Proc. Estonian Acad. Sci. Phys. Math., 2005, **54**, 1, 46–54

Languages recognized by two-sided automata of graphs

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Received 11 January 2002, in revised form 24 November 2004

Abstract. We introduce two-sided automata defined by directed graphs and describe all languages recognized by these automata.

Key words: labelled graphs, finite state automata, formal languages.

Throughout the paper the word *graph* means a finite directed graph without multiple edges but possibly with loops. Graphs and various objects derived from them are essential tools that have been actively used in different branches of modern mathematics and theoretical computer science. It makes sense to consider new ways of defining classical finite state acceptors using graph labellings, and determine how properties of the acceptors depend on the properties of the original graph labelling. The present paper continues the investigation initiated in [1−3], where the concept of a graph algebra has been used in order to define finite state automata. We introduce a common generalization of two earlier constructions of automata considered in the papers mentioned above and investigate properties of languages accepted by these automata.

We use standard concepts of automata and languages theory following [4−7]. A *language* over an alphabet X is a subset of the free monoid X^* generated by X. Let $D = (V, E)$ be a graph, $\ell : X \to \{+, -\}$ and $f : X \to V$ any mappings, and let T be a subset of $V \cup \{1\}$. The *two-sided automaton* $\mathrm{Atm}(D) = \mathrm{Atm}(D, T, f, \ell)$ of the graph D is the finite state acceptor with

(DA1) the set of states $V \cup \{1\}$;

(DA2) the initial state 1;

(DA3) the set of terminal states T;

(DA4) the next-state function given by

$$a \cdot x = \begin{cases} f(x) & \text{if } \ell(x) = + \text{ and } (a, f(x)) \in E, \text{ or if } a = 1, \\ a & \text{if } \ell(x) = - \text{ and } (f(x), a) \in E, \end{cases}$$

for a state a and $x \in X$.

Thus the edges of the transition diagram of $\mathrm{Atm}(D)$ are also edges or reversed edges of the graph D.

Our first main theorem describes all languages recognized by two-sided automata of graphs in terms of combinatorial properties satisfied by these languages.

Theorem 1. *For every language L over an alphabet X, the following conditions are equivalent*:

(i) *there exists a directed graph D such that L is recognized by a two-sided automaton of D*;

(ii) *there exist two disjoint subsets X_- and X_+ of X such that $X = X_- \dot\cup X_+$ and, for all $x \in X_+$, $y \in X_-$, $z \in X$, and $u, v \in X^*$, the following implications hold*:

(a) $zxu \in L$ *implies* $xu \in L$,

(b) $xu, zxv \in L$ *implies* $zxu \in L$,

(c) $zyu \in L$ *implies* $zy^*u \in L$,

(d) $zv, zyu \in L$ *implies* $zyv \in L$,

(a) $xyu \in L$ *if and only if* $yxu \in L$.

The second main theorem describes all languages recognized by two-sided automata of graphs in terms of regular expressions for their complements. Given a relation $G \subseteq X \times X$, put

$$G^{-1} = \{(x, y) \mid (y, x) \in G\}.$$

Theorem 2. *For every language L over an alphabet X, the following conditions are equivalent*:

(i) *there exists a directed graph D such that L is recognized by a two-sided automaton of D*;

(ii) *there exist a subset X_T of X, disjoint subsets X_- and X_+ of X, and relations $G_1 \subseteq X_+ \times X_+$, $G_2 \subseteq X_- \times X_-$, and $G_3 \subseteq X_- \times X_+$ such that $X = X_- \dot\cup X_+$ and the language $X^+ \setminus L$ has the following regular expression*:

$$(X_N \cap X_-)X_-^* + X^*(X_N \cap X_+)X_-^* \\ + \sum_{(x_i,x_j)\in G_1 \cup G_3^{-1}} X^* x_i X_-^* x_j X^* + \sum_{(x_i,x_j)\in G_2 \cup G_3} x_i X_-^* x_j X^*, \qquad (1)$$

where X_N stands for $X \setminus X_T$.

The concept of two-sided automata of graphs provides a common generalization of two earlier constructions of automata introduced in [1−3] and motivated by the study of graph algebras and their applications. Graph algebras have been investigated by many authors (see, for example, [8−15]) in relation to various problems of discrete mathematics and computer science. Let $D = (V, E)$ be a graph. The *graph algebra* $\mathrm{Alg}(D)$ associated with D is the set $V \cup \{0\}$ equipped with multiplication defined by the rule

$$xy = \begin{cases} x & \text{if } x, y \in V \text{ and } (x, y) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The languages recognized by these automata have interesting combinatorial properties (see [1,2]). Our main theorem strengthens and generalizes several earlier results of the papers cited above. A routine verification shows that the next-state function of the automaton $\mathrm{Atm}(D, T, f, \ell)$ can be defined by the graph algebra $\mathrm{Alg}(D')$, where $D' = (V, E^{-1})$, if we use the following condition equivalent to (DA4):

(DA5) for $a \in \mathrm{Alg}(D) \cup \{1\}$, $x \in X$,

$$a \cdot x = \begin{cases} f(x)a & \text{if } \ell(x) = + \\ af(x) & \text{if } \ell(x) = -. \end{cases}$$

This means that x acts as a left multiplication by the element $f(x)$ in $\mathrm{Alg}(D')$ if $\ell(x) = +$, and as a right multiplication by $f(x)$ if $\ell(x) = -$. It follows that our new definition embraces the automata involving only right multiplications (see [1]) and the ones using only left multiplications (see [2]).

Note that a language L is accepted by $\mathrm{Atm}(D, T, f, \ell)$ if and only if $L \setminus \{1\}$ is accepted by

$$\mathrm{Atm}(D, T \setminus \{1\}, f, \ell).$$

As usual, $X^+ = X^* \setminus \{1\}$ stands for the free semigroup generated by X. Since conditions (ii) of our theorems remain unchanged if we replace L by $L \setminus \{1\}$, in both proofs we may assume that $L \subseteq X^+$.

Proof of Theorem 1. (i)$\Rightarrow$(ii): Suppose that the language L is recognized by the automaton $\mathrm{Atm}(D, T, f, \ell)$ of some graph $D = (V, E)$. Define

$$X_+ = \{x \in X \mid \ell(x) = +\},$$
$$X_- = \{x \in X \mid \ell(x) = -\},$$

and consider arbitrary elements $x, y \in X$ with $\ell(x) = +$ and $\ell(y) = -$.

To verify implication (a), take any word zxu in L, where $u \in X^*$ and $z \in X$. Since $1 \cdot zxu$ is defined and belongs to T, it follows that $(f(z), f(x)) \in E$. Besides, $f(z) \cdot x = f(x)$, because $\ell(x) = +$. Therefore $1 \cdot xu = f(x) \cdot u = (f(z) \cdot x) \cdot u = 1 \cdot zxu \in T$. Hence $xu \in L$, as required.

Now, pick arbitrary elements xu and zxv in L such that $u, v \in X^*$ and $z \in X$. Since $1 \cdot zxv$ is defined, we get $(f(z), f(x)) \in E$. Therefore $1 \cdot zxu = (f(z) \cdot f(x)) \cdot u = f(x) \cdot u = 1 \cdot xu \in T$, and so $zxu \in L$. Thus implication (b) holds, too.

Next, we verify implication (c). To this end, take a word zyu in L, where $u \in X^*$ and $z \in X$. Since $1 \cdot zyu$ is defined and $\ell(y) = -$, we get $(f(y), f(z)) \in E$. Hence $1 \cdot zy^k u = f(z) \cdot u = (f(z) \cdot f(y)) \cdot u = 1 \cdot zyu \in T$, for all $k \in \mathbf{N}_0$. Therefore $zy^*u \in L$, as required.

Consider any elements zv and zyu in L, where $u, v \in X^*$ and $z \in X$. Clearly, $zyu \in L$ implies $(f(y), f(z)) \in E$, because $\ell(y) = -$. Hence $1 \cdot zyv = (f(z)f(y)) \cdot v = f(z) \cdot v = 1 \cdot zv \in T$, and so $zyv \in L$. We see that (d) holds.

Finally, let us prove the equivalence (e). Take an arbitrary $u \in X^*$ such that $xyu \in L$. It follows that $(f(y), f(x)) \in E$. Implication (c) yields us $xu \in L$. Therefore $1 \cdot yxu = (f(y) \cdot f(x)) \cdot u = f(x) \cdot u = 1 \cdot xu \in T$; whence $yxu \in L$. To verify the converse implication, suppose that $yxu \in L$. Hence $(f(y), f(x)) \in E$. Besides, $xu \in L$ in view of (a). Therefore $1 \cdot xyu = (f(x) \cdot f(y)) \cdot u = f(x) \cdot u = 1 \cdot xu \in T$. This means that $yxu \in L$. Thus the whole condition (ii) holds.

(ii)$\Rightarrow$(i): Condition (ii) involves the sets X_+ and X_-. Let us label x by $\ell(x) = +$ if $x \in X_+$, and by $\ell(x) = -$ otherwise. Introduce a graph $G = (V, E)$ with the set $V = X$ of vertices, and the set $E = E_1 \cup E_2$ of edges where

$$\begin{aligned} E_1 &= \{(x, y) \in X \times X_+ \mid xyw \in L \text{ for some } w \in X^*\}, \\ E_2 &= \{(x, y) \in X_- \times X \mid yxw \in L \text{ for some } w \in X^*\}. \end{aligned}$$

Let f be the mapping from X to V defined by $f(x) = x$. Put $T = X \cap L$.

By induction on the length n of a word u we show that $u \in L$ if and only if u is recognized by the automaton $\mathrm{Atm}(D, T, f, \ell)$. If $n = 1$, then this follows immediately from the definition of T. Suppose that the claim is true for all words v of length less than n, where $n > 1$. Take an arbitrary word $u = x_1 \cdots x_n$ in L. We claim that then $\mathrm{Atm}(D, T, f, \ell)$ accepts u.

First, consider the case where $x_2 \in X_+$. Implication (a) shows that $x_2 \cdots x_n \in L$, and by the induction hypothesis $x_2 \cdots x_n$ is accepted by $\mathrm{Atm}(D)$.

Note that $(f(x_1), f(x_2)) \in E_1$ by the definition of $E_1 \subseteq E$. It follows that $1 \cdot u = f(x_1) \cdot x_2 \cdots x_n = f(x_2) \cdots f(x_n) = 1 \cdot x_2 \cdots x_n \in T$. This means that the $\mathrm{Atm}(D)$ recognizes u.

Second, consider the case where $x_2 \in X_-$. Then (c) shows that $x_1 x_3 \cdots x_n \in L$. By the induction hypothesis $x_1 x_3 \cdots x_n$ is recognized by $\mathrm{Atm}(D)$. By the definition of E_2, we get $(f(x_2), f(x_1)) \in E_2$. Therefore $1 \cdot u = f(x_1) \cdot x_2 \cdots x_n = f(x_1) \cdot x_3 \cdots f(x_n) = 1 \cdot x_1 x_3 \cdots x_n \in T$. Hence $\mathrm{Atm}(D)$ recognizes u, in all cases. Thus the language accepted by $\mathrm{Atm}(D)$ contains L.

Conversely, suppose that our word $u = x_1 \cdots x_n$ is accepted by the automaton $\mathrm{Atm}(D, T, f, \ell)$. We are going to show that u belongs to L.

First, consider the case where $\ell(x_2) = +$. Clearly, $(f(x_1), f(x_2)) \in E$. Hence we get

$$
\begin{array}{rcl}
1 \cdot x_2 \cdots x_n & = & (f(x_2) \cdot f(x_3)) \cdots f(x_n) \\
& = & (f(x_1) \cdot f(x_2)) \cdots f(x_n) \\
& = & 1 \cdot u \in T.
\end{array}
$$

By the induction hypothesis $x_2 \cdots x_n \in L$. Since $x_2 \in X_+$, we get $(f(x_1), f(x_2)) \in E_1 \cup E_2$.

If $(f(x_1), f(x_2)) \in E_1$, then $x_1 x_2 w \in L$, for some $w \in X^*$. Therefore implication (b) gives us $x_1 x_2 \cdots x_n \in L$.

If, however, $(f(x_1), f(x_2)) \in E_2$, then $x_1 \in X_-$ and $x_2 x_1 w \in L$ for some $w \in X^*$. By the equivalence (e) we get $x_1 x_2 w \in L$, and (d) implies $x_1 x_2 \cdots x_n \in L$, again.

Second, suppose that $\ell(x_2) = -$. Then $(f(x_2), f(x_1)) \in E = E_1 \cup E_2$. We get $1 \cdot x_1 x_3 \cdots x_n = (f(x_1) \cdot f(x_3)) \cdots f(x_n) = (f(x_1) \cdot f(x_2)) \cdots f(x_n) = 1 \cdot u \in T$. By the induction hypothesis $x_1 x_3 \cdots x_n \in L$.

If $(f(x_2), f(x_1)) \in E_2$, then $x_1 x_2 w \in L$, for some $w \in X^*$. Hence implication (b) yields $x_1 x_2 \cdots x_n \in L$.

If, however, $(f(x_2), f(x_1)) \notin E_2$, then $(f(x_2), f(x_1)) \in E_1$. Therefore $x_1 \in X_-$ and we get $x_2 x_1 w \in L$, for some $w \in X^*$. Hence $x_1 x_2 w \in L$, in view of (e). Implication (d) gives us $x_1 x_2 \cdots x_n \in L$, again. Thus we have proved that L is precisely the language recognized by $\mathrm{Atm}(D)$. This completes the proof. $\square$

Proof of Theorem 2. (i)⇒(ii): Suppose that L is recognized by the automaton $\mathrm{Atm}(D)$ induced by a graph $D = (V, E)$ with functions ℓ and f. Let us introduce the sets

$$
\begin{array}{rcl}
X_N & = & \{x \in X \mid f(x) \notin T\}, \\
X_+ & = & \{x \in X \mid \ell(x) = +\}, \\
X_- & = & \{x \in X \mid \ell(x) = -\},
\end{array}
$$

and the relations

$$
\begin{aligned}
G_1 &= \{(x,y) \in X_+ \times X_+ \mid (f(x), f(y)) \notin E\}, \\
G_2 &= \{(x,y) \in X_- \times X_- \mid (f(y), f(x)) \notin E\}, \\
G_3 &= \{(x,y) \in X_- \times X_+ \mid (f(x), f(y)) \notin E\}.
\end{aligned}
$$

We claim that $X^+ \setminus L$ is equal to the language defined by the regular expression (1). To prove one inclusion, take any element $u \in X^+ \setminus L$, say $u = x_1 \cdots x_n$, where $n \geq 1$ and $x_1, \ldots, x_n \in X$, and consider two possible cases.

Case 1: $1 \cdot u$ is defined. If $u \in X_-^*$, then $1 \cdot u = f(x_1) \notin T$. We have $u \in x_1 X_-^* \subseteq (X_- \cap X_N) X_-^*$, and so u belongs to the language (1), as required. Further, we may assume that some letters in u are labelled by $+$. Let x_k be the last letter in u of this kind. We get $1 \cdot u = f(x_k) \notin T$, and hence $u \in X^* x_k X_-^* \subseteq X^*(X_+ \cap X_N) X_-^*$. Therefore u lies in the language defined by (1), again.

Case 2: $1 \cdot u$ is undefined. Let j be the largest integer such that $1 \leq i \leq n$ and $1 \cdot x_1 \cdots x_j$ is defined. This index j always exists because $1 \cdot x_1 = f(x_1)$.

Subcase 2.1: $x_1 \cdots x_j \in X_-^*$. Then $1 \cdot x_1 \cdots x_j = f(x_1)$. Since $1 \cdot u$ is undefined, it follows that $(f(x_k), f(x_1)) \notin E$ with $\ell(x_k) = -$, or $(f(x_1), f(x_k)) \notin E$ with $\ell(x_k) = +$, for some $j < k \leq n$. Therefore $u \in x_1 X_-^* x_k X^*$ with $(x_1, x_k) \in G_2 \cup G_3$.

Subcase 2.2: Some letters in $x_1 \cdots x_j$ are labelled by $+$. Let x_i be the last letter in $x_1 \cdots x_j$ labelled by $+$. We get $1 \cdot x_1 \cdots x_j = 1 \cdot x_1 \cdots x_i = f(x_i)$. Since $1 \cdot u$ is undefined, it follows that there exists $j < k \leq n$ such that $(f(x_k), f(x_i)) \notin E$ with $\ell(x_k) = -$, or $(f(x_i), f(x_k)) \notin E$ with $\ell(x_k) = +$. Hence $u \in X^* x_i X_-^* x_k X^*$ with $(x_i, x_k) \in G_3^{-1} \cup G_1$.

Thus in all cases u belongs to the language (1). This completes the proof of one inclusion.

To prove the reverse inclusion, consider an arbitrary element $u = x_1 \cdots x_n$ of the language defined by (1). Obviously, $u \neq 1$. If $1 \cdot u$ is undefined, then $u \in X^+ \setminus L$, and we are done. Further, we may assume that $1 \cdot u$ is defined.

If $u \in x X_-^* y X^*$, for some $(x,y) \in G_2 \cup G_3$, then $1 \cdot u$ is undefined. Similarly, if $u \in X^* x X_-^* y X^*$, for $(x,y) \in G_1 \cup G_3^{-1}$, then $1 \cdot u$ is undefined, too. Therefore we may exclude these two summands of (1) from further consideration: u does not belong to them.

If $u \in (X_- \cap X_N) X_-^*$, or $u \in X^*(X_+ \cap X_N) X_-^*$, then $1 \cdot u \notin T$, by the definition of X_-. Therefore $u \in X^+ \setminus L$, again. Thus $X^+ \setminus L$ coincides with the language given by the regular expression (1).

(ii)⇒(i): Let L be a language with the complement $X^+ \setminus L$ defined by the regular expression (1). Introduce a graph $G = (V, E)$ with the set $V = X$ of vertices, and the set E of edges consisting of all pairs (x,y) which are not in $G_1 \cup G_2^{-1} \cup G_3$. Let f be the mapping from X to V defined by $f(x) = x$. Put $\ell(x) = +$ for all $x \in X_+$, and $\ell(x) = -$ for all $x \in X_+$. Set $T = V \setminus X_N$. Denote by R the language recognized by $\operatorname{Atm}(G, T, f, \ell)$. Since $1 \notin T$, we get $1 \notin R$, and so further we have to deal only with words in X^+. In order to show that $L = R$, we verify the equality $X^+ \setminus L = X^+ \setminus R$.

Take an arbitrary element $u = x_1 \dots x_n \in X^+ \setminus R$, where $n \geq 1$ and $x_1, \dots, x_n \in X$. We are going to show that $u \in X^+ \setminus L$.

First, suppose that $1 \cdot u$ is defined. If $u \in X_-^*$, then $1 \cdot u = x_1$, and $x_1 \notin T$ because $u \notin R$. We have $u \in x_1 X_-^* \subseteq (X_- \cap X_N) X_-^*$, and so u belongs to the language (1), as required. Further, we may assume that some letters in u are labelled by $+$. Let x_k be a $+$-labelled letter in u of this kind. We get $1 \cdot u = x_k \notin T$, and so $u \in X^* x_k X_-^* \subseteq X^*(X_+ \cap X_N) X_-^*$. Thus we see that $u \in X^+ \setminus L$.

Second, suppose that $1 \cdot u$ is undefined. Let i be the last index such that $1 \cdot (x_1 \cdots x_i) = x_i$. If some letters on the right of x_i in u are labelled by $+$, then we consider the nearest letter x_{i+k} be of this kind. Since $1 \cdot (x_1 \cdots x_{i+k})$ is undefined, we get $(x_i, x_{i+k}) \notin E$ or $(x_{i+j}, x_i) \notin E$, for some $j < k$. If, however, all letters on the right of x_i in u are labelled by $-$, we get $(x_{i+j}, x_i) \notin E$, for some $j > 0$, because $1 \cdot u$ is undefined. Therefore, we always have $(x_i, x_{i'}) \in G_1 \cup G_3$, for some $i' > i$, or $(x_{i''}, x_i) \in G_2^{-1} \cup G_3$, for some $i'' > i$.

Assume that $\ell(x_i) = -$. Let j be the index of the first occurrence of x_i in u. We have $x_j = x_i$ and all letters between x_j and x_i are labelled by $-$, because $1 \cdot x_1 \cdots x_i = x_i$. It easily follows that $j = 1$, and so $(x_1, x_{i'}) \in G_3$, for some $i' > 1$ or $(x_{i''}, x_1) \in G_2^{-1}$, for some $i'' > 1$. Hence $u \in x_1 X_-^* y X^*$, for some $(x_1, y) \in G_2 \cup G_3$.

Assume now that $\ell(x_i) = +$, then $(x_i, x_{i'}) \in G_1$, for some $i' > i$ or $(x_{i''}, x_i) \in G_3$, for some $i'' > i$. Therefore we get $u \in X^* x_i X_-^* y X^*$, for some $(x, y) \in G_1 \cup G_3^{-1}$. Thus u is given by the regular expression (1) and so $u \in X^+ \setminus L$. Thus we see that $u \in X^+ \setminus L$ in any case.

We have proved that $X^+ \setminus L \supseteq X^+ \setminus R$. To verify the reverse inclusion, consider an arbitrary element $u = x_1 \cdots x_n$ of the language $X^+ \setminus L$ defined by (1).

First, suppose that $u \in x X_-^* y X^*$ with $(x, y) \in G_2 \cup G_3$. If $(x, y) \in G_2$, then $\ell(y) = +$ and there is no edge (x, y). If $(x, y) \in G_3$, then $\ell(y) = -$ and there is no edge (y, x). In both cases $1 \cdot u$ is undefined.

Second, suppose that $u \in X^* x X_-^* y X^*$ with $(x, y) \in G_1 \cup G_3^{-1}$. If $(x, y) \in G_1$, then $\ell(y) = +$ and there is no edge (x, y). If $(x, y) \in G_3^{-1}$, then $\ell(y) = -$ and there is no edge (y, x). It follows that $1 \cdot u$ is undefined in any case, and so u is not accepted by $\operatorname{Atm}(D, T)$.

Third, suppose that $u \in (X_- \cap X_N) X_-^*$. Then $1 \cdot u = f(x_1) \notin T$, and therefore $u \notin R$.

Finally, assume that $u \in X^*(X_+ \cap X_N) X_-^*$. If $1 \cdot u$ is defined, then $1 \cdot u = f(x_k) \notin T$, and so u is not accepted by $\operatorname{Atm}(D, T)$. If, however, $1 \cdot u$ is undefined, then u does not belong to R, either.

Thus $X^+ \setminus L \subseteq X^+ \setminus R$. This completes the proof. $\square$

Remark 3. Condition (ii) of Theorem 2 can be expressed in the equivalent form using a grammar generating the language $X^+ \setminus L$. Indeed, the standard method gives us a grammar which generates the language described by the regular expression of the form (1) (see, for example, the proof of Proposition 6.2.3 in [16]).

Remark 4. If one of the sets X_+ or X_- is empty, then our proof shows that condition (ii) of Theorem 2 remains in force, and all summands of (1) involving empty sets vanish.

Corollary 5. *It is decidable whether a regular language is recognizable by two-sided automata of graphs.*

Proof follows from condition (ii) of Theorem 2. Indeed, given a regular language L, we can find a finite automaton accepting it, and then find a regular expression for the language $X^+ \setminus L$. After that, it remains to consider all subsets X_T of X, partitions $X = X_- \dot{\cup} X_+$, and relations $G_1 \subseteq X_+ \times X$, $G_2 \subseteq X_- \times X$, and use well-known algorithms to verify whether $X^+ \setminus L$ is equal to the regular language defined by the regular expression (1). □

ACKNOWLEDGEMENTS

The third author was supported by grant No. 4912 of the Estonian Science Foundation and by the Fulbright fellow award. The authors are grateful to two referees for corrections and comments that have helped to improve the text.

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Graafide abil defineeritud kahepoolsete automaatide poolt äratuntavad keeled

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Käesolevas artiklis on väljas uuritud kahepoolseid automaate, mis defineeritakse orienteeritud graafide abil. Kirjeldatakse kõiki keeli, mis on nende automaatide abil äratuntavad.