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Controller and controllability canonical forms for discrete-time nonlinear systems*

Ülle Kotta

Institute of Cybernetics at Tallinn University of Technology, Akadeemia tee 21, 12618 Tallinn, Estonia; kotta@cc.ioc.ee

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Abstract. The paper studies the controller and controllability canonical forms for single-input discrete-time nonlinear systems applying the algebraic formalism of one-forms. Necessary and sufficient conditions are given under which the system can be transformed locally into the controller or controllability form by means of a coordinate transformation. These conditions are formulated in terms of integrability of certain subspaces of one-forms, classified according to their relative degree, and in terms of factorizability of a given function in a certain way.

Key words: nonlinear systems, discrete-time systems, algebraic methods, canonical forms, accessibility.

1. INTRODUCTION

It is well known that the issue of canonical forms plays an important role in the theory of linear systems. The controller form is frequently used for constructing the state feedback, and the controllability canonical form is determined by the fact that its controllability is given structurally, i.e. independently of the parameters of the system. This definition also means that controllable linear systems can be always transformed into an equivalent controllability canonical form. A comprehensive treatment of the nonlinear controllability and controller normal forms *in the continuous-time case* is given in [1] for scalar input systems of general form, and in [2] for systems, linear in control.

This paper concentrates on controller and controllability canonical forms for single-input *discrete-time* nonlinear systems. Though the results are similar to those

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of the continuous-time nonlinear case, the mathematical tools used are different. We apply the algebraic formalism of one-forms developed in [3] to study accessibility and state feedback linearizability. The conditions are given under which a nonlinear discrete-time system can be transformed into the controller and controllability form by means of a coordinate transformation. Basically, these conditions are given in terms of integrability of certain subspaces of one-forms and factorizability of given functions in a certain way.

Note that the transformation of nonlinear discrete-time systems into the observer canonical form is studied in [4]. Paper [5] presents necessary and sufficient conditions for a discrete-time nonlinear dynamical system to be equivalent to the so-called feedforward form.

2. ALGEBRAIC FORMALISM

Consider a discrete-time single-input single-output nonlinear system Σ described by the equation

$$x(t+1) = f(x(t), u(t)), \tag{1}$$

where $u \in U \subset I\!R$ is the input, $x \in X$, an open subset of $I\!R^n$, is the state, $f : X \times U \to X$ is a real analytic function. In order to be able to use mathematical tools from the algebraic framework, we assume that $f(x,u)$ is generically a submersion, i.e. generically $\mathrm{rank}[\partial f(x,u)/\partial(x,u)] = n$.

Below, we recall some material from [3]. Let $\mathcal{K}$ denote the field of meromorphic functions in a *finite* number of variables $\{x(0), u(t),\ t \geq 0\}$. The forward-shift operator $\delta : \mathcal{K} \to \mathcal{K}$ is defined by $\delta\zeta(x(0), u(0), \ldots, u(N)) = \zeta(f(x(0), u(0)), u(1), \ldots, u(N+1))$, i.e. the forward-shifts of a function can be obtained by substituting the variables by "forward-shifted" variables and, moreover, $x(1)$ is determined by the system equations (1). Under the submersivity assumption the pair $(\mathcal{K}, \delta)$ is a difference field, and up to an isomorphism, there exists a unique difference field $(\mathcal{K}^*, \delta^*)$, called the *inversive closure* of $(\mathcal{K}, \delta)$. Hereinafter we assume that the inversive closure $(\mathcal{K}^*, \delta^*)$ is given and use the same symbol to denote the difference field $(\mathcal{K}, \delta)$ and its inversive closure.

We denote by $\mathcal{E}$ the vector space spanned over $\mathcal{K}$ by the elements of $\mathrm{d}x_i(0)$, $i = 1, \ldots, n$ and $\mathrm{d}u_j(k)$, $j = 1, \ldots, m$, $k \geq 0$, namely $\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_i(0),\ \ i = 1, \ldots, n,\ \ \mathrm{d}u_j(k),\ \ j = 1, \ldots, m,\ \ k \geq 0\}$. Any element in $\mathcal{E}$ is a vector of the form

$$\omega = \sum_{i=1}^{n} \varphi_i \mathrm{d}x_i(0) + \sum_{k \geq 0} \sum_{j=1}^{m} \varphi_{jk} \mathrm{d}u_j(k),$$

where only a finite number of coefficients φ are nonzero elements of $\mathcal{K}$. The elements of $\mathcal{E}$ will be called one-forms. Denote $(\mathrm{d}x_1(0), \ldots, \mathrm{d}x_n(0)) = \mathrm{d}x(0)$. The operator δ induces a forward-shift operator $\Delta : \mathcal{E} \to \mathcal{E}$ by

$$
\begin{aligned}
\Delta\omega &= \sum_{i=1}^{n} \delta\varphi_i \mathrm{d}(\delta x_i(0)) + \sum_{k\geq 0}\sum_{j=1}^{m} \delta\varphi_{jk}\mathrm{d}(\delta u_j(k)) \\
&= \sum_{i=1}^{n} \delta\varphi_i \mathrm{d}f_i(x(0),u(0)) + \sum_{k\geq 0}\sum_{j=1}^{m} \delta\varphi_{jk}\mathrm{d}u_j(k+1),
\end{aligned}
$$

where $f_i(\cdot)$ is the ith component of $f(\cdot)$ in (1).

The relative degree of a one-form $\omega \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x(0)\}$ is defined to be $r = \min\{k \geq 0 \mid \Delta^k\omega \notin \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x(0)\}\}$. If such an integer does not exist, set $r = \infty$. The relative degree of a function $\varphi(x) \in \mathcal{K}$ is defined to be the relative degree of the one-form $\mathrm{d}\varphi(x)$.

Introduce the sequence of subspaces $\mathcal{H}_0 \supset \mathcal{H}_1 \supset \ldots \supset \mathcal{H}_k$ of $\mathcal{E}$ defined by

$$
\begin{aligned}
\mathcal{H}_0 &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x(0), \mathrm{d}u(0)\}, \\
\mathcal{H}_k &= \mathrm{span}_{\mathcal{K}}\{\omega \in \mathcal{H}_{k-1} \mid \Delta\omega \in \mathcal{H}_{k-1}\}, \quad k \geq 1.
\end{aligned}
\tag{2}
$$

There exists an integer $k^* \leq n$ such that $\mathcal{H}_0 \supset \mathcal{H}_1 \supset \ldots \supset \mathcal{H}_{k^*-1} \supset \mathcal{H}_{k^*} = \mathcal{H}_{k^*+1} = \mathcal{H}_\infty$. It is obvious that $\mathcal{H}_k$ is the space of one-forms whose relative degree is greater than or equal to k. The subspaces $\mathcal{H}_k$ are invariant under state diffeomorphism.

Definition 1 [3]. *A function φ is said to be an autonomous element for system* (1) *if there exists an integer ν and a nonzero function $F \in \mathcal{K}$ such that $F(\varphi, \delta\varphi, \ldots, \delta^\nu\varphi) = 0$. System* (1) *is said to be forward accessible if there does not exist any nonzero autonomous element in $\mathcal{K}$.*

Proposition 1 [3]. *System* (1) *is forward accessible if and only if $\mathcal{H}_\infty = \{0\}$.*

3. CONTROLLER CANONICAL FORM

The controller canonical form of a linear system is characterized by the fact that it simplifies the feedback design. The nonlinear controller canonical form

$$
x^*(t+1) = \begin{bmatrix} x_2^*(t) \\ \vdots \\ x_n^*(t) \\ f_n(x^*(t), u(t)) \end{bmatrix}
\tag{3}
$$

can be defined by analogy to the corresponding linear form and continuous-time nonlinear form. Since the function $f_n(x^*(t), u(t))$ contains $f_1 x_1^*(t) + \ldots + f_n x_n^*(t) + u(t)$ as a special case, the nonlinear controller canonical form is consistent with the linear one. The characteristic property of a system in the controller canonical form (3) is that those systems can be linearized by nonlinear static state feedback which obviously simplifies the controller design enormously. By direct computation one can prove the following Proposition.

Proposition 2. *For system* (3) $\mathcal{H}_\infty = \{0\}$, *or alternatively, system* (3) *is forward accessible.*

Proposition 3 [3]. *Suppose* $\mathcal{H}_\infty = \{0\}$. *Then there exists a basis* $\{\omega_i, 1 \le i \le n\}$ *of* $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}x(t)\}$ *such that tangent linearized system corresponding to* (1)

$$\mathrm{d}x(t+1) = \frac{\partial f(\cdot)}{\partial x(t)}\mathrm{d}x(t) + \frac{\partial f(\cdot)}{\partial u(t)}\mathrm{d}u(t)$$

yields the infinitesimal Brunovsky form

$$\begin{aligned}
\omega_1(t+1) &= \omega_2(t),\\
&\vdots\\
\omega_{n-1}(t+1) &= \omega_n(t),\\
\omega_n(t-1) &= \sum_{i=1}^{n} a_i\omega_i(t) + b\mathrm{d}u(t),
\end{aligned}$$

where $\omega_1 \in \mathcal{H}_n$, $a_i \in \mathcal{K}$, $b \in \mathcal{K}$, *and* $b \not\equiv 0$.

It is obvious that the infinitesimal form can be transformed into controller canonical form if and only if the one-form ω_1 can be integrated (perhaps after multiplying by an integrating factor). In [3], where the static state feedback linearizability problem was studied, the necessary and sufficient conditions were given implicitly for system (1) to be transformable into the nonlinear controller form and the constructive algorithm was given (up to finding the integrating factor and integrating the one-form) to compute the state transformations. These conditions are recalled in Proposition 4.

Proposition 4. *System* (1) *is transformable via state transformation into the controller canonical form if and only if*

(i) $\mathcal{H}_{n+1} = \mathcal{H}_\infty = \{0\}$, *or alternatively, system* (1) *is forward accessible*;

(ii) $\mathcal{H}_1, \ldots, \mathcal{H}_n$ *are completely integrable.*

It is obvious from the above theorem that in the nonlinear case, unlike the linear case, not every accessible system can be transformed into the controller canonical form. For this to be possible, additional restrictive integrability assumptions should be satisfied.

4. CONTROLLABILITY CANONICAL FORM

The notion "controllability canonical form" originates from the fact that the associated controllability matrix equals the unity matrix I_n, i.e. is given structurally. According to this, controllability no longer depends on the parameters of the system. This definition also means that controllable linear systems can be transformed into an equivalent controllability canonical form.

The discrete-time nonlinear controllability normal form

$$z(t+1) = \begin{bmatrix} & - & a_1(z_n(t), u(t)) \\ z_1(t) & - & a_2(z_n(t)) \\ & \vdots & \\ z_{n-1}(t) & - & a_n(z_n(t)) \end{bmatrix} \tag{4}$$

can be introduced by analogy to the corresponding linear and continuous-time forms. The functions $a_1(z_n, u)$, and $a_i(z_n),\ i = 2, \ldots, n$, which are assumed to be analytic, contain the expressions $a_1 z_n + u$ and $a_i z_n,\ i = 2, \ldots, n$, of the linear form as a special case. So the nonlinear controllability normal form meets the *consistency* criterion.

The structurally given accessibility of (4) follows from Proposition 5 below. It also reveals the fact that, unlike the linear case, accessibility is not enough to be able to transform the system into controllability canonical form.

Proposition 5. *For nonlinear systems in the controllability form* (4), *the following statements hold*:

(i) *the system is (forward) accessible*;

(ii) *The subspaces $\mathcal{H}_k$, $k = 1, \ldots, n$, are completely integrable.*

Proof. Compute the subspaces $\mathcal{H}_1, \ldots, \mathcal{H}_\infty$. By definition $\mathcal{H}_1 = \mathrm{span}_\mathcal{K}\{\mathrm{d}z_1(t), \ldots, \mathrm{d}z_n(t)\}$. To compute $\mathcal{H}_2$, we have to find

$$\begin{aligned}\mathcal{H}_1^+ = \mathrm{span}_\mathcal{K}\Big\{ -\frac{\partial a_1}{\partial z_n}\mathrm{d}z_n(t) - \frac{\partial a_1}{\partial u}\mathrm{d}u(t), \mathrm{d}z_1(t) - \frac{\partial a_2}{\partial z_n}\mathrm{d}z_n(t), \ldots, \mathrm{d}z_{n-1}(t) \\ - \frac{\partial a_n}{\partial z_n}\mathrm{d}z_n(t)\Big\}.\end{aligned}$$

By (2),

$$\begin{aligned}\mathcal{H}_2 &= \Delta^{-1}(\mathcal{H}_1 \cap \mathcal{H}_1^+) \\ &= \Delta^{-1}\mathrm{span}_\mathcal{K}\Big\{\mathrm{d}z_1(t) - \frac{\partial a_2}{\partial z_n}\mathrm{d}z_n(t), \ldots, \mathrm{d}z_{n-1}(t) - \frac{\partial a_n}{\partial z_n}\mathrm{d}z_n(t)\Big\} \\ &= \mathrm{span}_\mathcal{K}\{\mathrm{d}z_2(t), \ldots, \mathrm{d}z_n(t)\}.\end{aligned}$$

Continuing analogously, we get $\mathcal{H}_k = \Delta^{-1}(\mathcal{H}_{k-1} \cap \mathcal{H}_{k-1}^+) = \mathrm{span}_\mathcal{K}\{\mathrm{d}z_k(t), \ldots, \mathrm{d}z_n(t)\},\ k = 3, \ldots, n$, and finally, $\mathcal{H}_{n+1} = \{0\}$. □

A simple consequence of Proposition 5 is the following conclusion.

Conclusion 1. *The controllability normal form* (4) *can be transformed into the controller normal form* (3).

Below, we look for conditions under which it is possible to transform system (1) into the controllability canonical form, as well as for a state transformation $z = \zeta(x)$ that brings system (1) into the controllability form (4).

From inspecting Eqs. (4) it is clear that the crucial step is to find the last state coordinate z_n. The other coordinates can be defined recursively from the formulae

$$\begin{aligned} z_{n-1}(t) &= z_n(t+1) - a_n(z_n(t)), \\ &\vdots \\ z_1(t) &= z_2(t+1) - a_2(z_n(t)). \end{aligned}$$

The key factor in defining $z_n(t)$ is to note that its relative degree is n. So, to find $z_n(t)$, we consider one-form ω in $\mathcal{H}_n$, which contains one-forms of degree n iff $\mathcal{H}_{n+1} = \{0\}$, i.e. if the system is accessible. But to define z_n, we have to be able to integrate the one-form, i.e. $\mathcal{H}_n$ has to be integrable. In that case we have (perhaps after multiplying ω by an integrating factor) $\omega = \mathrm{d}\xi(x)$.

The converse of Conclusion 1 is not true. A nonlinear controller canonical form (again, unlike the linear case) cannot be necessarily transformed into the controllability canonical form. For this to be possible, an additional condition must be met.

Proposition 6. *The controller canonical form* (3) *is transformable into the controllability canonical form* (4) *if* $f_n(x^*, u)$ *is structurally additive*

$$f_n(x^*, u) = -a_1(x_1^*, u) - a_2(x_2^*) - \ldots - a_n(x_n^*). \tag{5}$$

Proof. If (5) holds, then the state transformation

$$\begin{aligned} z_n &= x_1^*, \\ z_{n-1} &= x_2^* + a_n(x_1^*), \\ z_{n-2} &= x_3^* + a_{n-1}(x_1^*) + a_n(x_2^*), \\ &\vdots \\ z_1 &= x_n^* + a_2(x_1^*) + \ldots + a_n(x_{n-1}^*) \end{aligned}$$

brings the state equations into the form (4). □

The following example shows that the condition in Proposition 6 is not a necessary condition.

Example 1.

$$\begin{aligned} x_1^*(t+1) &= x_2^*(t), \\ x_2^*(t+1) &= x_3^*(t), \\ x_3^*(t+1) &= \exp\{x_1^*(t)u(t) + x_2^*(t) + x_3^*(t)\}. \end{aligned} \tag{6}$$

Obviously, $f_n(x^*, u) = \exp\{x_1^* u + x_2^* + x_3^*\}$ does not satisfy condition (5). However, an equivalent controller canonical form which satisfies (5) is possible for (6) under the state transformation $\tilde{x}_i = \ln x_i^*$, $i = 1, 2, 3$:

$$\begin{aligned}
\tilde{x}_1(t+1) &= \tilde{x}_2(t), \\
\tilde{x}_2(t+1) &= \tilde{x}_3(t), \\
\tilde{x}_3(t+1) &= u(t)\exp\{\tilde{x}_1(t)\} + \exp\{\tilde{x}_2(t)\} + \exp\{\tilde{x}_3(t)\}.
\end{aligned}$$

Suppose we pick $\mathrm{d}\xi(x) \in \mathcal{H}_n$ to define $x_1^* = \xi(x)$. We may find that the corresponding $f_n(x^*, u)$ does not satisfy (5). This will not necessarily be so for other possible choices of $\tilde{x}_1 = g(x_1^*)$, as seen from the example above. This choice defines $\tilde{f}_n(\tilde{x}, u)$ as follows:

$$\tilde{f}_n(\tilde{x}, u) = g[f_n(g^{-1}(\tilde{x}_1), \ldots, g^{-1}(\tilde{x}_n), u)].$$

The question whether or not (3) can be transformed into controllability form may be reduced to the question whether or not

$$g[f_n(g^{-1}(\tilde{x}_1), \ldots, g^{-1}(\tilde{x}_n), u)] = -a_1(\tilde{x}_1, u) - a_2(\tilde{x}_2) - \ldots - a_n(\tilde{x}_n)$$

or, alternatively, the function f_n in (3) may be written in the special form

$$f_n(x^*, u) = g^{-1}[-a_1(g(x_1^*), u) - a_2(g(x_2^*)) - \ldots - a_n(g(x_n^*))].$$

Proposition 7. *The controller canonical form* (3) *is transformable into the controllability form* (4) *iff it is possible to find an invertible function* $g(\cdot)$ *such that the composition* $g \circ f_n$ *is structurally additive*

$$g[f_n(x^*, u)] = -a_1(g(x_1^*), u) - a_2 g(x_2^*)) - \ldots - a_n(g(x_n^*)). \tag{7}$$

Proof. Sufficiency. Under (7) the state transformation

$$\begin{aligned}
z_n &= g(x_1^*), \\
z_{n-1} &= g(x_2^*) + a_n(g(x_1^*)), \\
&\vdots \\
z_1 &= g(x_n^*) + a_2(g(x_1^*)) + \ldots + a_n(g(x_{n-1}^*))
\end{aligned}$$

brings the state equations into the form (4).

Necessity. Assume now that (3) is transformable into (4). Since $\mathrm{d}z_n \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1^*\} = \mathcal{H}_n$ and by (4), $\delta \mathrm{d}z_n = \mathrm{d}z_{n-1} - \mathrm{d}a_n(z_n)$, we have $\mathrm{d}z_{n-1} = \delta \mathrm{d}z_n + \mathrm{d}a_n(z_n) = \mathrm{d}g(x_2^*) + \mathrm{d}a_n(g(x_1^*))$. Following in a similar manner, we get for $k = 2, \ldots, n-1$:

$$\mathrm{d}z_{n-k} = \mathrm{d}g(x_{k+1}^*) + \mathrm{d}a_n(g(x_k^*)) + \ldots + \mathrm{d}a_{n-k+1}(g(x_1^*)). \tag{8}$$

Finally, calculating $\delta \mathrm{d}z_1$ from (8), we get

$$\delta \mathrm{d}z_1 = \mathrm{d}g(f_n(x^*, u)) + \mathrm{d}a_n(g(x_n^*)) + \ldots + \mathrm{d}a_2(g(x_2^*)),$$

which by (4) has to be equal to $\delta \mathrm{d}z_1 = -\mathrm{d}a_1(g(x_1^*), u)$, yielding (7).

5. CONCLUSIONS

Necessary and sufficient conditions are given under which discrete-time nonlinear systems can be transformed generically into a controller or controllability form by means of a state transformation. The first condition requires that, as in the linear case, the system is forward accessible. The second restrictive condition is formulated in terms of integrability of certain subspaces of one-forms, classified according to their relative degree, and indicates that every accessible system cannot be transformed into the controller or controllability form. Moreover, unlike the linear case, every controller form cannot be transformed into the controllability form. For this to be possible, in addition to the above mentioned conditions, a certain type of structural additivity is required from the controller form.

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Kontrolleri ja juhitavuse kanoonilised kujud mittelineaarsete diskreetsete süsteemide jaoks

Ülle Kotta

On leitud tarvilikud ja piisavad tingimused, mille täidetuse korral on diskreetse ajaga mittelineaarseid süsteeme võimalik olekuteisendusega viia kontrolleri või juhitavuse kanoonilisele kujule. Esimene tingimustest nõuab sarnaselt lineaarse juhuga, et süsteem oleks juhitav. Teine kitsendav tingimus on formuleeritud süsteemiga seotud teatud üksvormide integreeruvuse kaudu. Teisest tingimusest järeldub, et mitte iga juhitav süsteem ei ole viidav kanoonilistele kujudele. Erinevalt lineaarsest juhust ei ole ka iga kontrolleri kanoonilisel kujul olev süsteem viidav juhitavuse kanoonilisele kujule. Et see võimalikuks osutuks, peab lisaks eelnevale kahele tingimusele olema täidetud veel teatud aditiivse faktoriseeritavuse tingimus.