solution [19,20], which was found by another way and which reads for $\theta=\pi/2$ according to the notation in [18], p. 305, Eq. (10.58):

$$\mathrm{d}\bar{\sigma}'^2=-\frac{\Delta}{r^2}[c\mathrm{d}\bar{t}-a\mathrm{d}\bar{\varphi}]^2+\frac{1}{r^2}[(r^2+a^2)\mathrm{d}\bar{\varphi}-ac\mathrm{d}\bar{t}]^2+\frac{r^2}{\Delta}\mathrm{d}r^2+r^2\mathrm{d}\theta^2, \tag{44}$$

where $\Delta=r^2-2Mr+a^2$ and a contains Ω. Here it should be noted that the angular velocity intervenes in Eq. (43) in different places for two different reasons: (a) in the factor $\cos^2\psi$ because of Eqs. (41), (b) in the term $(\Omega r/c)^2$ because of Eqs. (42). If the central body has a nonspherical, e.g., ellipsoidal form in case of a rapid rotation, the calculation of $\cos^2\psi$ would be similar, but more complicated.

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Kõverpindade suurte deformatsioonide analüüs holograafilise interferomeetriaga ning mõned märkused mittesfääriliste gravitatsiooniväljade ja pöörlevate kehade kohta

Walter Schumann

On vaadeldud suurte deformatsioonide mõõtmist holograafilise interferomeetriaga. Seni nähtamatute ribade esiletoomiseks on vajalik rekonstrueerimisprotsessi modifitseerida. Ribade paigutust ja kontrastsust iseloomustatakse riba- ja nähtavusvektoritega. Käiguvahe esimene tuletis on seotud deformatsioonigradiendi ja afiinsete seoste polaarse dekompositsiooniga. Modifitseerimisel tuleb kujutise aberratsiooni vaadelda koos geodeetiliste kõveruste ja pinna kõveruste muutustega. See viib samasugustele probleemidele hüperpindade puhul ja ühele Schwarzschildi lahendi interpretatsioonile virtuaalsete deformatsioonide kohta. Mõned järeldused puudutavad ka mittesfäärilisi gravitatsioonivälju ja katset läheneda Kerri lahendile pöörlevate kehade juhtumil.