

COERCIVITY INEQUALITY FOR THE QUASILINEAR FINITE DIFFERENCE OPERATOR

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Abstract. The coercivity inequality is established for the finite difference operator which approximates in two- or three-dimensional unit cube the quasilinear elliptic operator for the second boundary value problem. The obtained results are based on special discrete analogues of the Sobolev imbedding theorems.

Key words: finite difference method.

1. INTRODUCTION: NOTATIONS AND AUXILIARY RESULTS

The coercivity inequalities have been used in the study of convergence of the finite difference method in strong Sobolev norms (see [1,2]). In the present paper we establish a coercivity inequality for the finite difference operator that approximates the m -dimensional ($m = 2, 3$) Neumann boundary value problem for the quasilinear elliptic operator of the 2nd order on the unit cube. In case $m = 3$ we assume that the differential operator contains no mixed derivatives. Note that in the case of the Dirichlet boundary value problem ($m = 2, 3$) the coercivity inequality holds without this restriction (see [1]).

Let

$$\Omega = \{0 < x_i < 1, i = 1, \dots, m\}$$

be the m -dimensional unit cube with the boundary $\partial\Omega$ and closure $\overline{\Omega}$. Introduce the grid

$$\overline{\Omega}_h = \{\xi = (k_1h, \dots, k_mh), \quad k_i = 0, 1, \dots, n; \quad i = 1, \dots, m\}, \quad h = \frac{1}{n},$$

and denote

$$\Omega_h = \bar{\Omega}_h \cap \Omega, \quad \partial\Omega_h = \bar{\Omega}_h \cap \partial\Omega.$$

Given a grid functions $y: \Omega_h \rightarrow R$, we prolong it to the boundary $\partial\Omega_h$ using the values of y at the nearest grid points of Ω_h , e.g.

$$y(k_1h, \dots, k_{l-1}h, 0, k_{l+1}h, \dots, k_mh) = y(k_1h, \dots, k_{l-1}h, h, k_{l+1}h, \dots, k_mh),$$

$$y(k_1h, \dots, k_{l-1}h, 1, k_{l+1}h, \dots, k_mh) = y(k_1h, \dots, k_{l-1}h, (n-1)h, k_{l+1}h, \dots, k_mh);$$

$$k_i = 1, \dots, n-1, \quad i = 1, \dots, m.$$

Into the remaining grid points of $\partial\Omega_h$ we prolong the grid function y also with the value at the nearest point of the set Ω_h . For the prolonged grid functions we define the discrete Laplace operator by the formula

$$-\Delta_h y(x) = -\sum_{i=1}^m \partial_i \bar{\partial}_i y(x), \quad x \in \Omega_h,$$

where

$$\begin{aligned} \partial_i y &= \frac{1}{h}(y^{+1i} - y), \quad \bar{\partial}_i y = \frac{1}{h}(y - y^{-1i}), \\ y^{+1i} &= y(x + he_i), \quad y^{-1i} = y(x - he_i), \quad e_i = (\delta_{i1}, \dots, \delta_{im}). \end{aligned}$$

For the prolonged $y, v: \Omega_h \rightarrow R$, we use the following notations:

$$\|y\|_{L_p(\Omega_h)} = \left(h^m \sum_{\xi \in \Omega_h} |y(\xi)|^p \right)^{1/p}, \quad 1 \leq p < \infty$$

(in the case $p = 2$ we also use the notation $|y|_0 = \|y\|_0 = \|y\|_{L_2(\Omega_h)}$),

$$\|y\|_{C(\Omega_h)} = \|y\|_{L_\infty(\Omega_h)} = \max_{\xi \in \Omega_h} |y(\xi)|,$$

$$(y, v) = h^m \sum_{\xi \in \Omega_h} y(\xi)v(\xi),$$

$$|y|_1 = \left(\sum_{i=1}^m \|\partial_i y\|_0^2 \right)^{1/2},$$

$$|y|_2 = \left(\sum_{i,j=1}^m \|\partial_i \bar{\partial}_j y\|_0^2 \right)^{1/2},$$

$$\|y\|_k = \left(\sum_{s=0}^k |y|_s^2 \right)^{1/2}, \quad k = 1, 2.$$

Further, by $H^k(\Omega_h)$ ($k = 1, 2$) we denote the discrete Sobolev space supplied with the norm $\|y\|_k$.

Using the formulas of summation by parts, we get

$$\begin{aligned}\|y\|_1^2 &= \|y\|_0^2 + (-\Delta_h y, y), \\ \|y\|_2^2 &= \|y\|_1^2 + (-\Delta_h y, -\Delta_h y).\end{aligned}$$

Consequently,

$$\|y\|_1^2 = ((-\Delta_h + I_h)y, y)$$

and

$$\frac{1}{2}((-\Delta_h + I_h)y, (-\Delta_h + I_h)y) \leq \|y\|_2^2 \leq ((-\Delta_h + I_h)y, (-\Delta_h + I_h)y), \quad (1)$$

where I_h is the identity operator. This means that the Sobolev norms $\|y\|_1$ and $\|y\|_2$ are equivalent to the norms $\|(-\Delta_h + I_h)^{1/2}y\|_0$ and $\|(-\Delta_h + I_h)y\|_0$, respectively.

Now with the help of the operator

$$\Phi_h = -\Delta_h + I_h$$

we define the interpolation space $H^{2\beta}(\Omega_h)$ ($\frac{1}{2} < \beta < 1$) with the norm

$$\|y\|_{2\beta} = \|\Phi_h^\beta y\|_0.$$

For the interpolation spaces $H^{2\beta}(\Omega_h)$ the following theorem [3] holds.

Theorem 1. *For the grid functions $y: \Omega_h \rightarrow R$, prolonged to the boundary with the value from the nearest inner grid point, the following inequalities hold:*

$$\|y\|_{C(\Omega_h)} \leq c\|y\|_{2\beta}, \quad \beta > \frac{m}{4}, \quad m = 2, 3,$$

$$\|\hat{\partial}_i y\|_{L_q(\Omega_h)} \leq c\|y\|_{2\beta}, \quad \beta \geq \frac{(q-2)m + 2q}{4q}, \quad m < \frac{2q}{q-2},$$

where $\hat{\partial}_i$ denotes either ∂_i , $\bar{\partial}_i$, or $\tilde{\partial}_i = \frac{1}{2}(\partial_i + \bar{\partial}_i)$.

Also the following result (see [1]) is valid.

Corollary 1. *For any $\varepsilon > 0$, we have*

$$\|y\|_{2\beta} \leq 2^{\beta/2}(\varepsilon^{1-\beta}\|y\|_2 + \varepsilon^{-\beta}\|y\|_0). \quad (2)$$

2. COERCIVITY INEQUALITY

Let $\partial u / \partial \nu$ denote the outer normal derivative of the boundary $\partial \Omega$. Consider the quasilinear elliptic operator of the Neumann boundary value problem defined by

$$Au = - \sum_{i,j=1}^m \frac{\partial}{\partial x_i} \left(a_{ij}(x, u) \frac{\partial u}{\partial x_j} \right) + b(x, u)u, \quad \frac{\partial u}{\partial \nu} \Big|_{\partial \Omega} = 0, \quad m = 2, 3,$$

$$a_{ij}(x, u) = a_{ji}(x, u),$$

and its discrete analogue in the following form:

$$A_h y = - \frac{1}{2} \sum_{i,j=1}^m [\partial_i (a_{ij}(x, y) \bar{\partial}_j y) + \bar{\partial}_i (a_{ij}(x, y) \partial_j y)] + b(x, y)y, \\ y \in H^2(\Omega_h).$$

In what follows we assume that the following conditions hold:

(I) For any $a > 0$ there exists a number $\kappa_a > 0$ such that, for all $\xi_i \in R$, $x \in \bar{\Omega}$, $u \in [-a, a]$, there hold

$$\sum_{i,j=1}^m a_{ij}(x, u) \xi_i \xi_j \geq \kappa_a \sum_{i=1}^m \xi_i^2, \quad b(x, u) \geq \kappa_a.$$

(II) The functions $a_{ij}(x, u)$, $b(x, u)$ are continuously differentiable, and for $x \in \bar{\Omega}$, $u \in [-a, a]$, there hold

$$\left| \frac{\partial^{\mu_1 + \mu_2} a_{ij}(x, u)}{\partial x_l^{\mu_1} \partial u^{\mu_2}} \right| \leq d_a, \quad \left| \frac{\partial^{\mu_3 + \mu_4} b(x, u)}{\partial x_l^{\mu_3} \partial u^{\mu_4}} \right| \leq d_a,$$

where $0 \leq \mu_1, \mu_2 \leq 3$, $\mu_1 + \mu_2 \leq 3$, $0 \leq \mu_3, \mu_4 \leq 2$, $\mu_3 + \mu_4 \leq 2$, $l = 1, \dots, m$. Here d_a is a positive constant dependent on a .

Theorem 2. *Let conditions (I), (II) be fulfilled. In case $m = 3$ assume also that $a_{ij} = 0$ for $j \neq i$. Then, for all $y, v \in H^2(\Omega_h)$ such that $\max(\|y\|_{C(\Omega_h)}, \|v\|_{C(\Omega_h)}, \|y\|_2, \|v\|_2) \leq a$, there holds*

$$(A_h y - A_h v, \Phi_h(y - v)) \geq \frac{\kappa_a}{2} \|y - v\|_2^2 - c_a \|y - v\|_1 \|y - v\|_2, \quad (3)$$

and together with it the following coercivity inequality is valid:

$$\|A_h y - A_h v\|_0 \geq \frac{\kappa_a}{2\sqrt{2}} \|y - v\|_2 - \frac{c_a}{\sqrt{2}} \|y - v\|_1. \quad (4)$$

Here

$$c_a = d_a(5m^2\sqrt{2} + c(a + a^2)) + \left(\frac{\kappa_a}{2}\right)^{\beta/(\beta-1)} [d_a(a + a^2)]^{1/(1-\beta)},$$

$\beta = \frac{3}{4}$ for $m = 2$, $\beta = \frac{7}{8}$ for $m = 3$, c is a positive constant.

Proof. It follows from condition (II) that the operator $A_h: H^2(\Omega_h) \rightarrow L_2(\Omega_h)$ possesses the Frechet derivative:

$$\begin{aligned} A'_h(y)w &= \left\{ -\frac{1}{2} \sum_{i,j=1}^m [\partial_i(a_{ij}(x, y)\bar{\partial}_j w) + \bar{\partial}_i(a_{ij}(x, y)\partial_j w)] + b(x, y)w \right\} \\ &+ \left\{ -\frac{1}{2} \sum_{i,j=1}^m \left[\partial_i \left(\frac{\partial a_{ij}(x, y)}{\partial u} \bar{\partial}_j y \cdot w \right) + \bar{\partial}_i \left(\frac{\partial a_{ij}(x, y)}{\partial u} \partial_j y \cdot w \right) \right] \right. \\ &\left. + \frac{\partial b(x, y)}{\partial u} y w \right\} \\ &= A_{h1}(y)w + A_{h2}(y)w, \quad y, w \in H^2(\Omega_h). \end{aligned}$$

Since

$$A_h y - A_h v = \int_0^1 A'_h(\theta y + (1 - \theta)v)(y - v) d\theta, \quad (5)$$

we have

$$\begin{aligned} (A_h y - A_h v, \Phi_h z) &= \int_0^1 (A'_h(\theta y + (1 - \theta)v)(y - v), \Phi_h z) d\theta \\ &= \int_0^1 (A_{h1}(\eta)z, \Phi_h z) d\theta + \int_0^1 (A_{h2}(\eta)z, \Phi_h z) d\theta, \\ z &= y - v, \quad \eta = \theta y + (1 - \theta)v. \end{aligned}$$

First we consider the scalar product

$$(A_{h1}(\eta)z, \Phi_h z) = (A_{h1}(\eta)z, -\Delta_h z) + (A_{h1}(\eta)z, z),$$

where

$$A_{h1}(\eta)z = -\frac{1}{2} \sum_{i,j=1}^m [\partial_i(a_{ij}(x, \eta)\bar{\partial}_j z) + \bar{\partial}_i(a_{ij}(x, \eta)\partial_j z)] + b(x, \eta)z. \quad (6)$$

Using the Leibniz formula for the difference of the product, we get

$$\begin{aligned}
A_{h1}(\eta)z &= -\frac{1}{2} \sum_{i,j=1}^m a_{ij}(x, \eta)(\partial_i \bar{\partial}_j z + \bar{\partial}_i \partial_j z) \\
&\quad - \frac{1}{2} \sum_{i,j=1}^m [\partial_i(a_{ij}(x, \eta))(\bar{\partial}_j z)^{+1_i} + \bar{\partial}_i(a_{ij}(x, \eta))(\partial_j z)^{-1_i}] \\
&\quad + b(x, \eta)z.
\end{aligned}$$

Thus,

$$\begin{aligned}
&(A_{h1}(\eta)z, -\Delta_h z) \\
&= \frac{1}{2} \left(\sum_{i,j=1}^m a_{ij}(x, \eta)(\partial_i \bar{\partial}_j z + \bar{\partial}_i \partial_j z), \sum_{k=1}^m \partial_k \bar{\partial}_k z \right) \\
&\quad + \frac{1}{2} \left(\sum_{i,j=1}^m [\partial_i(a_{ij}(x, \eta))(\bar{\partial}_j z)^{+1_i} + \bar{\partial}_i(a_{ij}(x, \eta))(\partial_j z)^{-1_i}], \sum_{k=1}^m \partial_k \bar{\partial}_k z \right) \\
&\quad + \left(b(x, \eta)z, -\sum_{k=1}^m \partial_k \bar{\partial}_k z \right) \\
&= S^{(1)} + S^{(2)} + S^{(3)}.
\end{aligned}$$

The addend $S^{(1)}$ occupies the central place in the estimation of the scalar product $(A_{h1}(\eta)z, -\Delta_h z)$:

$$\begin{aligned}
S^{(1)} &= \frac{1}{2} \left(\sum_{i,j=1}^m a_{ij}(x, \eta)(\partial_i \bar{\partial}_j z + \bar{\partial}_i \partial_j z), \sum_{k=1}^m \partial_k \bar{\partial}_k z \right) \\
&= \frac{1}{2} \sum_{\substack{i,j,k=1 \\ k=i \text{ or } k=j}}^m (a_{ij}(x, \eta)(\partial_i \bar{\partial}_j z + \bar{\partial}_i \partial_j z), \partial_k \bar{\partial}_k z) \\
&\quad + \frac{1}{2} \sum_{\substack{i,k=1 \\ k \neq i}}^m (a_{ii}(x, \eta)(\partial_i \bar{\partial}_i z + \bar{\partial}_i \partial_i z), \partial_k \bar{\partial}_k z) = S_1^{(1)} + S_2^{(1)},
\end{aligned}$$

because for $m = 3$, $a_{ij} = 0$ if $j \neq i$. Using the formulas of summation by parts (cf. [4]) and the way of prolongations to the boundary $\partial \Omega_h$, we get

$$S_2^{(1)} = \frac{1}{2} \sum_{\substack{i,k=1 \\ k \neq i}}^m [(a_{ii}(x, \eta) \partial_i \bar{\partial}_k z, \partial_i \bar{\partial}_k z) + (a_{ii}(x, \eta) \bar{\partial}_i \partial_k z, \bar{\partial}_i \partial_k z)] + S_{22}^{(1)}, \quad (7)$$

where

$$S_{22}^{(1)} = \frac{1}{2} \sum_{\substack{i,k=1 \\ k \neq i}}^m [-(\bar{\partial}_k(a_{ii}(x, \eta)))(\partial_i \bar{\partial}_i z)^{-1k}, \bar{\partial}_k z) + (\partial_i(a_{ii}(x, \eta)))(\bar{\partial}_k z)^{+1i}, \partial_i \bar{\partial}_k z) \\ - (\partial_k(a_{ii}(x, \eta)))(\bar{\partial}_i \partial_i z)^{+1k}, \partial_k z) + (\bar{\partial}_i(a_{ii}(x, \eta)))(\partial_k z)^{-1i}, \bar{\partial}_i \partial_k z] .$$

Thus,

$$S^{(1)} = S_1^{(1)} + S_2^{(1)} = S_0^{(1)} + S_{22}^{(1)} ,$$

where

$$S_0^{(1)} = \frac{1}{2} \sum_{k=1}^m \sum_{i,j=1}^m [(a_{ij}(x, \eta) \partial_i \bar{\partial}_k z, \partial_j \bar{\partial}_k z) + (a_{ij}(x, \eta) \bar{\partial}_i \partial_k z, \bar{\partial}_j \partial_k z)]$$

($a_{ij} = 0, j \neq i$ for $m = 3$). Summation by parts yields

$$S^{(3)} = \left(b(x, \eta) z, - \sum_{k=1}^m \partial_k \bar{\partial}_k z \right) \\ = \sum_{k=1}^m (b(x, \eta) \partial_k z, \partial_k z) + \sum_{k=1}^m (\partial_k (b(x, \eta)) z^{+1k}, \partial_k z) = S_1^{(3)} + S_2^{(3)} .$$

From (I) it follows that

$$S_1^{(3)} \geq \kappa_a |z|_1^2$$

and

$$S_0^{(1)} \geq \kappa_a \sum_{i,k=1}^m (\partial_i \bar{\partial}_k z, \partial_i \bar{\partial}_k z) = \kappa_a |z|_2^2 .$$

Using condition (II) and the mean-value theorem, we get

$$|S_2^{(3)}| \leq d_a (\sqrt{2m} |z|_0 |z|_1 + \|z\|_{C(\Omega_h)} |\eta|_1 |z|_1) .$$

Analogously

$$|S_{22}^{(1)}| \leq 2d_a \sqrt{2m} (|z|_1 |z|_2 + |\eta|_{1,4} |z|_{1,4} |z|_2) ,$$

$$|S^{(2)}| \leq d_a m^2 \sqrt{2} (|z|_1 |z|_2 + |\eta|_{1,4} |z|_{1,4} |z|_2) ,$$

where

$$|y|_{1,4} = \left(\sum_{i=1}^m \|\partial_i y\|_{L_4(\Omega_h)}^4 \right)^{1/4} .$$

Summing up, we obtain

$$\begin{aligned} & (A_{h1}(\eta)z, -\Delta_h z) \\ & \geq \kappa_a |z|_2^2 + \kappa_a |z|_1^2 - \sqrt{2}(2\sqrt{m} + m^2)d_a(|z|_1 + |\eta|_{1,4}|z|_{1,4})|z|_2 \\ & \quad - d_a(\sqrt{2m}|z|_0 + \|z\|_{C(\Omega_h)}|\eta|_1)|z|_1. \end{aligned}$$

Similarly, it follows from conditions (I), (II) that

$$(A_{h1}(\eta)z, z) \geq \kappa_a |z|_0^2 - d_a m(|z|_2|z|_0 + \sqrt{2m}|z|_1|z|_0 + \sqrt{2}\|z\|_{C(\Omega_h)}|\eta|_1|z|_1).$$

Thus,

$$\begin{aligned} (A_{h1}(\eta)z, \Phi_h z) & \geq \kappa_a \|z\|_2^2 - d_a m^2 \sqrt{2}(|z|_0|z|_2 + 2|z|_0|z|_1 + 2|z|_1|z|_2 \\ & \quad + 2|\eta|_{1,4}|z|_{1,4}|z|_2 + |\eta|_1\|z\|_{C(\Omega_h)}|z|_1). \end{aligned} \quad (8)$$

Now we estimate the scalar product $(A_{h2}(\eta)z, \Phi_h z)$. We have

$$\begin{aligned} A_{h2}(\eta)z & = -\frac{1}{2} \sum_{i,j=1}^m \left[\partial_i \left(\frac{\partial a_{ij}(x, \eta)}{\partial u} \bar{\partial}_j \eta \cdot z \right) + \bar{\partial}_i \left(\frac{\partial a_{ij}(x, \eta)}{\partial u} \partial_j \eta \cdot z \right) \right] \\ & \quad + \frac{\partial b(x, \eta)}{\partial u} \eta \cdot z. \end{aligned}$$

Using the Leibniz formula, we get

$$\begin{aligned} A_{h2}(\eta)z & = -\frac{1}{2} \sum_{i,j=1}^m \left\{ \frac{\partial a_{ij}(x, \eta)}{\partial u} [\partial_i(\bar{\partial}_j \eta \cdot z) + \bar{\partial}_i(\partial_j \eta \cdot z)] \right. \\ & \quad \left. + \partial_i \left(\frac{\partial a_{ij}(x, \eta)}{\partial u} \right) (\bar{\partial}_j \eta \cdot z)^{+1_i} + \bar{\partial}_i \left(\frac{\partial a_{ij}(x, \eta)}{\partial u} \right) (\partial_j \eta \cdot z)^{-1_i} \right\} \\ & \quad + \frac{\partial b(x, \eta)}{\partial u} \eta \cdot z. \end{aligned}$$

With the help of the mean-value theorem we obtain

$$\begin{aligned} & |(A_{h2}(\eta)z, \Phi_h z)| \\ & \leq d_a m^2 \sqrt{2} |z|_2 [|z|_{1,4} |\eta|_{1,4} + \|z\|_{C(\Omega_h)} (|\eta|_0 + |\eta|_1 + |\eta|_2 + |\eta|_{1,4}^2)] \\ & \quad + d_a m^2 \sqrt{2} \|z\|_{C(\Omega_h)} [|z|_1 |\eta|_1 + |z|_0 (|\eta|_0 + |\eta|_1 + |\eta|_2 + |\eta|_{1,4}^2)]. \end{aligned}$$

Thus,

$$\begin{aligned}
& (A'_h(\eta)z, \Phi_h z) \\
&= (A_{h1}(\eta)z + A_{h2}(\eta)z, \Phi_h z) \\
&\geq \kappa_a \|z\|_2^2 - d_a m^2 \sqrt{2} |z|_2 [3|z|_{1,4} |\eta|_{1,4} + \|z\|_{C(\Omega_h)} (|\eta|_0 + |\eta|_1 + |\eta|_2 + |\eta|_{1,4}^2)] \\
&\quad - d_a m^2 \sqrt{2} [|z|_0 |z|_2 + 2|z|_0 |z|_1 + 2|z|_1 |z|_2 + 2\|z\|_{C(\Omega_h)} |z|_1 |\eta|_1 \\
&\quad + \|z\|_{C(\Omega_h)} |z|_0 (|\eta|_0 + |\eta|_1 + |\eta|_2 + |\eta|_{1,4}^2)].
\end{aligned}$$

Further, using Theorem 1 and Corollary 1, we get

$$\begin{aligned}
& (A'_h(\eta)z, \Phi_h z) \\
&\geq \kappa_a \|z\|_2^2 - cd_a \|z\|_2 \|z\|_{2\beta} (\|\eta\|_2 + \|\eta\|_2^2) \\
&\quad - d_a m^2 \sqrt{2} [\|z\|_0 \|z\|_2 + 2\|z\|_0 \|z\|_1 + 2\|z\|_1 \|z\|_2 + c\|z\|_2 \|z\|_1 \|\eta\|_2 \\
&\quad + c\|z\|_2 \|z\|_0 (\|\eta\|_2 + \|\eta\|_2^2)],
\end{aligned}$$

where $c = \text{const} > 0$, $\beta = \frac{3}{4}$ for $m = 2$, $\beta = \frac{7}{8}$ for $m = 3$. Here we used the inequalities

$$\begin{aligned}
\|\hat{\partial}_i y\|_{L_4(\Omega_h)} &\leq c \|y\|_2, & i = 1, \dots, m; \quad m = 2, 3; \\
\|y\|_{C(\Omega_h)} &\leq c \|y\|_2, & m = 2, 3,
\end{aligned}$$

which follow from Theorem 1 and Corollary 1.

Using inequality (2) with $\|\eta\|_2 \leq a$, we get

$$\begin{aligned}
(A'_h(\eta)z, \Phi_h z) &\geq [\kappa_a - cd_a \varepsilon^{1-\beta} (a + a^2)] \|z\|_2^2 - d_a 5m^2 \sqrt{2} \|z\|_1 \|z\|_2 \\
&\quad - cd_a (a + a^2) (1 + \varepsilon^{-\beta}) \|z\|_1 \|z\|_2.
\end{aligned}$$

Choosing $\varepsilon > 0$ so that

$$cd_a \varepsilon^{1-\beta} (a + a^2) = \frac{\kappa_a}{2},$$

we obtain

$$(A'_h(\eta)z, \Phi_h z) \geq \frac{\kappa_a}{2} \|z\|_2^2 - c_a \|z\|_1 \|z\|_2,$$

where

$$c_a = d_a (5m^2 \sqrt{2} + c(a + a^2)) + \left(\frac{\kappa_a}{2}\right)^{\beta/(\beta-1)} [d_a (a + a^2)]^{1/(1-\beta)}.$$

From (5) it follows that

$$(A_h y - A_h v, \Phi_h(y - v)) \geq \frac{\kappa_a}{2} \|y - v\|_2^2 - c_a \|y - v\|_2 \|y - v\|_1,$$

i.e. (3) holds. Together with (1) this implies (4).

The next theorem shows that if A contains no mixed derivatives, then the corresponding operator A_{h1} (see (6)) is invertible.

Theorem 3. *In case of the difference operator*

$$\begin{aligned} \bar{A}_h y &= -\frac{1}{2} \sum_{i=1}^m [\partial_i(a_{ii}(x, y)\bar{\partial}_i y) + \bar{\partial}_i(a_{ii}(x, y)\partial_i y)] + b(x, y)y, \\ y &\in H^2(\Omega_h), \end{aligned}$$

under conditions (I), (II), the following inequality holds:

$$\|\bar{A}_{h1}(\eta)z\|_0 \geq \frac{\kappa_a^2}{2(\sqrt{2}\kappa_a + \tau_a)} \|z\|_2, \quad (9)$$

$$z, \eta \in H^2(\Omega_h), \quad \max(\|\eta\|_{C(\Omega_h)}, \|\eta\|_2) \leq a,$$

where (cf. (6))

$$\begin{aligned} \bar{A}_{h1}(\eta)z &= -\frac{1}{2} \sum_{i=1}^m [\partial_i(a_{ii}(x, \eta)\bar{\partial}_i z) + \bar{\partial}_i(a_{ii}(x, \eta)\partial_i z)] + b(x, \eta)z, \\ \tau_a &= d_a(5m^2\sqrt{2} + ca) + \left(\frac{\kappa_a}{2}\right)^{\beta/(\beta-1)} (cd_a a)^{1/(1-\beta)}, \end{aligned}$$

$\beta = \frac{3}{4}$ for $m = 2$, $\beta = \frac{7}{8}$ for $m = 3$, c is a positive constant.

Proof. From inequality (8) and Corollary 1 with $\|\eta\|_2 \leq a$ we get

$$\begin{aligned} &(\bar{A}_{h1}(\eta)z, \Phi_h z) \\ &\geq (\kappa_a - ca d_a \varepsilon^{1-\beta}) \|z\|_2^2 - d_a \|z\|_2 \|z\|_1 (5m^2\sqrt{2} + ca + ca\varepsilon^{-\beta}). \end{aligned}$$

Let us denote

$$\tau_a = d_a(5m^2\sqrt{2} + ca + ca\varepsilon^{-\beta}),$$

where we choose $\varepsilon > 0$ so that

$$ca d_a \varepsilon^{1-\beta} = \frac{\kappa_a}{2}.$$

Thus

$$(\bar{A}_{h1}(\eta)z, \Phi_h z) \geq \frac{\kappa_a}{2} \|z\|_2^2 - \tau_a \|z\|_2 \|z\|_1,$$

which together with (1) implies

$$\|\bar{A}_{h1}(\eta)z\|_0 \geq \frac{\kappa_a}{2\sqrt{2}}\|z\|_2 - \frac{\tau_a}{\sqrt{2}}\|z\|_1. \quad (10)$$

Using the formula of summation by parts and (I), we get

$$(\bar{A}_{h1}(\eta)z, z) \geq \kappa_a\|z\|_1^2.$$

Consequently,

$$\|\bar{A}_{h1}(\eta)z\|_0 \geq \kappa_a\|z\|_1,$$

or

$$\|z\|_1 \leq \frac{1}{\kappa_a}\|\bar{A}_{h1}(\eta)z\|_0.$$

Now it follows from (10) that

$$\begin{aligned} \frac{\kappa_a}{2\sqrt{2}}\|z\|_2 &\leq \|\bar{A}_{h1}(\eta)z\|_0 + \frac{\tau_a}{\sqrt{2}}\frac{1}{\kappa_a}\|\bar{A}_{h1}(\eta)z\|_0 \\ &= \frac{\sqrt{2}\kappa_a + \tau_a}{\sqrt{2}\kappa_a}\|\bar{A}_{h1}(\eta)z\|_0. \end{aligned}$$

Theorem 3 is proven.

It should be noted that in case of the Dirichlet boundary value problem the result like (9) is valid also for the finite difference operator (6) (see [1]) with mixed differences. In [5,6] one can find the coercivity inequalities for linear finite difference operators. The coercivity inequalities for the finite difference operator, which approximates a nonlinear monotone elliptic operator, can be found in [2,7].

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KOERTSITIIVSUSE VÕRRATUS KVAASILINEAARSE DIFERENTSOPERAATORI JAOKS

Malle FISCHER

On vaadeldud teist järku kvaasilineaarse diferentsiaaloperaatori Neumanni rajaülesande diskreetset analoogi kahe- ja kolmemõõtmelises ühikkuubis. Nime-
tatud operaatori jaoks on tõestatud koertsitiivsuse võrratus, kusjuures kolme-
dimensioonilisel juhul kehtib tulemus vaid siis, kui diferentsiaaloperaator ei sisalda
segaosatuletiisi.