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ON CONSERVATIVE AND COERCIVE SM-METHODS

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Abstract. We study the space $\mathcal{C}_e$ of double sequences (x_{kl}), satisfying $\lim_l \overline{\lim_k} |x_{kl} - a| = 0$ for some number a. In this note, using gliding hump arguments, we give necessary and sufficient conditions for a 3-dimensional matrix (i.e. SM-method) to transform every convergent or bounded sequence (x_k) into the space $\mathcal{C}_e$ or $\mathcal{C}_{be}$, the space of elements in $\mathcal{C}_e$ with bounded columns.

Key words: summability, SM-methods, gliding hump method, theorems of Toeplitz–Silverman type.

1. INTRODUCTION AND PRELIMINARIES

The best known and well-studied convergence notion for double sequence spaces is Pringsheim convergence. A double sequence (x_{kl}) of complex (or real) numbers is said to *converge to the limit a in the sense of Pringsheim* if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} : \; k, l > N \; \Rightarrow | \, x_{kl} - a \, | < \varepsilon.$$

In case of this convergence the row-index k and the column-index l tend independently to infinity.

Boos et al. [1] considered a more general notion of convergence, where, in contrast to Pringsheim's notion of convergence, the row-index k depends on the column-index l in tending to infinity. The space of all double sequences converging in this way is denoted by $\mathcal{C}_e$. More precisely,

$$\begin{aligned} \mathcal{C}_e := \Big\{ x \in \Omega \,|\, \exists a \in \mathbb{K} \; \forall \varepsilon > 0 \; \exists l_0 \in \mathbb{N} \; \forall l \geq l_0 \; \exists k_l \in \mathbb{N} : \\ k \geq k_l \; \Rightarrow \; |x_{kl} - a| \leq \varepsilon \Big\} \\ = \Big\{ x \in \Omega \,|\, \exists a \in \mathbb{K} : \; \lim_l \overline{\lim_k} |x_{kl} - a| = 0 \Big\}, \end{aligned}$$

where Ω denotes the linear space of all complex (or real) double sequences and $\mathbb{K}$ is the field of all complex (or real) numbers. In more detail the paper [1] deals with the subspace

$$\mathcal{C}_{be} := \left\{x \in \mathcal{C}_e |\ \forall l \in \mathbb{N}:\ (x_{kl})_k \in \mathfrak{m}\right\}$$

of $\mathcal{C}_e$, where $\mathfrak{m}$ is the space of all bounded sequences. Note that in [1] the notation $\widehat{\mathcal{C}}$ was used instead of $\mathcal{C}_{be}$.

We refer the reader to [2,3] for the basic terminology and notation concerning the theory of locally convex spaces and sequence spaces.

We call linear subspaces of Ω *double sequence spaces*. Let $\mathcal{V}$ be a space of double sequences converging with respect to a linear notion of convergence $\mathcal{V}\text{-}\lim:\ \mathcal{V} \to \mathbb{K}$. The sum of a double series $\sum_{k,l} u_{kl}$ with respect to this notion of convergence will be defined by $\mathcal{V}\text{-}\sum_{k,l} u_{kl} := \mathcal{V}\text{-}\lim_{m,n} \sum_{k=1}^{m} \sum_{l=1}^{n} u_{kl}$. Generally $\mathcal{V}$ will be omitted when no confusion may arise.

Let $B = (b_{mnk})$ be a 3-dimensional matrix. The summability method B induced by the summability domain

$$\mathcal{V}_B := \left\{ z \in \omega |\ Bz := \left(\sum_k b_{mnk} z_k \right)_{m,n} \text{ exists and } Bz \in \mathcal{V} \right\}$$

and the limit functional

$$\mathcal{V}\text{-}\lim_B:\ \mathcal{V}_B \to: \mathbb{K},\quad z \mapsto \mathcal{V}\text{-}\lim_{m,n} \sum_k b_{mnk} z_k$$

is called a *$\mathcal{V}$-SM-method* (cf. [1]). Following [1], a sequence of numbers $z = (z_k)$ is said to be *summable by a $\mathcal{V}$-SM-method B to a number s* if the limit $\mathcal{V}\text{-}\lim_B z$ exists and is equal to s.

In [1] the consistency and the structure of summability domains of $\mathcal{C}_{be}$-SM-methods are examined. Our aim is to give necessary and sufficient conditions for a $\mathcal{C}_e$-SM- ($\mathcal{C}_{be}$-SM-) method $B = (b_{mnk})$ to be *conservative* (i.e. to sum every convergent sequence) or *coercive* (i.e. to sum every bounded sequence).

Remark 1.1. The summation in Volkov's sense (cf. [4]) can be considered as a special $\mathcal{C}_e$-SM-method. Given a matrix $A = (a_{nk})$, we put $b_{mnk} := a_{nk}$ for $k = 1, \ldots, m$ and $b_{mnk} := 0$ otherwise $(m, n \in \mathbb{N})$. Then the summability domain $\mathcal{C}_{eB}$ of the $\mathcal{C}_e$-SM-method $B = (b_{mnk})$ coincides with the domain V_A of all sequences, summable by A in Volkov's sense, and $\mathcal{C}_e\text{-}\lim_B x$ equals $V\text{-}\lim_A x$ for all $x \in \mathcal{C}_{eB}$.

2. CONSERVATIVE SM-METHODS

In [1], Theorem 2.4, it was proved that $\mathcal{C}_e$ is an LFH-space (i.e. it can be written as a union of countably many FH-spaces, [3]) with $H = \Omega$. More precisely, $\mathcal{C}_e = \bigcup_n \mathcal{C}_e^n$, where

$$\mathcal{C}_e^n := \Big\{x \in \Omega \mid \sup_{l\geq n} \overline{\lim_k} |x_{kl}| < \infty \text{ and } \exists a \in \mathbb{K} : \Big(\overline{\lim_k} |x_{kl} - a|\Big)_{l \geq n} \in c_0\Big\}$$

is an FH-space with $H = \Omega$ $(n \in \mathbb{N})$. Note that $\mathcal{C}_e^1 = \mathcal{C}_{be}$. We will verify that for every conservative $\mathcal{C}_e$-SM-method B there exists $N \in \mathbb{N}$ such that B maps c into $\mathcal{C}_e^N$. Here we will make use of the following result.

Lemma 2.1 (cf. [3], Theorem 4.2.2). *Let Y be an FH-space, X an F-space, and $T : X \to Y$ a linear map. If $T : X \to H$ is continuous, then $T : X \to Y$ is continuous.*

Lemma 2.2. *Let E be an FK-space and suppose that $F = \bigcup_n F_n$ is an LFH-space with $H = \Omega$ and $F_n \subset F_{n+1}$ $(n \in \mathbb{N})$. If a 3-dimensional matrix $B = (b_{mnk})$ maps E into F, then there exists $N \in \mathbb{N}$ such that $B(E) \subset F_N$.*

Proof. By Lemma 2.1 the matrix map B is continuous, hence (cf. [5], 19.5 (4)) there exists $N \in \mathbb{N}$ such that $B(E) \subset F_N$. ▼

Theorem 2.3. *A 3-dimensional matrix $B = (b_{mnk})$ maps c into $\mathcal{C}_e$ if and only if each of the following conditions holds*:

(i) *for every $k \in \mathbb{N}$ the limit $b_k := \mathcal{C}_e\text{-}\lim_{m,n} b_{mnk}$ exists,*
(ii) $\sum_k |b_{mnk}| < \infty$ *for all $m, n \in \mathbb{N}$,*
(iii) *the limit $v := \mathcal{C}_e\text{-}\lim_{m,n} \sum_k b_{mnk}$ exists,*
(iv) *there exists $N \in \mathbb{N}$ such that $\sup_{m\in\mathbb{N}} \sum_k |b_{mnk}| < \infty$ for all $n \geq N$, and*
(v) *for every index sequence (L_n) there exists $N \in \mathbb{N}$ such that*

$$M := \sup_{n\geq N} \overline{\lim_m} \sum_{k=1}^{L_n} |b_{mnk}| < \infty.$$

Under these circumstances, $(b_k) \in \ell$ and

$$\lim_B x = \sum_k b_k x_k + \Big(v - \sum_k b_k\Big) \lim_k x_k \quad (x \in c).$$

Proof.

Necessity. The Necessity of (i)–(iii) is evident.

(iv) By Lemma 2.2 there exists $N \in \mathbb{N}$ such that $B(c) \subset \mathcal{C}_e^N$. For every $m, n \in \mathbb{N}$ we consider the operator $B_{mn} :\ c \to \mathbb{R}$, $B_{mn} :\ x \mapsto [Bx]_{mn}$. Since the sequence of operators $(B_{mn})_m$ is pointwise bounded for every $n \geq N$, (iv) follows from the Uniform Boundedness Principle.

(v) Since B is a continuous operator from c into $\mathcal{C}_e^N$ (cf. Lemma 2.1), there exists $K \in \mathbb{N}$ such that

$$\sup_{n\geq N} \overline{\lim_m} \Big|\sum_{k=1}^{\infty} b_{mnk} x_k\Big| \leq K \parallel x \parallel_\infty \text{ for every } x \in c.$$

Let (L_n) be an index sequence. By (iv)

$$\sup_m \sum_{k=1}^{L_n} |b_{mnk}| \le \sup_m \sum_k |b_{mnk}| =: M_n < \infty \quad \text{for } \ n \ge N.$$

Let (m_{in}) be a double sequence satisfying

$$\lim_i \sum_{k=1}^{L_n} |b_{m_{in}nk}| = \overline{\lim_m} \sum_{k=1}^{L_n} |b_{mnk}| \quad (n \ge N).$$

Passing to a subsequence of $(m_{in})_i$ if necessary $(n \in \mathbb{N})$, we may suppose that

$$\operatorname{sgn}\Re(b_{m_{i_1 n}nk}) = \operatorname{sgn}\Re(b_{m_{i_2 n}nk}) \text{ for } k = 1, \dots, L_n; \ i_1, i_2 \in \mathbb{N}, \ n \ge N.$$

For every fixed $n \ge N$ we put $y_k := \operatorname{sgn}\Re(b_{m_{1n}nk})$ for $1 \le k \le L_n$ and $y_k := 0$ otherwise. Then $\| y \|_\infty \le 1$ and

$$\overline{\lim_m} \sum_{k=1}^{L_n} |\Re(b_{mnk})| = \lim_i \Big| \Re\Big(\sum_k b_{m_{in}nk} y_k \Big) \Big| \le K.$$

Analogously, $\overline{\lim}_m \sum_{k=1}^{L_n} |\Im(b_{mnk})| \le K$. So $\sup_{n \ge N} \overline{\lim}_m \sum_{k=1}^{L_n} |b_{mnk}| \le 2K$.

Sufficiency. Note that (i) and (v) imply $\underline{(b_k)} \in \ell$. Really, by (i) for a fixed $s \in \mathbb{N}$ we may find $n \ge \max\{N, s\}$ such that $\overline{\lim}_m \sum_{k=1}^{s} |b_{mnk} - b_k| \le 1$. Hence by (v) we get $\sum_{k=1}^{s} |b_k| \le 1 + \sup_{i \ge N} \overline{\lim}_j \sum_{k=1}^{i} |b_{ijk}| < \infty$.

It is sufficient to verify that B maps c_0 into $\mathcal{C}_e$, since in this case by (iii) the limit

$$\lim_B x = \lim_i x_i \cdot \mathcal{C}_e\text{-}\lim_{m,n} \sum_k b_{mnk} + \mathcal{C}_e\text{-}\lim_{m,n} \sum_k b_{mnk}(x_k - \lim_i x_i)$$

exists for every $x \in c$.

So let $x \in c_0$ and $\varepsilon > 0$ be arbitrarily fixed. By (iv) we may find $N_1 \in \mathbb{N}$ such that $M_n := \sup_{m \in \mathbb{N}} \sum_k |b_{mnk}| < \infty$ $(n \ge N_1)$. Now we choose an index sequence (L_n) such that $|x_k| \le \varepsilon/(4M_n)$ for $k \ge L_n$. By (v) there exist $N_2 > N_1$, $M > 0$ and an index sequence (m_n) such that $\sum_{k=1}^{L_n} |b_{mnk}| \le M$ for all $n \ge N_2$, $m \ge m_n$. Select $K \in \mathbb{N}$ with $\sum_{k=K}^{\infty} |b_k| \le 1$ and $|x_k| \le \varepsilon/(4M)$ for $k \ge K$. By (i) we may find $N_3 > N_2$ and an index sequence (m'_n) with $m'_n > m_n$ $(n \ge N_3)$ such that $\sum_{k=1}^{K} |b_{mnk} - b_k||x_k| \le \varepsilon/4$ for all $n \ge N_3$ and $m \ge m'_n$. Now for every $n \ge N_3$ and $m \ge m'_n$ we get

$$\begin{aligned} &\Big| \sum_k b_{mnk} x_k - \sum_k b_k x_k \Big| \\ &\le \sum_{k=1}^{K} |b_{mnk} - b_k||x_k| + \frac{\varepsilon}{4M} \sum_{k=K}^{L_n} |b_{mnk}| + \frac{\varepsilon}{4} \sum_{k=K}^{\infty} |b_k| + \frac{\varepsilon}{4M_n} \sum_{k=L_n}^{\infty} |b_{mnk}| \le \varepsilon. \end{aligned}$$

Hence $\lim_B x = \sum_k b_k x_k$. ▼

Note that condition (ii) is independent of all others. The matrix $B = (b_{mnk})$ with $b_{11k} := (-1)^k/k$ and $b_{mnk} := 0$ $(m,n,k \in \mathbb{N};\ (m,n) \neq (1,1))$ satisfies all the hypotheses of Theorem 2.3 except (ii). At the same time it is possible to find $x \in c$ such that the series $\sum_k b_{11k}x_k$ diverges.

Theorem 2.4. *A* 3*-dimensional matrix* $B = (b_{mnk})$ *maps* c *into* $\mathcal{C}_{be}$ *if and only if each of the following conditions holds*:

(i) *for every* $k \in \mathbb{N}$ *the limit* $b_k := \mathcal{C}_{be}\text{-}\lim_{m,n} b_{mnk}$ *exists,*

(ii) $\sup_{m\in\mathbb{N}} \sum_k |b_{mnk}| < \infty$ *for all* $n \in \mathbb{N}$,

(iii) *the limit* $v := \mathcal{C}_{be}\text{-}\lim_{m,n} \sum_k b_{mnk}$ *exists, and*

(iv) $\sup_n \overline{\lim}_m \sum_{k=1}^{L_n} |b_{mnk}| < \infty$ *for every index sequence* (L_n).

Under these circumstances, $(b_k) \in \ell$ *and*

$$\lim_B x = \sum_k b_k x_k + \Big(v - \sum_k b_k\Big) \lim_k x_k \quad (x \in c).$$

Proof.

Necessity. (i) and (iii) are evident; (ii) and (iv) follow from Theorem 2.3, since for every fixed $n \in \mathbb{N}$ the matrix $(b_{mnk})_{m,k}$ maps c into $\mathfrak{m}$.

Sufficiency. By Theorem 2.3 the limit $\mathcal{C}_e\text{-}\lim_{m,n}[Bx]_{mn}$ exists for any fixed $x \in c$. Now (iv) implies that $([Bx]_{mn})_m \in \mathfrak{m}$ for every $n \in \mathbb{N}$. Hence for every $x \in c$ the limit $\mathcal{C}_{be}\text{-}\lim_{m,n}[Bx]_{mn}$ exists. ▼

3. COERCIVE SM-METHODS

Theorem 3.1. *A* 3*-dimensional matrix* $B = (b_{mnk})$ *maps* $\mathfrak{m}$ *into* $\mathcal{C}_e$ *if and only if each of the following conditions holds*:

(i) *for every* $k \in \mathbb{N}$ *the limit* $b_k := \mathcal{C}_e\text{-}\lim_{m,n} b_{mnk}$ *exists,*

(ii) $\sum_k |b_{mnk}| < \infty$ *for all* $m, n \in \mathbb{N}$,

(iii) *there exists* $N \in \mathbb{N}$ *such that* $\sup_{n\geq N} \overline{\lim}_m \sum_k |b_{mnk}| < \infty$, *and*

(iv) $\lim_n \overline{\lim}_m \sum_k |b_{mnk} - b_k| = 0$.

Under these circumstances, $(b_k) \in \ell$ *and*

$$\lim_B x = \sum_k b_k x_k \quad (x \in \mathfrak{m}).$$

In proving this proposition we make use of two nonsummability lemmas involving gliding hump arguments.

Let $\varphi:\ \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be a bijection defined inductively by

$$\varphi[(1,1)] = 1, \quad \varphi[(1,2)] = 2, \quad \varphi[(2,1)] = 3;$$
$$\varphi[(1,n)] = \frac{(n-1)n}{2} + 1, \quad \varphi[(2,n-1)] = \frac{(n-1)n}{2} + 2,\ \ldots,$$
$$\varphi[(n,1)] = \frac{n(n+1)}{2}.$$

Let $\pi_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $(a, b) \to a$ and $\pi_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $(a, b) \to b$ be the projection maps. We put $\lambda_i := \pi_i \varphi^{-1}$ $(i = 1, 2)$.

We say that a double sequence (m_{ij}) in $\mathbb{N}$ is *increasing* if $m_{i,j+1} > m_{ij}$ $(i, j \in \mathbb{N})$.

In proving the lemmas mentioned above we will use the following

Remark 3.2. Let a 3-dimensional matrix $B = (b_{mnk})$ and $x \in \omega$ be fixed. If there exists an index sequence (n_i) and an increasing double sequence (m_{ij}) in $\mathbb{N}$ such that $x \notin \mathcal{C}_{eD}$, where $D := (b_{m_{ij} n_i k})_{i,j,k}$, then $x \notin \mathcal{C}_{eB}$.

Lemma 3.3. *Let $B = (b_{mnk})$ be a 3-dimensional matrix such that*

$$\sup_m \sum_k |b_{mnk}| < \infty \; (n \in \mathbb{N}) \quad \textit{and} \quad \lim_n \lim_s \overline{\lim_m} \sum_{k=s}^{\infty} |b_{mnk}| \neq 0.$$

Then there exists an $x \in \mathfrak{m} \backslash \mathcal{C}_{eB}$.

Proof. Without loss of generality we may suppose that there exists an index sequence (n_r) such that

$$\lim_s \overline{\lim_m} \sum_{k=s}^{\infty} |\Re(b_{mn_rk})| > 5\gamma \quad (r \in \mathbb{N})$$

for some suitably chosen $\gamma > 0$.

Setting $s_{r1} := 0$ $(r \in \mathbb{N})$, we choose inductively increasing double sequences (μ_{rj}) and (s_{rj}) of indexes such that

$$\sum_{k=s_{rj}+1}^{\infty} |\Re(b_{\mu_{rj}n_rk})| > 4\gamma, \qquad \sum_{k=s_{r,j+1}+1}^{\infty} |b_{\mu_{rj}n_rk}| < \gamma \quad (r, j \in \mathbb{N}).$$

So

$$\sum_{k=s_{rj}+1}^{s_{r,j+1}} |\Re(b_{\mu_{rj}n_rk})| > 3\gamma \quad (r, j \in \mathbb{N}).$$

Setting $t_1 := s_{11}$ and putting $t_r := s_{\lambda_1(r)j_r}$, $m_{\lambda_1(r)\lambda_2(r)} := \mu_{\lambda_1(r)j_r}$ for $r > 1$, where $j_r \in \mathbb{N}$ is chosen such that $s_{\lambda_1(r)j_r} > s_{\lambda_1(r-1),j_{r-1}+1}$, we obtain an index sequence (t_i) and an increasing double sequence (m_{ij}) such that $(m_{ij})_j$ is a subsequence of $(\mu_{ij})_j$, $\sum_{k=t_i+1}^{t_{i+1}} |b_{m_{\lambda_1(i)\lambda_2(i)}n_{\lambda_1(i)}k}| > 3\gamma$ and $\sum_{k=t_{i+1}+1}^{\infty} |b_{m_{\lambda_1(i)\lambda_2(i)}n_{\lambda_1(i)}k}| < \gamma$ $(i \in \mathbb{N})$.

Fixing $x_k := 0$ for $k \leq t_1$, for $k = t_i + 1, \ldots, t_{i+1}$ we put

$$x_k := \begin{cases} \operatorname{sgn} \Re(b_{m_{\varphi^{-1}(i)} n_{\lambda_1(i)} k}) & \text{if } \lambda_2(i) = 1 \\ & \text{or } \sum_{l=1}^{t_i} \Re(b_{m_{\lambda_1(i),\lambda_2(i)-1} n_{\lambda_1(i)} l} x_l) < \sum_{l=1}^{t_i} \Re(b_{m_{\lambda_1(i)\lambda_2(i)} n_{\lambda_1(i)} l} x_l), \\ -\operatorname{sgn} \Re(b_{m_{\varphi^{-1}(i)} n_{\lambda_1(i)} k}) & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} &\Big| \sum_{k=1}^{t_i} \Re(b_{m_{\lambda_1(i),\lambda_2(i)-1} n_{\lambda_1(i)} k} x_k) - \sum_{k=1}^{t_{i+1}} \Re(b_{m_{\lambda_1(i)\lambda_2(i)} n_{\lambda_1(i)} k} x_k) \Big| \\ &\qquad \geq \sum_{k=t_i+1}^{t_{i+1}} |\Re(b_{m_{\lambda_1(i)\lambda_2(i)} n_{\lambda_1(i)} k})| \ > \ 3\gamma \quad (i \in \mathbb{N} : \ \lambda_2(i) > 1). \end{aligned}$$

Hence

$$\begin{aligned} &\Re\Big([Bx]_{m_{\lambda_1(i),\lambda_2(i)-1} n_{\lambda_1(i)}} - [Bx]_{m_{\lambda_1(i)\lambda_2(i)} n_{\lambda_1(i)}} \Big) \\ &\qquad = \ \Big| \Re\Big(\sum_k b_{m_{\lambda_1(i),\lambda_2(i)-1} n_{\lambda_1(i)} k} x_k \Big) - \Re\Big(\sum_k b_{m_{\lambda_1(i)\lambda_2(i)} n_{\lambda_1(i)} k} x_k \Big) \Big| \\ &\qquad \geq \ \sum_{k=t_i+1}^{t_{i+1}} |\Re(b_{m_{\lambda_1(i)\lambda_2(i)} n_{\lambda_1(i)} k})| - \sum_{k=t_i+1}^{\infty} |b_{m_{\lambda_1(i),\lambda_2(i)-1} n_{\lambda_1(i)} k}| \\ &\qquad\qquad - \sum_{k=t_{i+1}+1}^{\infty} |b_{m_{\lambda_1(i)\lambda_2(i)} n_{\lambda_1(i)} k}| \\ &\qquad \geq \ 3\gamma - \gamma - \gamma \ = \ \gamma \end{aligned}$$

for every $i \in \mathbb{N}$ with $\lambda_2(i) > 1$. Therefore, by Remark 3.2, $x \notin \mathcal{C}_{eB}$. ▼

Lemma 3.4. *Let $B = (b_{mnk})$ be a 3-dimensional matrix such that*

$$\mathcal{C}_e\text{-}\lim_{m,n} b_{mnk} = 0 \ (k \in \mathbb{N}) \quad \textit{and} \quad \lim_n \lim_s \overline{\lim_m} \sum_{k=s}^{\infty} |b_{mnk}| = 0.$$

If $\lim_n \overline{\lim}_m \sum_k |b_{mnk}| \neq 0$, then there exists an $x \in \mathfrak{m} \backslash \mathcal{C}_{eB}$.

Proof. Without loss of generality we may assume that there exist a $\gamma > 0$ and an index sequence $(n^{(i)})$ such that

$$\overline{\lim_m} \sum_k |\Re(b_{mn^{(i)}k})| > 5\gamma \quad (i \in \mathbb{N}).$$

Fixing $k_1 := 1$, we construct inductively two index sequences (k_i) and (n_i) choosing the second sequence as a subsequence of $(n^{(i)})$.

Suppose that $k_1, \dots, k_r$ and $n_1, \dots, n_{r-1}$ are fixed. Then we may choose n_r from $(n^{(j)})$ such that $n_r > n_{r-1}$ and

$$\overline{\lim_m} \sum_{k=1}^{k_r} |b_{mn_rk}| < \gamma \quad \text{and} \quad \lim_s \overline{\lim_m} \sum_{k=s}^{\infty} |b_{mn_rk}| < \gamma.$$

Now we take k_{r+1} with $k_{r+1} > k_r$ such that $\overline{\lim}_m \sum_{k=k_{r+1}+1}^{\infty} |b_{mn_rk}| < \gamma$. Hence

$$\overline{\lim_m} \sum_{k=k_r+1}^{k_{r+1}} |\Re(b_{mn_rk})| > 3\gamma \quad (r \in \mathbb{N}).$$

Then we find an increasing double sequence (m_{rj}) such that

$$\operatorname{sgn}\Re(b_{m_{ri}n_rk}) = \operatorname{sgn}\Re(b_{m_{rj}n_rk}) \ \text{ for } \ k = k_r + 1, \dots, k_{r+1},$$

$$\sum_{k=k_{r+1}+1}^{\infty} |b_{m_{rj}n_rk}| < \gamma, \quad \sum_{k=1}^{k_r} |b_{m_{rj}n_rk}| < \gamma, \quad \sum_{k=k_r+1}^{k_{r+1}} |\Re(b_{m_{rj}n_rk})| > 3\gamma$$

for all $r, i, j \in \mathbb{N}$. We put $x_k := 0$ for $k \leq k_1$ and $x_k := (-1)^r \operatorname{sgn}\Re(b_{m_{rj}n_rk})$ for $k_r < k \leq k_{r+1}$ $(r \in \mathbb{N})$. Then $x \in \mathfrak{m}$ and for all $r, i, j \in \mathbb{N}$ we get

$$\begin{aligned}
&\Re\big([Bx]_{m_{rj}n_r} - [Bx]_{m_{r+1,i}n_{r+1}}\big) \\
&\quad = \Big|\sum_k \Re(b_{m_{rj}n_rk}x_k) - \sum_k \Re(b_{m_{r+1,i}n_{r+1}k}x_k)\Big| \\
&\quad \geq \sum_{k=k_r+1}^{k_{r+1}} |\Re(b_{m_{rj}n_rk})| + \sum_{k=k_{r+1}+1}^{k_{r+2}} |\Re(b_{m_{r+1,i}n_{r+1}k})| - \sum_{k=1}^{k_r} |b_{m_{rj}n_rk}| \\
&\qquad - \sum_{k=1}^{k_{r+1}} |b_{m_{r+1,i}n_{r+1}k}| - \sum_{k=k_{r+1}+1}^{\infty} |b_{m_{rj}n_rk}| - \sum_{k=k_{r+2}+1}^{\infty} |b_{m_{r+1,i}n_{r+1}k}| \\
&\quad > 3\gamma + 3\gamma - 4\gamma = 2\gamma.
\end{aligned}$$

Hence, by Remark 3.2, $x \notin \mathcal{C}_{eB}$. ▼

Proof of Theorem 3.1.

Necessity. (i) and (ii) are evident.

(iii) By Lemma 2.2 there exists $N \in \mathbb{N}$ such that $B(\mathfrak{m}) \subset \mathcal{C}_e^N$. Applying Lemma 3.3 to the matrix $(b_{m,n+N,k})$, we get $\lim_n \lim_s \overline{\lim}_m \sum_{k=s}^{\infty} |b_{mnk}| = 0$. Hence there exists an index sequence (L_n) such that

$$\sup_{n \geq N} \lim_s \overline{\lim_m} \sum_{k=L_n+1}^{\infty} |b_{mnk}| < \infty.$$

By Theorem 2.3 (v) $\sup_{n \geq N} \overline{\lim}_m \sum_{k=1}^{L_n} |b_{mnk}| < \infty$. Hence (iii) follows.

(iv) By (i) and (iii) we get $(b_k) \in \ell$. We may assume that $b_k = 0$ $(k \in \mathbb{N})$. So (iv) follows by Lemma 3.4.

Sufficiency. From (i) and (iii) it follows that the series $\sum_k |b_k x_k|$ converges for every $x \in \mathfrak{m}$. Let $\gamma_{mn} := \sum_k |b_{mnk} - b_k|$ $(m, n \in \mathbb{N})$. By (iv) $\lim_n \overline{\lim}_m |\gamma_{mn}| = 0$. For every $x \in \mathfrak{m}$ we get

$$\Big|\sum_k b_{mnk} x_k - \sum_k b_k x_k\Big| \leq \gamma_{mn} \|x\|_\infty \quad (m, n \in \mathbb{N}).$$

Hence $\mathcal{C}_e$-$\lim_{m,n} \sum_k b_{mnk} x_k = \sum_k b_k x_k$. So $\mathfrak{m} \subset \mathcal{C}_{eB}$. ▼

Theorem 3.5. *A* 3*-dimensional matrix* $B = (b_{mnk})$ *maps* $\mathfrak{m}$ *into* $\mathcal{C}_{be}$ *if and only if* B *satisfies* (iv) *of Theorem* 3.1 *and*

(i') *for every* $k \in \mathbb{N}$ *the limit* $b_k := \mathcal{C}_{be}$-$\lim_{m,n} b_{mnk}$ *exists,*

(ii') $\sup_n \overline{\lim}_m \sum_k |b_{mnk}| < \infty$.

Under these circumstances, $(b_k) \in \ell$ *and*

$$\lim_B x = \sum_k b_k x_k \quad (x \in \mathfrak{m}).$$

Proof. It may be obtained in the same way as the proof of Theorem 3.1. ▼

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KOONDUVUST SÄILITAVAD JA TEKITAVAD SM-MENETLUSED

Maria ZELTSER

On vaadeldud topeltjadade ruumi

$$C_e := \{x = (x_{kl}) | \exists a \in \mathbb{K} : \lim_{l\to\infty} \overline{\lim_{k\to\infty}} |x_{kl} - a| = 0\}.$$

Libiseva küüru meetodi abil on leitud tarvilikud ja piisavad tingimused selleks, et kolmemõõtmeline maatriks (ehk SM-menetlus) teisendaks iga koonduva või tõkestatud jada (x_k) ruumi $\mathcal{C}_e$ või tema alamruumi

$$\mathcal{C}_{be} := \left\{x \in \mathcal{C}_e | \ \forall l \in \mathbb{N}: \ \ (x_{kl})_k \in \mathfrak{m}\right\}.$$