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TWO-BASED DUPLICATE-CLONES

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Abstract. The notion "two-based clone" on a pair of sets (universes) is defined. Some properties of two-based duplicate-clones are proved. The lattice of all double-dually closed duplicate-clones on a pair of 2-element sets is described.

Key words: two-based clone, lattice, two-based duplicate-clone, double-dually closed clone.

1. INTRODUCTION

The notion "clone on A" was introduced for classifying algebras on a fixed universe A. Two algebras on A are term equivalent if and only if the clones generated by all fundamental operations of them coincide. A review of the results on clones is given by Sichler and Trnková ([1]). Under inclusion the set of all clones on A forms the lattice $\mathcal{L}_A$. The structure of the lattice $\mathcal{L}_A$ has been studied in general (see, e.g., [2]) and for some $k = |A|$. The lattice $\mathcal{L}_A$ is completely known for Boolean functions, i.e. for $|A| = 2$ (see [3]). As for $|A| \geqslant 3$ the lattice $\mathcal{L}_A$ is uncountable, it seems hopeless to find a satisfactory description of $\mathcal{L}_A$ in general. Special parts of $\mathcal{L}_A$ (with $|A| = k > 2$) are described, for example, by Burle [4] and Hoa [5].

Let S_A be the full symmetric group on A. The notion "S_A-clone" was introduced in [6]. From [6,7] we know that the lattice of all S_A-clones is finite if $|A| = 2, 3$. In the present paper we define the notion "two-based clone". This notion is justified by the fact that many algebraic structures (acts, modules, linear spaces, etc.) are two-based (called also "two-sorted"). The set of all two-based clones on a fixed pair $\mathbf{A}$ forms (with respect to the set inclusion) the lattice $\mathcal{L}_{\mathbf{A}}$. As expected, the lattice $\mathcal{L}_{\mathbf{A}}$ has a very complicated structure.

In Section 3 we define the notions "two-based duplicate-clone", "1-component" and "2-component" of a two-based clone. We prove some properties of two-based duplicate-clones and describe the 1-component and 2-component of a two-based duplicate-clone.

In Section 4 we apply these results to the 2-Boolean clones, i.e. to two-based clones on a pair of 2-element sets. The first results in this direction were obtained by Kudrjavcev and Burosch [8] who studied generating sets of closed classes of the two-based full iterative algebra on a pair of 2-element sets. Here, for any double-dually closed 2-Boolean clone the subset of all unary functions is described (see Proposition 3.3).

The main result of the present paper is a full description of the sublattice consisting of all duplicate-dd'-clones in $\mathcal{L}_{2\times 2}$ (see Theorem 4.1).

2. NOTATIONS AND PRELIMINARIES

Let $\mathbf{A} := (A_1, A_2)$ be a pair of (finite) disjoint sets containing at least two elements each. The sets A_1, A_2 will be called *the first* and *the second universe*, respectively. Let us denote

$$O_{\mathbf{A}} := \left\{ f : A_{i_1} \times \ldots \times A_{i_n} \to A_{i_{n+1}} \mid i_1, \ldots, i_{n+1} \in \{1,2\}, \quad n \in \mathbf{N}^+ \right\}$$

and let $\tau = (i_1, \ldots, i_n; i_{n+1})$ be called the *signature* of the mapping f. We denote by $J_{\mathbf{A}}$ the set of all projections

$$e_k^{i_1 \cdots i_n} : A_{i_1} \times \ldots \times A_{i_n} \to A_{i_k} : (x_1, \ldots, x_n) \mapsto x_k$$

with $k \in \{1, \ldots, n\},\ i_1, \ldots, i_n \in \{1,2\}$. Let all five Mal′tsev's operations (see [9]) be acting on $O_{\mathbf{A}}$. Then superposition, composition, and linearized composition of mappings are defined on $O_{\mathbf{A}}$ too.

Definition 2.1. *If a subset $F \subseteq O_{\mathbf{A}}$ contains $J_{\mathbf{A}}$ and is closed under composition, then we write $F \leqslant O_{\mathbf{A}}$ and call F a* two-based clone on $\mathbf{A}$. *We denote by $\langle F \rangle_{O_{\mathbf{A}}}$ (or simply by $\langle F \rangle$) the two-based clone generated by $F \subseteq O_{\mathbf{A}}$.*

For any subset $F \subseteq O_{\mathbf{A}}$ and any signature $\tau \in \{1,2\}^{n+1}$ we introduce the set

$$F^\tau = \{ f^\tau \in F \mid f \ \text{ is of signature } \tau \} .$$

Functions with values in A_1 (or in A_2) are called 1-functions (*or* 2-functions). *We denote by F_1 and F_2 the subsets of all 1-functions and 2-functions of a set $F \subseteq O_{\mathbf{A}}$. Let O_{A_1} and O_{A_2} be the sets of all functions on the first universe A_1 and on the second universe A_2, respectively. Then*

$$F_{A_1} := F \bigcap O_{A_1}, \qquad F_{A_2} := F \bigcap O_{A_2}$$

will be called the 1-component *and the* 2-component *of $F \subseteq O_{\mathbf{A}}$, respectively.*

Example 2.1. Let both universes be 2-element sets:

$$A_1 = E_2 := \{0, 1\}, \quad A_2 = E_2' := \{0', 1'\}.$$

The set of all functions on this pair will be denoted by $O_{\mathbf{2}\times\mathbf{2}}$. A two-based clone $F \leqslant O_{\mathbf{2}\times\mathbf{2}}$ will be called a 2-*Boolean clone*. The 1-component (2-component) of a 2-Boolean clone is the clone of Boolean functions over $\{0, 1\}$ (over $\{0', 1'\}$, respectively).

Let $\neg$ be the negation on E_2, i.e. $\neg(0) = 1$, $\neg(1) = 0$, and $\neg'$ be the negation on E_2'. Besides the identity functions and negations there are only four other unary nonconstant functions

$$d_1(0') = 0,\ \ d_1(1') = 1; \quad d_2(0) = 0',\ d_2(1) = 1'$$

and their negations $\neg d_1,\ \neg' d_2$. An $n$ary function ($n \geqslant 1$) is called *essentially unary* if it depends only on one of the variables.

Kudrjavcev and Burosch [8] investigated closed under composition classes of functions over a pair of 2-element sets. They found the subset of all unary nonconstant functions for all closed classes. Let us remark that all closed classes containing $J_{\mathbf{A}}$, and only such classes, are 2-Boolean clones. The results about 2-Boolean clones contained in [8] can be systematized and represented as in the next Proposition 2.1.

Proposition 2.1. *There are* 19 2-*Boolean clones generated by a subset of unary nonconstant functions in* $O_{\mathbf{2}\times\mathbf{2}}$:

$J_{\mathbf{A}} = \langle G_4 \rangle$	(projections),
$\langle \neg \rangle = \langle G_2 \rangle$	(1-negations of projections),
$\langle \neg' \rangle = \langle G_3 \rangle$	(2-negations of projections),
$\langle \neg, \neg' \rangle = \langle G_1 \rangle$	(negations of projections),
$\langle d_1 \rangle = \langle F_{14} \rangle$	(1-duplicates of 2-projections),
$\langle d_2 \rangle = \langle F_{12} \rangle$	(2-duplicates of 1-projections),
$\langle \neg d_1 \rangle = \langle F_{15} \rangle$	(neg-1-duplicates of 2-projections),
$\langle \neg' d_2 \rangle = \langle F_{13} \rangle$	(neg-2-duplicates of 1-projections),
$\langle d_1, \neg d_1 \rangle = \langle F_{11} \rangle$	(1-duplicates and neg-1-duplicates of 2-projections),
$\langle d_2, \neg' d_2 \rangle = \langle F_{10} \rangle$	(2-duplicates and neg-2-duplicates of 1-projections),
$\langle d_1, d_2 \rangle = \langle F_8 \rangle$	(duplicates of projections),
$\langle \neg d_1, \neg' d_2 \rangle = \langle F_9 \rangle$	(neg-duplicates of projections),
$\langle \neg, d_1, (\neg d_1) \rangle = \langle F_5 \rangle$	(all (essentially) unary 1-functions),
$\langle \neg, d_2, (\neg' d_2) \rangle = \langle F_4 \rangle$	(negations, 2-duplicates and neg-2-duplicates of 1-projections),
$\langle \neg', d_2, (\neg' d_2) \rangle = \langle F_7 \rangle$	(all (essentially) unary 2-functions),
$\langle \neg', d_1, (\neg d_1) \rangle = \langle F_6 \rangle$	(negations, 1-duplicates and neg-1-duplicates of 2-projections),

$\langle\neg,\neg',d_1,(\neg d_1)\rangle=\langle F_3\rangle$ (negations; all (essentially) unary 1-functions),
$\langle\neg,\neg',d_2,(\neg' d_2)\rangle=\langle F_2\rangle$ (negations; all (essentially) unary 2-functions),
$\langle\neg,\neg',d_1,(\neg d_1),d_2,(\neg' d_2)\rangle=\langle F_1\rangle$ (all (essentially) unary functions).

Remark 2.1. Here the functions in round brackets may be omitted (for example, the parts $(\neg d_1)$ and $(\neg d_2)$ in the last line).

3. DUPLICATE-CLONES AND dd'-CLONES

We define the notion of duplication over a pair $\mathbf{A}=(A_1,A_2)$ as follows.

Definition 3.1. *Let both universes have the same power, i.e.* $|A_1|=|A_2|$ *and assume that a two-based clone* F *contains bijections* $d_1:A_2\to A_1,\ d_2:A_1\to A_2$ *which are inverses of each other. Then we say that* F *is a* two-based duplicate-clone (*for short,* d_1d_2-clone). *The functions* d_1 *and* d_2 *will be called* 1-duplication *and* 2-duplication, *respectively.*

Proposition 3.1. *The* 1*-component* F_{A_1} *and the* 2*-component* F_{A_2} *of a* d_1d_2*-clone* $F\leqslant O_{\mathbf{A}}$ *are clones on* A_1 *and* A_2*, respectively, and they are isomorphic.*

Proof. The 1-component F_{A_1} and the 2-component F_{A_2} are both closed under composition. So they are clones on A_1 and on A_2, respectively. An isomorphism from F_{A_1} to F_{A_2} can be given by the correspondence

$$f\mapsto f^{d_2},\quad\text{where}\quad f^{d_2}(y_1,\dots,y_n)=d_2(f(d_1(y_1),\dots,d_1(y_n))).\tag{1}$$

Proposition 3.2. *For any two signatures*

$$\tau_1=(i_1,\dots,i_n;i_{n+1}),\ \tau_2=(j_1,\dots,j_n;j_{n+1})$$

of the same length, and for any d_1d_2*-clone* F *we have*

$$|F^{\tau_1}|=|F^{\tau_2}|,$$

where both sets determine each other uniquely.

Proof. Let F be a d_1d_2-clone and let

$$\tau_1=(i_1,\dots,i_n;i_{n+1}),\ \tau_2=(j_1,\dots,j_n;j_{n+1})$$

be signatures of the same length. Let

$$u=\begin{cases}\mathrm{id}A_l & \text{if } i_{k+1}=j_{k+1}=l,\\ d_{j_{k+1}} & \text{if } i_{k+1}\neq j_{k+1};\end{cases}\qquad v=\begin{cases}\mathrm{id}A_l & \text{if } i_{k+1}=j_{k+1}=l,\\ d_{i_{k+1}} & \text{if } i_{k+1}\neq j_{k+1}\end{cases}$$

and for all $k = 1, \ldots, n$ let us have the mappings

$$u_k = \begin{cases} \mathrm{id}A_l & \text{if } i_k = j_k = l, \\ d_{i_k} & \text{if } i_k \neq j_k; \end{cases} \qquad v_k = \begin{cases} \mathrm{id}A_l & \text{if } i_k = j_k = l, \\ d_{j_k} & \text{if } i_k \neq j_k. \end{cases}$$

For any $f \in F^{\tau_1}$ and any $g \in F^{\tau_2}$ we define functions $f' \in F^{\tau_2}$, $g' \in F^{\tau_1}$ as follows:

$$f'(y_1, \ldots, y_n) = u(f(u_1(y_1), \ldots, u_n(y_n))), \tag{2}$$

$$g'(x_1, \ldots, x_n) = v(g(v_1(x_1), \ldots, v_n(x_n))) \tag{3}$$

for all $y_1 \in A_{j_1}, \ldots, y_n \in A_{j_n}$, $x_1 \in A_{i_1}, \ldots, x_n \in A_{i_n}$.

The correspondences $f \mapsto f'$ and $g \mapsto g'$, defined by formulas (2) and (3), respectively, are bijections between the sets F^{τ_1} and F^{τ_2}. $\square$

Let F be again a 2-Boolean clone and let $\mathbf{s}$ denote the pair of negations, i.e. $\mathbf{s} := (\neg, \neg')$. For the functions

$$f: \; E_2^m \times {E_2'}^k \to E_2 \text{ and } g: \; E_2^m \times {E_2'}^k \to E_2'$$

the $\mathbf{s}$-dual functions are defined by the formulas

$$f^{\mathbf{s}}(x_1, \ldots, x_n, y_1, \ldots, y_m) := \neg f(\neg x_1, \ldots, \neg x_n, \neg' y_1, \ldots, \neg' y_m)$$

and

$$g^{\mathbf{s}}(x_1, \ldots, x_n, y_1, \ldots, y_m) := \neg' g(\neg x_1, \ldots, \neg x_n, \neg' y_1, \ldots, \neg' y_m).$$

For functions f and g with a different order of variables the functions $f^{\mathbf{s}}$ and $g^{\mathbf{s}}$ are defined similarly. For a set $F \subseteq O_{\mathbf{2}\times\mathbf{2}}$, let $F^{\mathbf{s}} := \{f^{\mathbf{s}} \mid f \in F\}$.

Definition 3.2. *A two-based clone $F \leqslant O_{\mathbf{2}\times\mathbf{2}}$ is called a* double-dually closed 2-Boolean clone (*in short*, dd'-clone) *if* $F^{\mathbf{s}} = F$.

Proposition 3.3. *The subset of all unary nonconstant functions of a dd'-clone has one of the* 19 *forms* ($Q_1, \ldots, Q_4, F_1, \ldots, F_{15}$) *listed in Proposition* 2.1. *The subset of all unary nonconstant functions of a duplicate-dd'-clone is* F_1, F_8, or F_9.

Proof. Any dd'-clone contains a minimal two-based clone listed in Proposition 2.1, because any unary nonconstant function is $\mathbf{s}$-dual to itself. It is easy to verify that just the subset F_8 is closed under both duplications (d_1 and d_2), the subset F_9 is closed under negations of both duplications ($\neg d_1$ and $\neg' d_2$), and F_1 is closed under all four of these functions. $\square$

4. LATTICE OF DUPLICATE-dd'-CLONES

Now we will focus on the most interesting part of the lattice $\mathcal{L}_{2\times 2}$ consisting of all duplicate-dd'-clones F. In such a dd'-clone F the set F_1 depends on F_2 and vice versa. Gorlov and Pöschel described in [6] the lattice $\mathcal{L}_{2,S_2}$ of all dually closed clones (i.e. S_2-clones) of Boolean functions (on one universe). This lattice consists of 14 elements and has the structure pictured in Fig. 1.

The list of clones shown in Fig. 1 and sets generating them (in notations of [10] and [6]) is as follows:

$\mathbf{O_1} = J_{\mathbf{A}}$ (projections),
$\mathbf{O_4} = \langle\neg\rangle$ (projections and their negations),
$\mathbf{O_8} = \langle c_0, c_1\rangle$ (constants),
$\mathbf{O_9} = \langle c_0, c_1, \neg\rangle$ (essentially unary functions),
$\mathbf{L_1} = \langle c_1, +\rangle$ (all linear functions),
$\mathbf{L_4} = \langle g\rangle$ (linear idempotent functions (where $g(x,y,z) := x+y+z$)),
$\mathbf{L_3} = \langle g, \neg\rangle$ (linear self-dual functions),
$\mathbf{D_2} = \langle h\rangle$ (self-dual monotone functions (where $h(x,y,z) := (x\wedge y)\vee(y\wedge z)\vee(z\wedge x)$)),
$\mathbf{D_1} = \langle g, h\rangle$ (self-dual idempotent functions),
$\mathbf{D_3} = \langle h, \neg\rangle$ (self-dual functions),
$\mathbf{A_4} = \langle\wedge, \vee\rangle$ (monotone idempotent functions),
$\mathbf{A_1} = \langle c_0, c_1, \wedge, \vee\rangle$ (monotone functions),
$\mathbf{C_4} = \langle g, \wedge, \vee\rangle$ (idempotent functions),
$\mathbf{C_1} = O_{\mathbf{A}}$ (all functions).

Theorem 4.1. *There are exactly* 22 *duplicate-*dd'*-clones in* $\mathcal{L}_{2\times 2}$*. Together with the minimal* 2*-Boolean clone* $\mathbf{O_1}$ *they form a lattice pictured in Fig.* 2.

Proof. Let F be a duplicate-dd'-clone. There are three possibilities for the duplication functions: 1) d_1 and d_2, 2) $\neg d_1$ and $\neg' d_2$, 3) d_1, d_2, $\neg d_1$, and $\neg' d_2$. In case of 1, 2, or 3 we will say that F has type 1, 2, or 3, respectively. By Proposition 3.1 the 1-component F_{E_2} and the 2-component $F_{E_2'}$ of the duplicate-dd'-clone F are clones of Boolean functions on E_2 and E_2', respectively, and these clones are isomorphic. It follows immediately from the definitions of dd'-clones and S_2-clones that F_{E_2} and $F_{E_2'}$ are S_2-clones. The set of 1-components F_{E_2} (2-components $F_{E_2'}$) of all duplicate-dd'-clones F of type 1 (or 2 or 3) under inclusion forms a lattice which is isomorphic to a sublattice of the lattice $\mathcal{L}_{2,S_2}$ (given in Fig. 1).

An immediate calculation shows that any of these 14 clones on E_2 is the 1-component for some duplicate-dd'-clone of type 1. Namely, we get from a fixed clone $\mathbf{C}$ on E_2 a duplicate-dd'-clone F of type 1 if we construct all subsets F^{τ_2} for all signatures $\tau_2 \in \{1,2\}^{n+1}$ by the formulas (2), (3) with the condition $F_{A_1} = \mathbf{C}$. The duplicate-dd'-clone of type 1, just constructed, will be denoted by $\mathbf{C_d}$. It follows from Proposition 3.2 that $\mathbf{C_d}$ is uniquely determined by $\mathbf{C}$. It is easy to

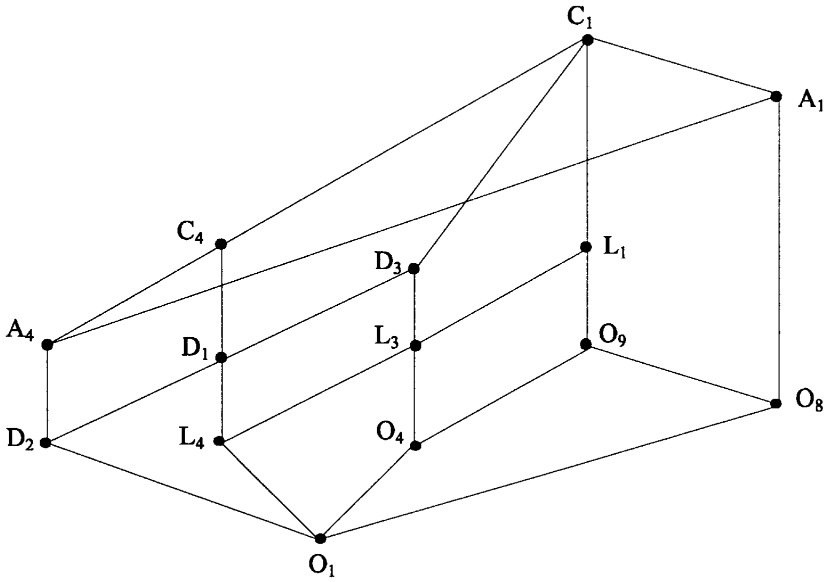

Fig. 1. The lattice $\mathcal{L}_{2,S_2}$ of S_2-clones.

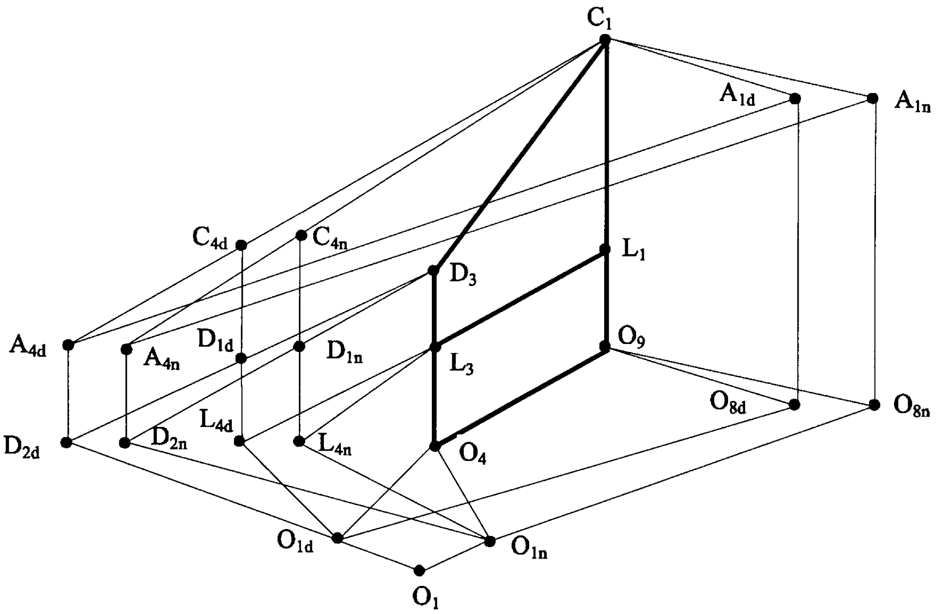

Fig. 2. The lattice of duplicate-dd'-clones.

verify that $\mathbf{C_d}$ is a dd'-clone. Hence the lattice of all duplicate-dd'-clones of type 1 is isomorphic to the lattice $\mathcal{L}_{2,S_2}$.

Similarly, in order to describe all duplicate-dd'-clones of type 2, we have to use the functions $\neg d_1$ and $\neg' d_2$ instead of the functions d_1 and d_2, respectively, in the formulas (2), (3). By $\mathbf{C_n}$ we denote the duplicate-dd'-clone of type 2, constructed from a fixed clone $\mathbf{C}$ on E_2 in the same way as $\mathbf{C_d}$ (but using $\neg d_1$ and $\neg' d_2$). We see that the lattice of all duplicate-dd'-clones of type 2 is also isomorphic to the lattice $\mathcal{L}_{2,S_2}$.

Now we consider duplicate-dd'-clones F of type 3. First we notice that the set of unary nonconstant functions of F consists of all such functions. In particular it contains the negation $\neg$. Thus the 1-component of a duplicate-dd'-clone of type 3 can be one of the following: $\mathbf{O_4}$, $\mathbf{O_9}$, $\mathbf{L_1}$, $\mathbf{L_3}$, $\mathbf{D_3}$, and $\mathbf{C_1}$. If we take all duplicates (or all neg-duplicates) of all functions of these clones, then we get a uniquely determined duplicate-dd'-clone of type 3. The duplicate-dd'-clone of type 3, just constructed, we denote also by $\mathbf{O_4}$, $\mathbf{O_9}$, $\mathbf{L_1}$, $\mathbf{L_3}$, $\mathbf{D_3}$, and $\mathbf{C_1}$, respectively. Hence they form the lattice of all duplicate-dd'-clones of type 3, which is shown in Fig. 2 by bold lines.

By an easy checking we see that the equations $\mathbf{O_{4d}} = \mathbf{O_{4n}} = \mathbf{O_4}$, $\mathbf{O_{9d}} = \mathbf{O_{9n}} = \mathbf{O_9}$, $\mathbf{L_{1d}} = \mathbf{L_{1n}} = \mathbf{L_1}$, $\mathbf{L_{3d}} = \mathbf{L_{3n}} = \mathbf{L_3}$, $\mathbf{D_{3d}} = \mathbf{D_{3n}} = \mathbf{D_3}$, and $\mathbf{C_{1d}} = \mathbf{C_{1n}} = \mathbf{C_1}$ hold. Altogether we got 22 different duplicate-dd'-clones. We have to add $\mathbf{O_1}$ to the set of all duplicate-dd'-clones to get a lattice because $\mathbf{O_{1d}} \cap \mathbf{O_{1n}} = \mathbf{O_1}$. But the minimal 2-Boolean clone $\mathbf{O_1} = J_{\mathbf{A}}$ is not a duplicate-clone. Hence we got the lattice graphed in Fig. 2. This completes our proof. $\square$

Two-based clones that are not duplicate-clones will be considered in a forthcoming paper.

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KAHEALUSELISED DUBLIKAATKLOONID

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On defineeritud kahealuselise klooni mõiste ühisosata hulkade paaril, tehtud kindlaks kahealuseliste dublikaatkloonide omadusi ja esitatud topeltduaalsete dublikaatkloonide võre täielik kirjeldus.