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On informational properties of covers

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Abstract. Any partition on a finite set gives us partial information about an object represented by this set. This algebraic approach to information has proved very useful for solving various problems concerning structural complexity of finite objects. Like a partition, a cover on a finite set can be interpreted as information about structural complexity of an object represented by this cover. But unlike partitions, the quantitative measure of the information content of covers is not yet established. In this paper an effort is made to provide covers with a quantitative measure of their information content from the axiomatic point of view, by finding an extropy for a special class of covers which reflect the information difference between two partitions. This result breaks ground in providing all relevant for practice covers with a quantitative measure, which would make it possible to evaluate the structural complexity for a wide range of finite objects.

Key words: entropy, extropy, partition, cover.

Information and energy are two of the most fundamental concepts of modern science, forming the notional basis of a theoretical background for almost any branch of human activity, but there is a notion linking together energy and information. This notion is extropy[1], measuring both the quality of energy and the quantity of information, and on top of it playing the central role in ecology. The broad extropy concept has very many different applications in various fields of science. The classical statistical foundation of extropy is well known. However, the probabilistic approach for the interpretation of extropy as a measure of information about structural complexity is unnatural and logically inconsistent. We need an algebraic axiomatic approach for these purposes, showing that extropy satisfies some basic intuitive properties of an evaluation function for information content. In [1] it was shown that the partition as information has a quantitative measure in

[1] Instead of the rather widespread notion of *negentropy* we use the term *extropy*, as *neg en* means in Greek *not into*, which is equivalent to *out*, or *ex* in Greek.

the form of extropy derived entirely from the algebraic properties of the partition through an axiomatic approach. The lattice-based operations on partitions are limited to multiplication (representing the total information content of both partitions) and addition (representing the common part for the information content of these partitions), but rather often it is needed to find the difference between information contents of objects, represented by partitions. This problem cannot be solved in the class of partitions and requires switching over to a more general notion of structural complexity, represented by covers on a finite set. In this paper we analyse the basic properties of covers and show that the algebraic extropy concept can be extended further for covers on a finite set as well. We extend the concept of extropy for a special class of covers which can be characterized by a pair of comparable partitions and reflect the difference between information contents represented by these. It is shown that the extropy for this class of covers is equal to the difference in extropy of these partitions.

1. BASIC INFORMATIONAL PROPERTIES OF COVERS

Let us define a cover $\tau_i(X)$ on a finite set $X = \{x_1, x_2, ..., x_m\}$ as a class of its subsets (blocks of cover) $\{B_i^{(1)}, B_i^{(2)}, ..., B_i^{(\alpha)}, ..., B_i^{(m_i)}\}$, satisfying the following conditions:

a) $\bigcup_{\alpha=1}^{m_i} B_i^{(\alpha)} = X$;

b) for any arbitrary $B_i^{(\alpha)}, B_i^{(\beta)} \in \tau_i(X)$ we have $B_i^{(\alpha)} \subset B_i^{(\beta)} \Rightarrow \alpha = \beta$.

A block $B^{(\alpha)} \in \tau(X)$, consisting of elements $x_{\alpha 1}, x_{\alpha 2}, ..., x_{\alpha m} \in X$, will be denoted by $\overline{x_{\alpha 1}, x_{\alpha 2}, ..., x_{\alpha m}}$. Extreme covers are a zero cover (denoted by 0_X) having in each block no more than one element and a unit cover (denoted by 1_X) having only one block. A cover $\tau_i(X)$ will be called a partition iff for any of its different blocks $B_i^{(\alpha)}, B_i^{(\beta)} \in \tau_i$ we have $B_i^{(\alpha)} \bigcap B_i^{(\beta)} = \varnothing$. It is clear that 0_X and 1_X are partitions. In the following a partition on a set X will be denoted by $\pi(X)$. If for any arbitrary $B_i^{(\alpha)} \in \tau_i$ there exists $B_j^{(\beta)} \in \tau_j$ such that $B_i^{(\alpha)} \subseteq B_j^{(\beta)}$, we will denote it by $\tau_i(X) \le \tau_j(X)$. It is not hard to show that if $\tau_i(X) \le \tau_j(X)$ and $\tau_j(X) \le \tau_i(X)$, then $\tau_i(X) = \tau_j(X)$. Let $\mathcal{G} = \{S^{(\alpha)} \mid \alpha = 1, ..., m\}$ be an arbitrary collection of ordered objects. We will introduce an operation

$$\max(\mathcal{G}) \underset{\mathrm{Df}}{=} \{S^{(\alpha)} \mid S^{(\alpha)} \in \mathcal{G} \wedge (\forall S^{(\beta)} \in \mathcal{G})(S^{(\alpha)} \le S^{(\beta)} \Rightarrow S^{(\alpha)} = S^{(\beta)})\}.$$

We will agree for any arbitrary $\tau_i(X)$ and $\tau_j(X)$ that $\tau_i \cdot \tau_j \underset{\mathrm{Df}}{=} \max\{B_i^{(\alpha)} \bigcap B_j^{(\beta)} \mid B_i^{(\alpha)} \in \tau_i \ \wedge \ B_j^{(\beta)} \in \tau_j\}$ and $\tau_i + \tau_j \underset{\mathrm{Df}}{=} \max\{\tau_i \bigcup \tau_j\}$. The restriction of a cover $\tau(X)$ onto $X' \subset X$ will be denoted by $\overline{\tau}(X) \underset{\mathrm{Df}}{=} \max\{B^{(\alpha)} \bigcap X' \mid B^{(\alpha)} \in \tau\}$. Let us define now the operations $\tau_i \oplus \tau_j \underset{\mathrm{Df}}{=} \bigcap\{\pi_k \mid \pi_k \ge \tau_i, \tau_j\}$ and $\chi(\tau) \underset{\mathrm{Df}}{=} \bigcap\{\pi_i \mid \pi_i \ge \tau\}$. It is easy to see that $\tau_i + \tau_j = \bigcap\{\tau_k \mid \tau_k \ge \tau_i, \tau_j\}$ and $\tau_i \oplus \tau_j = \chi(\tau_i + \tau_j) \ge \tau_i + \tau_j$. With respect to the multiplication "$\cdot$" and addition "+" operations defined above, all possible covers on

X will build up a distributive lattice [2], which will be denoted by $\mathcal{L}(X)$. For any arbitrary subset of covers $\mathfrak{T} \subset \mathcal{L}(X)$ we will introduce notations $m(\mathfrak{T}) \underset{\mathrm{Df}}{=} \bigcap_{\tau_i \in \mathfrak{T}} \tau_i$ and $M(\mathfrak{T}) \underset{\mathrm{Df}}{=} \sum_{\tau_i \in \mathfrak{T}} \tau_i$. Let us define for each cover $\tau_i(X)$ its shell

$$k(\tau_i) \underset{\mathrm{Df}}{=} \max\{B^{(\alpha)} \mid B^{(\alpha)} \subset X \ \wedge\ (\forall B_i^{(\beta)} \in \tau_i)(B^{(\alpha)} \subset B_i^{(\beta)} \ \vee\ B^{(\alpha)} \bigcap B_i^{(\beta)} = \varnothing\}.$$

It is obvious that for every $\tau(X)$ its shell $k(\tau)$ is a partition. For any cover we will denote $P(\tau) \underset{\mathrm{Df}}{=} \max\{\pi_\alpha \mid k(\tau) \le \pi_\alpha \le \tau\}$.

Lemma 1. *For any arbitrary cover* $\tau(X)$ *we have*:
(i) $k(\tau) = m(P(\tau))$;
(ii) $\tau = M(P(\tau))$.

Proof.
(i) Let us have $B \in \tau(X)$. Denote $\pi_B(X) \underset{\mathrm{Df}}{=} \{B, X \backslash B\}$. It is obvious that $\bigcap_{B\in\tau} \pi_B(X) \le k(\tau)$. Due to this, in view of $k(\tau) \le \pi_B(X)$, we obtain $\bigcap_{B\in\tau} \pi_B(X) = k(\tau)$. On the other hand, it is clear that $k(\tau) \le m(P(\tau))$. Since for any $\pi(X) \in P(\tau)$ with $B \in \pi(X)$ we have $\pi_B(X) \ge \pi(X)$, we get $k(\tau) = \bigcap_{B\in\tau} \pi_B(X) \ge m(P(\tau))$. Hence $k(\tau) = m(P(\tau))$.
(ii) As one can easily see, $\tau \ge M(P(\tau))$. Let us now assume that $B \in \tau$. Then there exists $\pi(X) \in P(\tau)$ such that $B \in \pi(X)$ and therefore $B \in M(P(\tau))$. Hence $\tau \le M(P(\tau))$, and in view of that we have $\tau = M(P(\tau))$. □

For any subset $X' \subseteq X$ we will define its weight $q_X(X')$ as a ratio $q_X(X') \underset{\mathrm{Df}}{=} \|X'\| / \|X\|$ (as a rule, the subscript next to q will be omitted). Covers $\tau_i(X')$ and $\tau_j(X'')$ will be called quasi-equivalent iff there exists a bijection $\varphi : \tau_i \to \tau_j$ such that for any $B_i^{(\alpha)} \in \tau_i$ we have $q_{X'}(B_i^{(\alpha)}) = q_{X''}(\varphi(B_i^{(\alpha)}))$. Covers $\tau_i(X')$ and $\tau_j(X'')$ will be called equivalent (the corresponding denotation is $\tau_i(X') \equiv \tau_j(X'')$) if there exist bijections $\varphi : \tau_i \to \tau_j$ and $\psi : k(\tau_i) \to k(\tau_j)$ establishing quasi-equivalence between τ_i, τ_j and $k(\tau_i), k(\tau_j)$, respectively, such that for any $B^{(\alpha)} \in k(\tau_i)$ and $B^{(\beta)} \in \tau_i$ from $B^{(\alpha)} \subset B^{(\beta)}$ it follows that $\psi(B^{(\alpha)}) \subset \varphi(B^{(\beta)})$. It is obvious that for partitions the notions of quasi-equivalence and equivalence coincide. Let us call $\tau_i(X)$ quasi-independent with respect to $\tau_j(X)$ (denoted by $\tau_i \top \tau_j$) iff for any $B \in \tau_i \oplus \tau_j$ and $B_j^{(\alpha)} \in \tau_j$ with $B_j^{(\alpha)} \subset B$ the condition $\overline{\tau}_i(B) \equiv \overline{\tau}_i(B_j^{(\alpha)})$ is satisfied. In general, the quasi-independence relation is not symmetric. A cover $\tau(X)$ will be called complete iff for any $B^{(\alpha)}, B^{(\beta)}, B^{(\gamma)} \in \tau(X)$, $X', X'', X''' \subset X$ with $X', X'' \subset B^{(\alpha)}$, $X'', X''' \subset B^{(\beta)}$ and $X''', X' \subset B^{(\gamma)}$ there always exists $B^{(\delta)} \in \tau(X)$ such that $X', X'', X''' \subset B^{(\delta)}$. Complete covers represent the tolerance relation, which plays an outstanding role in the theory of classification [3,4]. It is obvious that all partitions are complete.

Lemma 2. *For any complete cover* $\tau(X)$ *we have*

$$(\forall B^{(\alpha)} \in \tau)(\forall x_i \notin B^{(\alpha)})(\exists x_j \in B^{(\alpha)})((\forall B^{(\beta)} \in \tau)(x_i \in B^{(\beta)} \ \Rightarrow\ x_j \notin B^{(\beta)})).$$

Proof. Let us assume that there exist $B^{(\alpha)} \in \tau$ and $x_i \notin B^{(\alpha)}$ such that for any $x_j \in B^{(\alpha)}$ $(x_j \neq x_i)$ we get $B^{(\beta)} \in \tau$ with $x_i, x_j \in B^{(\beta)}$. Then $x_i, x_{j1} \in B^{(\beta 1)}$, $x_i, x_{j2} \in B^{(\beta 2)}, \ldots,$ $x_i, x_{jm_\alpha} \in B^{(\beta m_\alpha)}$, $x_{j1}, x_{j2}, ..., x_{jm_\alpha} \in B^{(\alpha)}$. Due to the completeness of τ from this follows that there exists a block $B^{(\alpha)} \in \tau$ such that $x_i, x_{j1}, x_{j2}, ..., x_{jm} \in B^{(\gamma)}$. Hence $x_i \in B^{(\gamma)}$ and $B^{(\alpha)} \subsetneq B^{(\gamma)}$, which contradicts the assumption $B^{(\alpha)} \in \tau$. Therefore $\tau(X)$ satisfies the condition of this lemma. ☐

For each $\tau(X)$ we will define a relation $R(\tau) \subset X^2$ as follows:

$$(\forall x_i x_j \in X)(x_i \; R(\tau) \; x_j \underset{\text{Df}}{\Leftrightarrow} (\exists B \in \tau)(x_i, x_j \in B)).$$

It is easy to see that the relation R is a tolerance relation, i.e. it is reflective and symmetric. Let us denote by $\tau_R(X)$ and $\mathfrak{T}_F(X)$, respectively, the set of all maximal subclasses of the relation R and the subset of all complete covers belonging to $\mathcal{L}(X)$. Indeed, it directly follows from the definition of $\tau_R(X)$ that it is a cover. Let us define for each $\tau(X)$ an operator of complementation as follows:

$$F(\tau) \underset{\text{Df}}{=} \bigcap \{\tau_i \mid \tau_i \in \mathfrak{T}_F(X) \wedge \tau \leq \tau_i\}.$$

Proposition 1. *For any* $\tau, \tau', \tau'' \in \mathcal{L}(X)$ *we have*

(i) $\tau', \tau'' \in \mathfrak{T}_F(X) \Rightarrow \tau' \cdot \tau'' \in \mathfrak{T}_F(X)$;

(ii) $F(\tau) \geq \tau$ *and* $F(F(\tau)) = F(\tau)$;

(iii) $\tau' \leq \tau'' \Rightarrow F(\tau') \leq F(\tau'')$;

(iv) $F(\tau' \cdot \tau'') \leq F(\tau') \cdot F(\tau'')$ *and* $F(\tau' + \tau'') \geq F(\tau') \geq F(\tau') + F(\tau'')$;

(v) $\tau_{R(\tau)}(X) = F(\tau(X))$.

Proof.

(i) Let $X', X'' \subset B_i^{(\alpha)} \bigcap B_j^{(\alpha)} \in \tau_i \cdot \tau_j$, $X'', X''' \subset B_i^{(\beta)} \bigcap B_j^{(\beta)} \in \tau_i \cdot \tau_j$, $X', X''' \subset B_i^{(\gamma)} \bigcap B_j^{(\gamma)} \in \tau_i \cdot \tau_j$. Then $X', X'' \subset B_i^{(\alpha)}$ and $X', X'' \subset B_j^{(\alpha)}$, $X'', X''' \subset B_i^{(\beta)}$ and $X'', X''' \subset B_j^{(\beta)}$, $X', X''' \subset B_i^{(\gamma)}$ and $X', X''' \subset B_j^{(\gamma)}$. Hence there exist $B_i^{(\delta)} \in \tau_i$, $B_j^{(\delta)} \in \tau_j$ such that $X', X'', X''' \subset B_i^{(\delta)}$, $X', X'', X''' \subset B_j^{(\delta)}$ and consequently, $X', X'', X''' \subset B_i^{(\delta)} \bigcap B_j^{(\delta)} \Rightarrow$ $(\exists B^{(\delta)} \in \tau_i \cdot \tau_j)(X', X'', X''' \subset B^{(\delta)})$.

(ii)–(iv) Are derived directly from the definition of the operator F.

(v) It is easy to see that $\tau_{R(\tau)}(X) \geq \tau(X)$ and $\tau_{R(\tau)}(X) \in \mathfrak{T}_F(X)$. Hence $\tau_{R(\tau)}(X) \geq F(\tau(X))$. On the other hand, we will assume that $\tau_{R(\tau)}^{(n)}(X)$ is a cover consisting of all blocks of $\tau_{R(\tau)}$, whose potency is less than or equal to n and all existing subsets of potency n of the blocks $\tau_{R(\tau)}$ having potency greater than n. Thus by the definition $\tau_{R(\tau)}^{(n)}(X) \leq \tau_{R(\tau)}(X)$. It is clear that $\tau_{R(\tau)}^{(1)} \leq F(\tau)$ and $\tau_{R(\tau)}^{(2)} \leq F(\tau)$. Let us assume that the formulas $\tau_{R(\tau)}^{(n)} \leq F(\tau)$ and $X_{n-1},\{x_i\},\{x_j\} \subset X_{n-1},\{x_i\},\{x_j\} \subset B \in \tau_{R(\tau)}^{(n+1)}$ with $\| X_{n-1} \| = n-1$ hold. Therefore, due to the assumption we have:

(a) $X_{n-1}, \{x_i\} \subset B \Rightarrow (\exists B' \in F(\tau))(X_{n-1}, \{x_i\} \subset B')$;

(b) $X_{n-1}, \{x_j\} \subset B \Rightarrow (\exists B'' \in F(\tau))(X_{n-1}, \{x_j\} \subset B'');$
(c) $x_i, x_j \in B \Rightarrow (\exists B''' \in F(\tau))(x_i, x_j \in B''').$

From the definition of $F(\tau)$, taking into consideration the result given above, we get that there exists $B^{(\alpha)} \in F(\tau)$ such that $X_{n-1}, \{x_i\}, \{x_j\} \subset B^{(\alpha)}$. Hence $\tau_{R(\tau)}^{(n+1)} \leq F(\tau)$. Thus, by induction we have proved that for any $\tau_{R(\tau)}^{(n)}$ the inequality $\tau_{R(\tau)}^{(n)} \leq F(\tau)$ holds. It is clear that if the number n is considerably large, then the condition $\tau_{R(\tau)}^{(n)}(X) = \tau_{R(\tau)}(X)$ is satisfied. Hence $\tau_{R(\tau)}(X) \leq F(\tau)$ and therefore $\tau_{R(\tau)}(X) = F(\tau(X))$. This completes the proof of the proposition. □

From the first statement of Proposition 1 it directly follows that for any $\tau(X)$ the formula $F(\tau) \in \mathfrak{T}_F(X)$ holds.

Let us assume now that the real value extropy function for covers $H(\tau)$ satisfies the following axioms:

(A1) from $\tau_i(X') \equiv \tau_j(X'')$ it follows that $H(\tau_i) = H(\tau_j)$;
(A2) if $\tau_i(X) \geq \tau_j(X)$, then $H(\tau_i) \leq H(\tau_j)$;
(A3) $H(\tau_i(X)) + H(\tau_j(X)) \geq H(\tau_i \cdot \tau_j) + H(\tau_i \oplus \tau_j)$ with the equality achieved in case $\tau_i \mathrm{T} \tau_j$;
(A4) $H(\tau) = H(F(\tau))$.

In [1] it is shown that for any partition $\pi_i(X)$ its extropy up to an arbitrary positive constant equals $H(\pi_i) = -\sum_{\alpha=1}^{m_i} q(B_i^{(\alpha)}) \ln q(B_i^{(\alpha)})$. Let $\pi_i(X) \leq \pi_j(X)$. In the following the notation $\tau(X) = \pi_i / \pi_j$ will mean $\tau(X) \underset{\text{Df}}{=} \sum\{\pi^{(\alpha)} \mid \pi_j \cdot \pi^{(\alpha)} = \pi_i\}$. As $(\pi_i / \pi_j) \cdot \pi_j = \pi_i$ holds, the above notation is justified. A cover $\tau(X)$ will be called a simple one iff there exist $\pi_i(X)$ and $\pi_j(X)$ with $\pi_i \leq \pi_j$ such that $\tau(X) = \pi_i / \pi_j$. It is not hard to see that each simple cover $\tau = \pi_i / \pi_j$ is expressed by the formula

$$\tau = \{B_i^{(\alpha 1)} \cup B_i^{(\alpha 2)} \cup \ldots \cup B_i^{(\alpha k)} \cup \ldots \cup B_i^{(\alpha m_j)} \mid B_i^{(\alpha k)} \in \overline{\pi}_i(B_j^{(k)})\}.$$

For any simple cover $\tau(X) = \pi_i / \pi_j$ the notation $X_\tau \underset{\text{Df}}{=} X \setminus \bigcup_{B^{(\alpha)} \in \pi_i, \pi_j} B^{(\alpha)}$ is introduced. It is not hard to understand that if $\tau(X) = \pi_i / \pi_j$, then $k(\tau) = \{X / X_\tau\} \cup \overline{\pi}_i(X_\tau) \geq \pi_i$. Therefore, if π_i and π_j have no more than one common block, then $k(\tau) = \pi_i$. Any pair of partitions $\langle \pi_{i1}(X), \pi_{i2}(X) \rangle$ will be denoted by $\rho_i(X)$. It is clear that ρ_i corresponds to a simple cover $(\pi_{i1} \cdot \pi_{i2}) / \pi_{i1}$. Let us take $H(\rho_i) \underset{\text{Df}}{=} H(\pi_{i1} \cdot \pi_{i2}) - H(\pi_{i1})$. For each simple cover $\tau(X)$ we will introduce a notation $\Omega(\tau) \underset{\text{Df}}{=} \{\langle \pi_j, \pi_i \rangle \mid \tau = \pi_i / \pi_j\}$. Any simple cover τ is called an elementary one iff it can be expressed in the form $\tau = \{B_1 \cup B_2, B_1 \cup B_3, \ldots, B_1 \cup B_n\}$ with $B_i \cap B_j = \varnothing$ $(i \neq j; i, j = 1, \ldots, n)$. It goes without saying that all two-block covers are elementary.

Lemma 3.

(i) *If* $\tau' = \pi_i' / \pi_j'$ *and* $\tau'' = \pi_i'' / \pi_j''$, *then* $\tau' \cdot \tau'' \leq \pi_i' \cdot \pi_i'' / \pi_j' \cdot \pi_j''$.
(ii) *For any* $\tau(X) = \pi_i / \pi_j$ *we have* $\tau = F(\tau' + \pi_i)$, *where* $\tau' = 0_X / \pi_j$.
(iii) *Let* $\tau(X) = \pi_i' / \pi_j' = \pi_i'' / \pi_j''$. *Then* $\bigcup_{B' \in \pi_i' \cap \pi_j'} B' = \bigcup_{B'' \in \pi_i'' \cap \pi_j''} B''$, $\overline{\pi}_i'(X_\tau) = \overline{\pi}_i''(X_\tau)$, *and* $\overline{\pi}_j'(X_\tau) = \overline{\pi}_j''(X_\tau)$.

(iv) *Each simple cover can be expressed as a multiplication of elementary covers*.

Proof.

(i) As a matter of fact, by the definition $\tau' = \sum\{\pi_\alpha \mid \pi'_j \cdot \pi_\alpha = \pi'_i\}$, $\tau'' = \sum\{\pi_\beta \mid \pi''_j \cdot \pi_\beta = \pi''_i\}$ and therefore

$$\tau' \cdot \tau'' = \sum\{\pi_\alpha\} \cdot \sum\{\pi_\beta\} = \sum\{\pi_\alpha \cdot \pi_\beta \mid \pi'_j \cdot \pi_\alpha = \pi'_i \wedge \pi''_j \cdot \pi_\beta = \pi''_i\}$$
$$\leq \sum\{\pi_\alpha \cdot \pi_\beta \mid \pi'_j \cdot \pi''_j \cdot \pi_\alpha \cdot \pi_\beta = \pi'_i \cdot \pi''_i\}$$
$$\leq \sum\{\pi_\gamma \mid \pi'_j \cdot \pi''_j \cdot \pi_\gamma = \pi'_i \cdot \pi''_i\} = \pi'_i \cdot \pi''_i / \pi'_j \cdot \pi''.$$

(ii) In view of

$$\tau = \{B_i^{(\alpha 1)} \cup B_i^{(\alpha 2)} \cup \ldots \cup B_i^{(\alpha k)} \cup \ldots \cup B_i^{(\alpha m_j)} \mid B_i^{(\alpha k)} \in \overline{\pi}_i(B_j^{(k)})\},$$

the definition of completeness, and the last statement of Proposition 1 it is not hard to see that the statement of the lemma holds.

(iii) Due to $k(\tau) = \{X \setminus X_\tau\} \cup \overline{\pi}_i(X_\tau)$, it is easy to prove the statement.

(iv) Let $\tau = \pi_i / \pi_j$. Assume that $\tau^{(k)} = \pi_i^{(k)} / \pi_j^{(k)}$ $(k = 1, \ldots, m_j)$, where

$$\pi_i^{(1)} = \left\{ B^{(\alpha)} \mid B^{(\alpha)} \in \overline{\pi}_i(B_j^{(1)}) \vee B^{(\alpha)} = \bigcup_{\beta=2}^{m_j} B_j^{(\beta)} \right\};$$

$$\pi_i^{(2)} = \left\{ B^{(\alpha)} \mid B^{(\alpha)} \in \overline{\pi}_i(B_j^{(2)}) \vee B^{(\alpha)} = \bigcup_{\beta=1, \beta\neq 2}^{m_j} B_j^{(\beta)} \right\};$$

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$$\pi_i^{(m_j)} = \left\{ B^{(\alpha)} \mid B^{(\alpha)} \in \overline{\pi}_i(B_j^{(m_j)}) \vee B^{(\alpha)} = \bigcup_{\beta=1}^{m_j-1} B_j^{(\beta)} \right\}$$

and

$$\pi_j^{(k)} = \left\{ B_j^{(k)}, \bigcup_{\beta=1}^{m_j} B_j^{(\beta)} \right\} \quad (k = 1, \ldots, m_j).$$

As one can see, the covers $\tau^{(k)}$ $(k = 1, \ldots, m_j)$ are elementary. It is obvious that $\tau^{(1)} \cdot \tau^{(2)} \cdot \ldots \cdot \tau^{(m_j)} = \tau$.

Corollary.

(i) *All simple covers are complete*.

(ii) *For any given* $\rho_i, \rho_j \in \Omega(\tau)$ *the equality* $H(\rho_i) = H(\rho_j)$ *holds*.

Let us define now for any given covers $\tau_i(X')$ and $\tau_j(X'')$ a Cartesian product $\tau_i(X')\otimes\tau_j(X'')$ as follows:

$$\tau_i(X')\otimes\tau_j(X'') \underset{\mathrm{Df}}{=} \tau(X'\times X'')=\{B_i^{(\alpha)}\times B_j^{(\beta)} \mid B_i^{(\alpha)}\in\tau_i \ \wedge\ B_j^{(\beta)}\in\tau_j\}.$$

It is not hard to prove the following statement.

Lemma 4. *For any given covers* $\tau_i(X')$, $\tau_j(X'')$, *and* $\tau_k(X''')$ *we have*:

(i) $\tau_i(X')\otimes(\tau_j(X'')\otimes\tau_k(X'''))=(\tau_i(X')\otimes\tau_j(X''))\otimes\tau_k(X''')$;

(ii) $\tau_i(X')\otimes\tau_j(X'')=\tau_j(X'')\otimes\tau_i(X')$;

(iii) $(\tau_{i1}(X')\le\tau_{i2}(X')\wedge\tau_{j1}(X'')\le\tau_{j2}(X''))\Rightarrow\tau_{i1}\otimes\tau_{j1}(X'\times X'')$
$\le\tau_{i2}\otimes\tau_{j2}(X'\times X'')$;

(iv) $(\tau_{i1}(X')\otimes\tau_{j1}(X''))\cdot(\tau_{i2}(X')\otimes\tau_{j2}(X''))=\tau_{i1}\cdot\tau_{i2}(X')\otimes\tau_{j1}\cdot\tau_{j2}(X'')$;

(v) $H(\tau_i(X')\otimes\tau_j(X''))=H(\tau_i(X'))+H(\tau_j(X''))$.

It should be noticed that if τ_i and τ_j are simple covers, then, in general, $\tau_i\otimes\tau_j$ is not simple. On the basis of Lemma 4 we will use the formula $\tau_1(X')\otimes\tau_2(X'')\otimes...\otimes\tau_n(X^{(n)})$ in the following without any further comment. Let us introduce the notation

$$(\tau(X))^n \underset{\mathrm{Df}}{=} \underbrace{\tau(X)\otimes\tau(X)\otimes...\otimes\tau(X)}_{n\text{ times}}.$$

Corollary. *For any* $k\ge1$ *we have*:

(i) $\tau_i\le\tau_j\Rightarrow(\tau_i)^k\le(\tau_j)^k$;

(ii) $(\tau_i)^k\cdot(\tau_j)^k=(\tau_i\cdot\tau_j)^k$ *and* $(\tau_i)^k+(\tau_j)^k\le(\tau_i+\tau_j)^k$;

(iii) $H(\tau^k)=kH(\tau)$.

For any natural numbers $\alpha_1,\alpha_2,...,\alpha_m$ with $\alpha_1+\alpha_2+...+\alpha_m=n$ and an arbitrary cover $\tau(X)$ of rank m (i.e. $\|\tau(X)\|=m$) we will introduce the definitions

$$C_n(\alpha_1,\alpha_2,...,\alpha_m)\underset{\mathrm{Df}}{=}\frac{n!}{\alpha_1!\cdot\alpha_2!\cdot...\cdot\alpha_m!}$$

and

$$Q_\tau^{(n)}(\alpha_1,\alpha_2,...,\alpha_m)\underset{\mathrm{Df}}{=}(q_X(B^{(1)}))^{\alpha1}\cdot(q_X(B^{(2)}))^{\alpha2}\cdot...\cdot(q_X(B^{(m)}))^{\alpha m}.$$

It is easy to prove the following statement:

Lemma 5. *Let* $\tau(X)$ *be an arbitrary cover of rank* m. Then τ^n *is a cover of rank* m^n, *where for every* $B^{(\alpha)}$ *there exists a set of natural numbers* $\alpha_1,\alpha_2,...,\alpha_m$ *with* $\alpha_1+\alpha_2+...+\alpha_m=n$ *such that* $q_{X^n}(B^{(\alpha)})=Q_\tau^{(n)}(\alpha_1,\alpha_2,...,\alpha_m)$ *and the number of blocks of this weight is not less than* $C_n(\alpha_1,\alpha_2,...,\alpha_m)$.

In the following we will use the general combinatorial formula

$$\frac{n!}{\alpha_1!\cdot\alpha_2!\cdot\ldots\cdot\alpha_m!} = \sum_{k=1}^{m}\frac{(n-1)!}{\alpha_1!\cdot\alpha_2!\cdot\ldots\cdot\alpha_{k-1}!\cdot\alpha_{k+1}!\cdot\ldots\cdot\alpha_m!}. \tag{1}$$

If $m=2$, the left side of formula (1) will be denoted by $\binom{n}{\alpha,\, n-\alpha}$. For any covers $\tau'(X)$ and $\tau''(X)$ we will define

$$\tau'/\tau'' \underset{\mathrm{Df}}{=} \sum\{\tau_\alpha \mid \tau''\cdot\tau_\alpha = \tau'\cdot\tau''\}.$$

Denote $\mathcal{C}(\tau) \underset{\mathrm{Df}}{=} 0_X/\tau$ and $\mathcal{C}(\{\tau^{(1)},\tau^{(2)},\ldots,\tau^{(m)}\}) \underset{\mathrm{Df}}{=} \{\mathcal{C}(\tau^{(1)}),\mathcal{C}(\tau^{(2)}),\ldots,\mathcal{C}(\tau^{(m)})\}$. It is easy to see that for any cover $\tau(X)$ we have $\mathcal{C}(\tau)\in\mathfrak{T}_F(X)$. Likewise, for any arbitrary covers $\tau_i(X)$ and $\tau_j(X)$ we have $\tau_i\cdot\tau_j = 0_X \Rightarrow F(\tau_i)\cdot F(\tau_j) = 0_X$.

Lemma 6. *For any covers τ,τ_i,τ_j we have*
(i) $\mathcal{C}(\mathcal{C}(\tau)) = F(\tau)$ *and* $\tau_i\le\tau_j \Rightarrow \mathcal{C}(\tau_i)\ge\mathcal{C}(\tau_j)$;
(ii) $\mathcal{C}(\tau_i\cdot\tau_j) = F(\mathcal{C}(\tau_i)+\mathcal{C}(\tau_j))$, $\mathcal{C}(\tau_i+\tau_j) = \mathcal{C}(\tau_i)\cdot\mathcal{C}(\tau_j)$;
(iii) $\mathcal{C}(\tau) = m(\mathcal{C}(P(\tau)))$;
(iv) *If* $\tau'\le\tau''$, *then* $\tau'/\tau'' = \mathcal{C}(\tau'')/\mathcal{C}(\tau')$;
(v) $\tau\cdot\mathcal{C}(\tau) = 0_X$ *and* $F(\tau+\mathcal{C}(\tau)) = 1_X$;
(vi) $\mathcal{C}(\tau_i\times\tau_j)\ge\mathcal{C}(\tau_i)\times\mathcal{C}(\tau_j)$.

Proof.

(i) It is not hard to see that for any $x_i,x_j\in X$ we get either $x_i,x_j\in R(\tau)$ or $x_i,x_j\in R(\mathcal{C}(\tau))$. Therefore $R(\mathcal{C}(\tau)) = \neg R(\tau)$ and $\mathcal{C}(\mathcal{C}(\tau)) = F(\tau)$ (indeed, $x_i,x_j\in R(\mathcal{C}(\tau))$ holds only if there does not exist a block $B_\tau^{(\alpha)}$ such that $x_i,x_j\in B_\tau^{(\alpha)}$). The definition of $\mathcal{C}(\tau)$ implies directly the statement $\tau_i\le\tau_j \Rightarrow \mathcal{C}(\tau_i)\ge\mathcal{C}(\tau_j)$.

(ii) As a matter of fact,

$$\begin{aligned}\mathcal{C}(\tau_i)+\mathcal{C}(\tau_j) &= \sum\{\tau_\alpha \mid \tau_i\cdot\tau_\alpha = 0_X \ \vee\ \tau_j\cdot\tau_\alpha = 0_X\}\\ &\le \sum\{\tau_\alpha \mid \tau_i\cdot\tau_j\cdot\tau_\alpha = 0_X\} = \mathcal{C}(\tau_i\cdot\tau_j)\end{aligned}$$

and therefore

$$\mathcal{C}(\tau_i\cdot\tau_j)\ge\mathcal{C}(\tau_i)+\mathcal{C}(\tau_j) \Rightarrow \mathcal{C}(\tau_i\cdot\tau_j)\ge F(\mathcal{C}(\tau_i)+\mathcal{C}(\tau_j)).$$

On the other hand, let $B\in\mathcal{C}(\tau_i\cdot\tau_j)$. Then for each $B'\subset B$ with $\|B'\|=2$ we have

$$\begin{aligned}&B'\not\subset B_i\in\tau_i \vee B'\not\subset B_j\in\tau_j\\ &\quad\Rightarrow (\exists B''\in\mathcal{C}(\tau_i))\,(B'\subset B'')\vee(\exists B'''\in\mathcal{C}(\tau_j))(B'\subset B''')\\ &\quad\Rightarrow (\exists B^{(\alpha)}\in(\mathcal{C}(\tau_i)+\mathcal{C}(\tau_j)))(B'\subset B^{(\alpha)})\\ &\quad\Rightarrow (\exists B^{(\beta)}\in F(\mathcal{C}(\tau_i)+\mathcal{C}(\tau_j)))(B\subset B^{(\beta)}).\end{aligned}$$

Hence $\mathcal{C}(\tau_i \cdot \tau_j) \le F(\mathcal{C}(\tau_i) + \mathcal{C}(\tau_j))$ and as a result we obtain $\mathcal{C}(\tau_i \cdot \tau_j) = F(\mathcal{C}(\tau_i) + \mathcal{C}(\tau_j))$. In the same way we get

$$\begin{aligned}\mathcal{C}(\tau_i + \tau_j) &= \sum\{\tau_\alpha \mid (\tau_i + \tau_j)\cdot\tau_\alpha = 0_X\} = \sum\{\tau_\alpha \mid \tau_i\cdot\tau_\alpha + \tau_j\cdot\tau_\alpha = 0_X\} \\ &= \sum\{\tau_\alpha \mid \tau_i\cdot\tau_\alpha = 0_X \ \wedge \ \tau_j\cdot\tau_\alpha = 0_X\} \\ &= \sum[\{\tau_\alpha \mid \tau_i\cdot\tau_\alpha = 0_X\} \bigcap \{\tau_\beta \mid \tau_j\cdot\tau_\beta = 0_X\}] = \mathcal{C}(\tau_i)\cdot\mathcal{C}(\tau_j).\end{aligned}$$

(iii) Indeed, $m(\mathcal{C}(P(\tau))) = \mathcal{C}\left(\sum P(\tau)\right) = \mathcal{C}(\tau)$ (on the basis of Lemma 1).

(iv) It is obvious that $x_i, x_j \in R(\tau'/\tau'')$ iff $x_i, x_j \notin R(\tau'')$ or $x_i, x_j \in R(\tau')$. But as $x_i, x_j \notin R(\tau'') \Leftrightarrow x_i, x_j \in \neg R(\tau'')$ and $x_i, x_j \in R(\tau') \Leftrightarrow x_i, x_j \notin \neg R(\tau')$, we get $x_i, x_j \in R(\tau'/\tau'') \Leftrightarrow x_i, x_j \in R(\mathcal{C}(\tau'')/\mathcal{C}(\tau'))$. Therefore, $\tau'/\tau'' = \mathcal{C}(\tau'')/\mathcal{C}(\tau')$.

(v)–(vi) The definition of $\mathcal{C}(\tau)$ yields directly the needed equalities. □

Proposition 2. *Any complete cover $\tau(X)$ can be represented as a multiplication of simple covers on X.*

Proof. Indeed, let $P(\mathcal{C}(\tau)) = \{\pi^{(1)}, \pi^{(2)}, ..., \pi^{(r)}\}$ and $\tau^{(1)} = \mathcal{C}(\pi^{(1)})$, $\tau^{(2)} = \mathcal{C}(\pi^{(2)}), ..., \tau^{(r)} = \mathcal{C}(\pi^{(r)})$. Then, due to the assumption we obtain $\tau = \mathcal{C}(\mathcal{C}(\tau)) = m(\mathcal{C}(P(\mathcal{C}(\tau)))) = \tau^{(1)}\cdot\tau^{(1)}\cdot ...\cdot\tau^{(r)}$. □

On the basis of Lemma 3 we get:

Corollary. *Each complete cover $\tau(X)$ can be represented as a multiplication of elementary covers on X.*

It is not hard to see that $\mathcal{C}(\tau)$ can be calculated by the following rules:

Algorithm 1.

(i) Let us arrange the blocks of a cover τ in some arbitrary way $B_\tau^{(1)}, B_\tau^{(2)}, ..., B_\tau^{(i)}, ..., B_\tau^{(m)}$, so that for any $i = 1, ..., m-1$ the condition $B_\tau^{(i)} \bigcap B_\tau^{(i+1)} = \varnothing \Rightarrow B_\tau^{(i)} \bigcap B_\tau^{(i+\alpha)} = \varnothing$ $(\alpha = 1, ..., m-i)$ is satisfied.

(ii) Let us write down the ordered blocks of τ in the form of an incompletely filled matrix in the following way:
   (a) write down the block $B_\tau^{(1)}$ in the first row from left to right;
   (b) write down the block $B_\tau^{(2)}$ in the second row, so that the elements in common with $B_\tau^{(1)}$ would be situated in the same column, but different elements in different columns;
   (c) write down $B_\tau^{(3)}$ in the third row, so that the elements in common with $B_\tau^{(1)}$ and $B_\tau^{(2)}$ would be in the same column, but different elements again in different columns;
   (d) proceed up to the moment when all blocks of τ will be written down into the incomplete matrix having its columns formed only by equal elements.

(iii) For the sake of clarity all columns consisting of more than one element will be framed by a box.

(iv) Find all possible sets of disjoint columns (i.e. not having common rows with non-void entries). Each such set represents a block of a cover $\mathcal{C}(\tau)$.

For any $\pi_i(X) \le \pi_j(X)$ we will define

$$\{\pi_i : \pi_j\} \underset{\mathrm{Df}}{=} \{\pi_\alpha \mid \pi_j \cdot \pi_\alpha = \pi_i \wedge (\forall \pi_\beta(X))(\pi_j \cdot \pi_\beta = \pi_i \Rightarrow H(\pi_\beta) \ge H(\pi_\alpha))\}.$$

2. MAIN RESULTS

Theorem 1. *If* $\pi^{(n)} \in \{(\pi_h)^n : (\pi_j)^n\}$, *then* $\lim_{n\to\infty} 1/n\ H(\pi^{(n)}) = H(\pi_h) - H(\pi_j)$.

Proof. On the basis of Lemma 5 every block $B^{(\alpha)} \in (\pi_j)^n$ is characterized by a vector $\langle i_1^{(\alpha)}, i_2^{(\alpha)}, ..., i_{mj}^{\alpha} \rangle$ with $i_1^{(\alpha)} + i_2^{(\alpha)} + ... + i_{mj}^{(\alpha)} = n$ and the number of blocks $(\pi_j)^n$ characterized by $\{i_1^{(\alpha)}, i_2^{(\alpha)}, ..., i_{mj}^{(\alpha)}\}$ is equal to $C_n(i_1^{(\alpha)}, i_2^{(\alpha)}, ..., i_{mj}^{(\alpha)})$. We claim that blocks $B^{(\alpha)}, B^{(\beta)} \in (\pi_j)^n$ belong to the same class λ iff the equality $\langle i_1^{(\alpha)}, i_2^{(\alpha)}, ..., i_{mj}^{\alpha} \rangle = \langle i_1^{(\beta)}, i_2^{(\beta)}, ..., i_{mj}^{(\beta)} \rangle$ holds. As a result we get a set of classes $\lambda_1, \lambda_2, ..., \lambda_k, ..., \lambda_r$. It is clear that $B^{(\alpha)}, B^{(\beta)} \in \lambda_k \Rightarrow q(B^{(\alpha)}) = q(B^{(\beta)})$. It is not hard to see that the number of such classes r equals $\binom{m_j+n-1}{n}$. Each $B^{(\alpha)} \in (\pi_j)^n$ can be characterized by the subset of blocks $\mathfrak{B}(B^{(\alpha)}) \underset{\mathrm{Df}}{=} \{B^{(\beta)} \mid B^{(\beta)} \in (\pi_h)^n \wedge B^{(\beta)} \subset B^{(\alpha)}\}$. For any class λ_k define $N(\lambda_k) \underset{\mathrm{Df}}{=} \|\mathfrak{B}(B^{(\alpha)})\|$ with $B^{(\alpha)} \in \lambda_k$. From the definition of classes λ it directly follows that such a definition of $N(\lambda_k)$ is correct (indeed, $N(\lambda_k) = \prod_{t=1}^{m_i} (n(B_j^{(t)})^{i_t^{(k)}}$, where $n(B_j^{(t)})$ is the number of π_h blocks in $B_j^{(t)} \in \pi_j$ and λ_k is characterized by a vector $\langle i_1^{(k)}, i_2^{(k)}, ..., i_t^{(k)}, ..., i_{mj}^{(k)} \rangle$). Let us arrange linearly blocks of $\mathfrak{B}(\lambda_\alpha)$ for each class λ_α and classes $\lambda_1 \prec \lambda_2 \prec ... \prec \lambda_k \prec ... \prec \lambda_r$ by the numbers $N(\lambda_k)$ in the increasing order (i.e. $\lambda_i \prec \lambda_j \underset{\mathrm{Df}}{\Leftrightarrow} N(\lambda_i) \le N(\lambda_j)$). Let us define now a partition $\pi_{0n}(X^n)$ as follows:

(a) In the first block $B_{0n}^{(1)}$ we will assemble by layers blocks of the partition $(\pi_h)^n$ from each block $(\pi_j)^n$ one at a time, so that in the first stratum we have the blocks of $(\pi_h)^n$ belonging to the union of all minimal blocks $B^{(1k)}$ of $\mathfrak{B}(B^{(\alpha)})$ $(k = 1, ..., r)$, in the second stratum we have all blocks of the partition $(\pi_h)^n$ from the union of all blocks $B^{(2k)}$ next to $B^{(1k)}$ from the subsets $\mathfrak{B}(B^{(\alpha)})$ $(k = 1, ..., r)$, and we proceed likewise until $\mathfrak{B}(B^{(\alpha)})$ with $B^{(\alpha)} \in \lambda_1$ is exhausted.

(b) Into the second block $B_{0n}^{(2)}$ we will unite again by layers blocks of the partition $(\pi_h)^n$ from each block $B_j^{(\alpha)} \backslash B_{0n}^{(1)}$ $(\alpha = 1, ..., m_j)$ one at a time in the increasing order of blocks $B^{(\alpha k)}$ $(\alpha = 1, ..., m_j)$ from subsets $\mathfrak{B}(\lambda_k)$ until $\mathfrak{B}(\lambda_2)$ comes to an end.

(c) We will proceed in the same way until $\mathfrak{B}(\lambda_r)$ is exhausted.

Figure 1 shows the configuration of the partition π_{0n}, where the blocks of $(\pi_h)^n$ are depicted by cells, the blocks of $(\pi_j)^n$ by columns of these cells, and

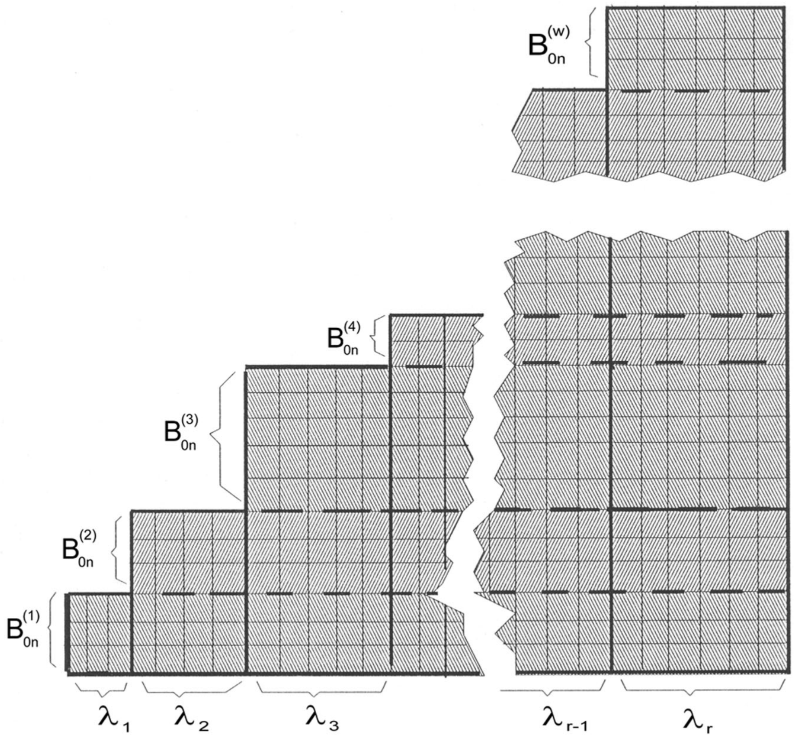

Fig. 1. Partition π_{0n}.

the blocks of π_{0n} by diagonal hatching. From the definition of π_{0n} it directly follows that

$$\|\pi_{0n}\| \leq \binom{m_j + n}{n}.$$

As for $\sum_{\alpha=1}^{m_j} N_\alpha = N \quad (0 \leq N_\alpha \leq N)$ the inequality

$$\frac{N!}{\prod_{\alpha=1}^{m_j} N_\alpha!} \leq \frac{N^N}{\prod_{\alpha=1}^{m_j} (N_\alpha)^{N^\alpha}}$$

holds [5], we get

$$\lim_{n\to\infty} 1/n\, H(\pi_{0n}) \leq \lim_{n\to\infty} 1/n \ln\left(\frac{(m_j+n)^{m_j+n}}{m_j^{m_j} \cdot n^n}\right)$$
$$= \lim_{n\to\infty} (m_j/n \ln(1+n/m_j) + \ln(1+m_j/n)) = 0$$
$$\Rightarrow \lim_{n\to\infty} 1/n\; H(\pi_{0n}) = 0.$$

Therefore,

$$\lim_{n\to\infty} 1/n(H((\pi_j)^n) + H(\pi_{0n})) \geq \lim_{n\to\infty} 1/n\,H((\pi_j)^n \cdot \pi_{0n})$$

$$\Rightarrow \lim_{n\to\infty} 1/n\,H((\pi_j)^n) \geq \lim_{n\to\infty} 1/n\,H((\pi_j)^n \cdot \pi_{0n})$$

$$\Rightarrow \lim_{n\to\infty} 1/n\,H((\pi_j)^n) = \lim_{n\to\infty} 1/n\,H((\pi_j)^n \cdot \pi_{0n}).$$

It is easy to see that one can choose π_α with $\pi_\alpha \cdot (\pi_j)^n = (\pi_h)^n$, so that $(\pi_j)^n \cdot \pi_{0n} \top \pi_\alpha$ and $\pi_{0n} = ((\pi_j)^n \cdot \pi_{0n}) \oplus \pi_\alpha$. From the above it follows that the equality $H((\pi_j)^n \cdot \pi_{0n}) + H(\pi_\alpha) = H((\pi_h)^n + H(\pi_{0n}))$ holds. Hence

$$\lim_{n\to\infty} 1/n(H((\pi_j)^n) + H(\pi_\alpha)) = \lim_{n\to\infty} 1/n(H((\pi_h)^n) + H(\pi_{0n}))$$

$$\Rightarrow H(\pi_j) + \lim_{n\to\infty} 1/n\,H(\pi_\alpha) = H(\pi_h) \Rightarrow \lim_{n\to\infty} 1/n\,H(\pi_\alpha) = H(\pi_h) - H(\pi_j).$$

If now $\pi^{(n)} \in \{(\pi_h)^n : (\pi_j)^n\}$, then by the definition $H(\pi^{(n)}) + nH(\pi_j) \geq nH(\pi_h)$ and $H(\pi^{(n)}) \leq H(\pi_\alpha)$, from which directly follows the statement of the theorem. □

To resolve problems connected with finding an entropy measure for covers we need to introduce some new notions. Let us define on an arbitrary finite set $N \underset{\text{Df}}{=} \{a_1, a_2, ..., a_n\}$ a commutative free group $\mathfrak{P} \underset{\text{Df}}{=} \langle N, \cdot, 1_N \rangle$, where "$\cdot$" is the group operation and "1_N" is a unit element of the group satisfying the condition $(\forall a_i \in N)(a_i \cdot 1_N = a_i \wedge a_i \cdot a_i = 1_N)$. From the definition of the group $\mathfrak{P}$ it directly follows that the set of its elements $\mathfrak{P}(N)$ can be represented by the formula $\mathfrak{P}(N) = \{a_{i_1} \cdot a_{i_2} \cdot ... \cdot a_{i_k} \mid k \leq n; a_{i_1}, a_{i_2}, ..., a_{i_k} \in N\}$. For the sake of convenience in the following the elements of the group $\mathfrak{P}$ will be denoted by $a_{i_1} a_{i_2} ... a_{i_k}$ instead of $a_{i_1} \cdot a_{i_2} \cdot ... \cdot a_{i_k}$. From the definition of the group $\mathfrak{P}$ it directly follows that $(a_{i_1} a_{i_2} ... a_{i_j} ... a_{i_k}) \cdot a_{i_j} = a_{i_1} a_{i_2} ... a_{i_{j-1}} a_{i_{j+1}} a_{i_{j+2}} ... a_{i_k}$. It is obvious that $\|\mathfrak{P}(N)\| = 2^n$. As each element of $\mathfrak{P}(N)$ can be represented in a unique way by a multiplication of $k \leq n$ elements from N, we associate with each element $b_i \in \mathfrak{P}(N)$ its length $n(b_i)$ as the number of elements from N by which multiplication b_i is determined. We introduce for the elements of $\mathfrak{P}(N)$ a partial order relation "$\leq$" as follows: $(\forall b_i, b_j \in \mathfrak{P}(N))(b_i \leq b_j \underset{\text{Df}}{\Leftrightarrow} n(b_i b_j) = n(b_j) - n(b_i))$. For any subset $\mathfrak{N} \subset \mathfrak{P}(N)$ we define its basis $B(\mathfrak{N}) \underset{\text{Df}}{=} \{a_i \in N \mid (\exists b_j \in \mathfrak{N})(a_i \leq b_j)\}$ and rank $r(\mathfrak{N}) \underset{\text{Df}}{=} \|B(\mathfrak{N})\|$. A subset $\mathfrak{N}' \underset{\text{Df}}{\subset} \mathfrak{P}(N)$ will be called rooted if $B(\mathfrak{N}') \subset \mathfrak{N}'$. As one can easily see, for any $\mathfrak{N} \subset \mathfrak{P}(N)$ always $r(\mathfrak{N}) \leq \sum_{b_i \in \mathfrak{N}} n(b_i)$. It is rather convenient to describe any subset $\mathfrak{N}' \subset \mathfrak{P}(N)$ by means of an incomplete matrix $\mathfrak{M}(\mathfrak{N}')$, where each row corresponds to an element of $\mathfrak{N}'$ (so the number of rows is equal to $\|\mathfrak{N}'\|$) and each column corresponds to an element of its basis $B(\mathfrak{N}')$ (it means that the number of columns is equal to $r(\mathfrak{N}')$). The matrix $\mathfrak{M}(\mathfrak{N}')$ can be interpreted as a set of nonzero coefficients for a linear equation system. A subset $\mathfrak{N} \subset \mathfrak{P}(N)$ will be called

(a) basic [complete] iff there exists $N' \subset N$ $[N' = N]$ such that $\mathfrak{P}(\mathfrak{N}) = \mathfrak{P}(N')$;
(b) compatible iff $\|\mathfrak{N}\| = r(\mathfrak{N})$;
(c) even iff $\prod_{b_x \in \mathfrak{N}} b_X = 1_N$.

It should be noticed that for any $\mathfrak{N}' \subset \mathfrak{P}(N)$ with $\|\mathfrak{N}'\| \geq 3$, $\mathfrak{P}(\mathfrak{N}')$ is always even, and if $\mathfrak{N}$ is complete, then $\|\mathfrak{N}\| \geq \|N\|$. If a subset $\mathfrak{N} \subset \mathfrak{P}(N)$ is not even, then it is called an uneven one. A basic and compatible subset $\mathfrak{N} \subset \mathfrak{P}(N)$ will be called perfect. Every perfect subset $\mathfrak{N} \subset \mathfrak{P}(N)$ is presented by a square matrix $\mathfrak{M}(\mathfrak{N})$.

Lemma 7. *The following statements about a compatible* $\mathfrak{N} \subset \mathfrak{P}(N)$ *are equivalent*:

(i) $\mathfrak{N}$ is *perfect*.
(ii) *For any different subsets* $\mathfrak{N}', \mathfrak{N}'' \subset \mathfrak{N}$ *we have* $\prod_{b_x \in \mathfrak{N}'} b_x \neq \prod_{b_y \in \mathfrak{N}''} b_y$.
(iii) *There exist no* $b_i \in \mathfrak{N}$ *and* $\mathfrak{N}' \subset \mathfrak{N}$ *with* $\{b_i\} \neq \mathfrak{N}'$ *such that* $b_i = \prod_{b_x \in \mathfrak{N}'} b_x$.
(iv) $\mathfrak{P}(\mathfrak{N})$ *does not have even subsets* $\mathfrak{N}' \subset \mathfrak{N}$.

Proof. Let $\mathfrak{N}$ be an arbitrary subset of $\mathfrak{P}(\mathfrak{N})$ with $\|\mathfrak{N}\| = m'$ and $r(\mathfrak{N}) = m''$. It is obvious that $\|\mathfrak{P}(\mathfrak{N})\| \leq 2^{m'}$. If now $\mathfrak{N}$ is compatible, then $m' = m''$ and therefore $\|\mathfrak{P}(\mathfrak{N})\| \leq 2^{m'}$.

(i) $\Rightarrow$ (ii). If $\mathfrak{N}$ is perfect, then $\|\mathfrak{P}(\mathfrak{N})\| = 2^{m'} = 2^{m''}$ and therefore for any different subsets $\mathfrak{N}', \mathfrak{N}'' \subset \mathfrak{N}$ we have $\prod_{b_x \in \mathfrak{N}'} b_x \neq \prod_{b_y \in \mathfrak{N}''} b_y$.
(ii) $\Rightarrow$ (iii). Obvious.
(iii) $\Rightarrow$ (iv). Indeed, $\prod_{b_x \in \mathfrak{N}'} b_x = 1_N \Rightarrow b_i = \prod_{b_y \in \mathfrak{N}' \setminus \{b_i\}} b_y$.
(iv) $\Rightarrow$ (ii). Let us assume that there exist subsets $\mathfrak{N}', \mathfrak{N}'' \subset \mathfrak{N}$ such that $\prod_{b_x \in \mathfrak{N}'} b_x = \prod_{b_y \in \mathfrak{N}''} b_y$. Then $\prod_{b_z \in \mathfrak{N}'''} b_z = 1_{B(\mathfrak{N})}$, where $\mathfrak{N}''' = (\mathfrak{N}' \setminus \mathfrak{N}'') \bigcup (\mathfrak{N}'' \setminus \mathfrak{N}')$, and hence $\mathfrak{N}$ is perfect.
(ii) $\Rightarrow$ (i). It is clear that if for any different subsets $\mathfrak{N}', \mathfrak{N}'' \subset \mathfrak{N}$ we have $\prod_{b_x \in \mathfrak{N}'} b_x \neq \prod_{b_y \in \mathfrak{N}''} b_y$, then $\|\mathfrak{P}(\mathfrak{N})\| = 2^{m'}$. Due to the compatibility of $\mathfrak{N}$, we have $m' = m''$ and therefore $\|\mathfrak{P}(\mathfrak{N})\| = 2^{m''}$. From this it directly follows that $\mathfrak{N}$ is perfect. □

We introduce for the subsets of $\mathfrak{P}(N)$ a relation of partial order "$\prec$", writing $\mathfrak{N}'' \prec \mathfrak{N}'$ for any arbitrary $\mathfrak{N}', \mathfrak{N}'' \subset \mathfrak{P}(N)$ iff $\mathfrak{N}''$ can be represented by a matrix $\mathfrak{M}(\mathfrak{N}'')$ derived from $\mathfrak{M}(\mathfrak{N}')$ by deleting some of its rows and columns. The subsets $\mathfrak{N}' \subset \mathfrak{P}(N')$ and $\mathfrak{N}'' \subset \mathfrak{P}(N'')$ are said to be ρ-isomorphic iff there exists a one-to-one mapping $\varphi : \mathfrak{N}' \to \mathfrak{N}''$, which is a perfect subset preserving function, i.e. φ generates a one-to-one mapping between the perfect subsets of $\mathfrak{N}'$ and $\mathfrak{N}''$. In the following we will refer to a subset $\mathfrak{N}' \subset \mathfrak{P}(N)$ with the cardinality m as an m-subset. Let us define $q \underset{\mathrm{Df}}{=} \lim_{n \to \infty} \prod_{k=1}^{\infty} (1 - 1/2^k) \approx 0.289$.

Lemma 8.

(i) *For any arbitrary* $m \le n$ *the probability of finding by a casual choice a perfect m-subset in* $\mathfrak{P}(N)$ *is not less than* q.

(ii) *If a subset* $\mathfrak{N}_x \subset \mathfrak{P}(N)$ *with the rank* m *consists of a perfect m-subset* $\mathfrak{N}' \subset \mathfrak{N}_x$, *then there exists a rooted subset* $\mathfrak{N}_y \subset \mathfrak{P}(N)$, *which is* ρ*-isomorphic to* $\mathfrak{N}_x$.

Proof.

(i) Let $\mathfrak{N}$ be an arbitrary subset of $\mathfrak{P}(N)$ having the rank $r(\mathfrak{N}) = m$. The first condition that $\mathfrak{N}$ should satisfy to be perfect is that the equity $\| \mathfrak{N} \| = m$ should hold. Now we are going to compose $\mathfrak{N}$ by selecting its elements one by one and consider the other conditions that $\mathfrak{N}$ should satisfy to be perfect. The first element of $\mathfrak{N}$ can be chosen arbitrarily with the only restriction that it is not equal to 1_N. Hence the probability of a positive choice for the first element equals $1 - 1/2^m$. While choosing the second element, we have to take into consideration the fact that it should not be equal to the first one. Therefore the probability of a positive choice for the second element is $1 - 1/2^{m-1}$. The third element, in addition to not being equal to the first two elements, has to be different from the product of the first two elements. Hence the probability of a positive choice for the third element is $1 - 1/2^{m-2}$. For the fourth element a positive choice is guaranteed if it equals neither any of the previous elements nor any product of arbitrary combinations of them. Thus the probability of a positive choice for the fourth element is $1 - 1/2^{m-3}$. Proceeding in an analogous way, we get for the kth and mth elements the probabilities $1 - 2^{m-k+1}$ and $1/2$, respectively. Hence the probability of finding a perfect subset $\mathfrak{N}$ equals $\prod_{k=1}^{m} (1 - 1/2^k) > q$, which completes the proof of part (i) of the lemma.

(ii) As $\mathfrak{N}'$ is a perfect subset with the rank m, in view of Lemma 7 there exists for any $b \in B(\mathfrak{N}_x)$ a subset $\mathfrak{N}'' \subset \mathfrak{N}'$ such that $b = \prod_{\bar{a}_i \in \mathfrak{N}''} \bar{a}_i$. Let us define now the mapping $\varphi : \mathfrak{N}_x \to \mathfrak{N}_y$:

(a) to any element $\bar{a} \in \mathfrak{N}'$ there corresponds a $b \in B(\mathfrak{N}_y)$;

(b) for each $b \in B(\mathfrak{N}_x)$ we have $\varphi(b) \underset{\text{Df}}{=} \prod_{\bar{a}_i \in \mathfrak{N}''} \varphi(\bar{a}_i)$;

(c) $\varphi(a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot a_{i_k}) \underset{\text{Df}}{=} \varphi(a_{i_1}) \cdot \varphi(a_{i_2}) \cdot \ldots \cdot \varphi(a_{i_k})$ for any $a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot a_{i_k} \in \mathfrak{N}_k$ with $a_{i_1}, a_{i_2}, \ldots, a_{i_k} \in B(\mathfrak{N}_x)$.

It is not hard to see, indeed, that the above defined mapping φ is an r-isomorphism between $\mathfrak{N}_x$ and $\mathfrak{N}_y$.

Corollary. *For any pair of integers* m *and* ρ *satisfying the condition* $m < \rho \le 2^n$ *there exists a rooted* ρ*-subset* $\mathfrak{N}_x \subset \mathfrak{P}(N)$ *such that the probability of finding a perfect m-subset* $\mathfrak{N}' \prec \mathfrak{N}_x$ *among all m-subsets in* $\mathfrak{N}_x$ *is not less than* q.

Proof. Indeed, as we have:

(a) the minimum number of perfect m-subsets in $\mathfrak{P}(N)$ is by the appropriate choice of columns greater than $q\binom{2^n}{m}$,

(b) each perfect m-subset belongs to $\binom{2^n-m}{\rho-m}$ ρ-subsets $\mathfrak{N}_x \subset \mathfrak{P}(N)$,
(c) the number of m-subsets in $\mathfrak{N}_x$ is $\binom{\rho}{m}$,
(d) $\mathfrak{P}(N)$ contains $\binom{2^n}{\rho}$ ρ-subsets $\mathfrak{N}_x$,

the average relative number q' of perfect m-subsets in $\mathfrak{N}_x$ over all possible choices for $\mathfrak{N}_x$ equals q, for

$$\binom{2^n}{m}\binom{2^n-m}{\rho-m}/\binom{2^n}{\rho}\binom{\rho}{m}=1.$$

Hence, in view of Lemma 8 (ii), there exists a rooted ρ-subset $\mathfrak{N}_x$ such that the probability of finding a perfect m-subset among all of its m-subsets is not less than q. □

Let a set of natural numbers $N=\{1,2,...,n\}$ be given. Define $\mathbb{N}_k \underset{\mathrm{Df}}{=} \{\{i_1,i_2,...,i_\alpha,...,i_k\}\,|\,(\forall i_\alpha)(i_\alpha \in N \wedge \alpha<\beta \Rightarrow i_\alpha < i_\beta)\}$. It is obvious that $\|\mathbb{N}_k\|=\binom{n}{k}$. Let us denote the permutation group on the set N by $F_n \underset{\mathrm{Df}}{=} \{\varphi_\alpha \,|\, \varphi_\alpha : N \to N\}$. For any $\varphi \in F_n$ and $\mathbb{N}'_k \subset \mathbb{N}_k$, define $\varphi(\mathbb{N}'_k) \underset{\mathrm{Df}}{=} \{\{\varphi(i_1),\varphi(i_2),...,\varphi(i_k)\}\,|\,\{i_1,i_2,...,i_k\}\in\mathbb{N}'_k\}$. Subsets $\mathbb{N}'_k$ and $\mathbb{N}''_k$ will be called isomorphic iff there exists a permutation $\varphi \in F_n$ with $\mathbb{N}'_k=\varphi(\mathbb{N}''_k)$. We put each $\mathbb{N}'_k \subset \mathbb{N}_k$ into correspondence with a class of subsets $\mathfrak{S}(\mathbb{N}'_k)$ containing $\mathbb{N}'_k$ and all subsets of $\mathbb{N}_k$ isomorphic to $\mathbb{N}'_k$. Define for each $\mathbb{N}'_k \subset \mathbb{N}_k$ a ratio $q(\mathbb{N}'_k) \underset{\mathrm{Df}}{=} \|\mathbb{N}'_k\|/\|\mathbb{N}_k\|$.

Lemma 9. *For any* $\mathbb{N}^0_k, \mathbb{N}' \subset \mathbb{N}_k$ *there exists a permutation* $\varphi \in F_n$ *such that* $\|\varphi(\mathbb{N}^0_k)\cap\mathbb{N}'_k\|/\|\mathbb{N}^0_k\| \geq q(\mathbb{N}'_k)$.

Proof. Let us define $\varphi'_m \in F_n$ as such a permutation that for any $\varphi \in F_n$ we have $\|\varphi'_m(\mathbb{N}^0_k)\cap\mathbb{N}'_k\| \geq \|\varphi(\mathbb{N}^0_k)\cap\mathbb{N}'_k\|$. Denote $\mathbb{N}''_k \underset{\mathrm{Df}}{=} \varphi'_m(\mathbb{N}^0_k)\cap\mathbb{N}'_k$. As one can easily see, for any $\bar{a}_k \in \mathbb{N}_k$ we get $\|\{\varphi' \,|\, \varphi' \in F_n \wedge \varphi'(\bar{a}_k)=\bar{a}_k\}\| = k!(n-k)!$. In an analogous way we have for any $\bar{a}_k \in \mathbb{N}^0_k$ and $\bar{b}_k \in \mathbb{N}'_k$ a permutation $\varphi \in F_n$ with $\varphi(\bar{a}_k)=\bar{b}_k$. Hence

$$\|\{\langle\bar{a}_k,\bar{b}_k,\varphi\rangle\,|\,\bar{a}_k\in\mathbb{N}^0_k \wedge \bar{b}_k\in\mathbb{N}'_k \wedge \varphi\in F_n \wedge \varphi(\bar{a}_k)=\bar{b}_k\}\|$$
$$= k!(n-k)!\|\mathbb{N}^0_k\|\|\mathbb{N}'_k\|.$$

As $\|\mathbb{N}_k\|=\binom{n}{k}=\dfrac{n!}{k!((n-k)!}$ and for any $\varphi \in F_n$ the equity

$$\|\varphi(\mathbb{N}^0_k)\cap\mathbb{N}'_k\| = \|\{\langle\bar{a}_k,\bar{b}_k,\varphi\rangle\,|\,(\exists\bar{a}_k,\bar{b}_k)(\bar{a}_k\in\mathbb{N}^0_k \wedge \bar{b}_k\in\mathbb{N}'_k \wedge \varphi(\bar{a}_k)=\bar{b}_k)\}\|$$

holds, we get

$$1/\|F_n\|\sum_{\varphi\in F_n}\|\varphi(\mathbb{N}_k^0)\cap\mathbb{N}'_k\| = \frac{k!(n-k)!}{n!}\|\mathbb{N}_k^0\|\,\|\mathbb{N}'_k\| = \|\mathbb{N}_k^0\|\|\mathbb{N}'_k\|/\|\mathbb{N}_k\|.$$

Thus $\|\mathbb{N}''_k\| \geq \|\mathbb{N}_k^0\|\|\mathbb{N}'_k\|/\|\mathbb{N}_k\|$ and therefore $\|\mathbb{N}''\|/\|\mathbb{N}_k^0\| \geq \|\mathbb{N}'_k\|/\|\mathbb{N}_k\| = q.$ The lemma is proved. □

Lemma 10. *Let* $X=\{x_1, x_2, x_3\}$, $\pi_j(X)=\{\overline{x_2, x_3}; \overline{x_1}\}$ *and* $\tau(X)=\{\overline{x_1, x_2}; \overline{x_1, x_3}\}$. *Then* $H(\tau)=H(0_X)-H(\pi_j)$.

Proof. It is not hard to see that $\pi_j \in P(\mathcal{C}(\tau))$. Assuming that $\pi^{(n)}\in P(\tau^n)$, we get for $H(\tau)$ with $n\geq 1$ the following evaluation:

$$\begin{aligned}&H((0_X)^n)-H((\pi_j)^n)\leq H(\tau^n)\leq H(\pi^{(n)})\\ &\qquad\Rightarrow \lim_{n\to\infty}1/n[H((0_X)^n)-H((\pi_j)^n)]\leq \lim_{n\to\infty}1/n\,H(\tau^n)\leq \lim_{n\to\infty}1/n\,H(\pi^{(n)})\\ &\qquad\Rightarrow H(0_X)-H(\pi_j)\leq H(\tau)\leq \lim_{n\to\infty}1/n\,H(\pi^{(n)}).\end{aligned}$$

We will show that $\lim_{n\to\infty}1/n\,H(\pi^{(n)})=H(0_X)-H(\pi_j)$. Let τ' be a cover, having blocks with the equal cardinality q, which along with Algorithm 1 can be represented as a square matrix, where the rows correspond to blocks of the cover τ' and the columns correspond to the blocks of the partition $\pi_\alpha \in P(\mathcal{C}(\tau'))$. We denote the matrix associated with τ' by

$$\mathfrak{M}(\tau') \underset{\mathrm{Df}}{=} [a'_{ij}]_{\substack{i=1,\dots,m\\ j=1,\dots,q}}.$$

Hence $\mathfrak{M}(\tau)=\begin{vmatrix}x_2 & x_1\\ x_3 & x_1\end{vmatrix}$. We will show now that there exists $\pi_{0n}(X^n)$ with $\|\pi_{0n}\|\leq 4n(n+1)$ such that $\pi^{(n)}\cdot\pi_{0n}\mathsf{T}(\pi_j)^n$ and $(\pi^{(n)}\cdot\pi_{0n})\oplus(\pi_j)^n=\pi_{0n}$. To reach this goal, one has to find a partition of columns of $\mathfrak{M}(\tau^n)$ corresponding to π_{0n}. First of all let us divide the set of columns of $\mathfrak{M}(\tau^n)$ into $n+1$ subsets with the cardinalities $\binom{n}{\alpha}$ $(\alpha=0,1,\dots,n)$. Indeed, in view of the Cartesian product definition each element of X^n can be represented by a vector $I \underset{\mathrm{Df}}{=} \langle i_1, i_2, \dots, i_k, \dots, i_n\rangle$, where $i_k\in\{x_1, x_2, x_3\}$ $(k=1,\dots,n)$. From this it directly follows that the set of columns of $\mathfrak{M}(\tau^n)$ can be divided into $n+1$ subsets with the cardinalities $\binom{n}{\alpha}$ $(\alpha=0,1,\dots,n)$ depending on the number of characters x_1 as components of the vector which represents an element of X^n. The elements of submatrices of $\mathfrak{M}(\tau^n)$, corresponding to these subsets, could be interpreted as a collection of binary numbers with length n containing element x_1 α times and elements x_2 or x_3 $n-\alpha$ times. These submatrices will be denoted in the following by $\begin{bmatrix}n\\ \alpha\end{bmatrix}$, having ρ rows and q columns; we will apply the notion $\begin{bmatrix}n\\ \alpha\end{bmatrix}(p,q)$. Hence $\begin{bmatrix}n\\ \alpha\end{bmatrix} = \begin{bmatrix}n\\ \alpha\end{bmatrix}\left(2^n,\binom{n}{\alpha}\right)$. A submatrix $\begin{bmatrix}n\\ \alpha\end{bmatrix}(p,q)$ will be called disjoint iff all its rows are pairwise disjoint (rows of a matrix are disjoint iff being interpreted as vectors with different components). For example, in Fig. 2 matrices $\mathfrak{M}(\tau^2)$, $\mathfrak{M}(\tau^3)$, and $\mathfrak{M}(\tau^4)$ are given (for mathematical shorthand each column $\langle a_{1i}, a_{2i}, ..., a_{ni} \rangle$ is replaced by a sequence of natural numbers $M = \{1, 2, ..., m\}$ according to the rules: (a) if by the lexicographic ordering $a_{hi} \geq a_{ki}$, then $j_h \geq j_k$ $(j_h, j_k \in M)$, (b) the order of columns in $\mathfrak{M}(\tau^n)$ is determined by the decreasing number of different elements in them). Non-hatched elements in the row of $\mathfrak{M}(\tau^n)$ give us the blocks of $\pi^{(n)}$. The blocks of $(\pi_j)^n$ are represented by the columns of $\mathfrak{M}(\tau^n)$.

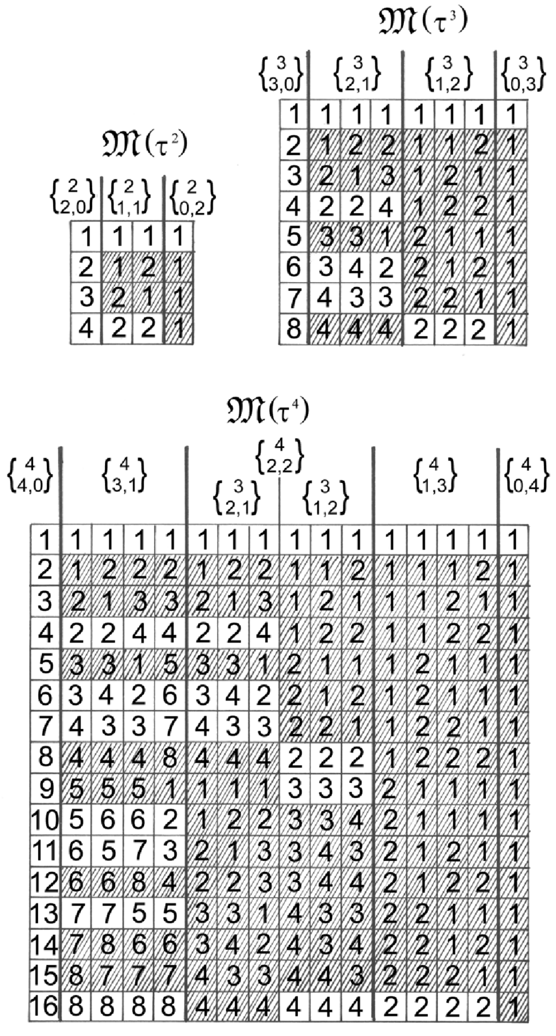

Fig. 2. Matrices $\mathfrak{M}(\tau^2)$, $\mathfrak{M}(\tau^3)$, and $\mathfrak{M}(\tau^4)$.

It is obvious that $\begin{bmatrix} n \\ 0 \end{bmatrix}$ consists of only one column, where the element vectors are characterized by the lack of x_1 as a component. Evidently, all these element vectors are different and their total number equals 2^n. The submatrix $\begin{bmatrix} n \\ 1 \end{bmatrix}$ is made up of columns, where the element vectors are characterized by the fact that their αth $(\alpha = 1, ..., n)$ component is x_1 and other components are x_2 or x_3. The number of different vectors in each column of this submatrix equals 2^{n-1}. The problem is how to select from 2^n potential rows 2^{n-1} disjoint rows. Next, the submatrix $\begin{bmatrix} n \\ 2 \end{bmatrix}$ is made up of columns, where two of n components of element vectors are characters x_1 and the other components are x_2 or x_3. It is clear that the number of different vectors in each column of this submatrix equals 2^{n-2} and the problem we are concerned with is how to select from 2^n potential rows 2^{n-2} disjoint rows.

In the same way one can characterize all other submatrices $\begin{bmatrix} n \\ \alpha \end{bmatrix}$ with $\alpha = 3, 4, ..., n$ as well. A common problem for all these subsets is how to select $2^{n-\alpha}$ disjoint rows from 2^n potential rows, i.e. how to find for each matrix $\begin{bmatrix} n \\ \alpha \end{bmatrix}$ with $\alpha = 0, 1, ..., n$ a disjoint submatrix $\begin{bmatrix} n \\ \alpha \end{bmatrix} \left(2^{n-\alpha}, \binom{n}{\alpha} \right)$. For this purpose, we will represent every row of a submatrix $\begin{bmatrix} n \\ \alpha \end{bmatrix}$ in the following way:

(a) for the given row there exists a generating binary digit with length n, having for its components characters “0” and “1”, corresponding to x_2 and x_3, respectively;

(b) each element in the given row is formed by deleting in this binary digit arbitrary α components.

Denote by $\mathscr{M}(n, n-\alpha)$ a binary matrix of characters “0” and “1”, having $2^{n-\alpha}$ rows and n columns. We associate with any submatrix $\begin{bmatrix} n \\ \alpha \end{bmatrix} \left(2^{n-\alpha}, \binom{n}{\alpha} \right)$ a matrix $\mathscr{M}(n, n-\alpha)$, which is characterized by a one-to-one correspondence between the rows of $\begin{bmatrix} n \\ \alpha \end{bmatrix} \left(2^{n-\alpha}, \binom{n}{\alpha} \right)$ and the $n-\alpha$ columns submatrices of $\mathscr{M}(n, n-\alpha)$. Thus, to each column of a matrix $\begin{bmatrix} n \\ \alpha \end{bmatrix} \left(2^{n-\alpha}, \binom{n}{\alpha} \right)$ there corresponds a submatrix of $\mathscr{M}(n, n-\alpha)$, which is formed as a result of deleting some α columns from $\mathscr{M}(n, n-\alpha)$. The problem of the choice of $2^{n-\alpha}$ disjoint rows from a matrix $\begin{bmatrix} n \\ \alpha \end{bmatrix}$ formulated above can be now reformulated as follows: to find a submatrix $\begin{bmatrix} n \\ \alpha \end{bmatrix} \left(2^{n-\alpha}, \binom{n}{\alpha} \right)$ such that for the corresponding matrix $\mathscr{M}(n, n-\alpha)$ we have:

(a) all of its rows are different;

(b) if we delete arbitrary α columns of the matrix $\mathscr{M}(n, n-\alpha)$, then the probability of getting again a matrix with different rows is the greatest compared with other submatrices $\begin{bmatrix} n \\ \alpha \end{bmatrix} \left(2^{n-\alpha}, \binom{n}{\alpha} \right)$.

Considering columns of $\mathscr{M}(n, n-\alpha)$ as vectors, one can apply to these vectors the operations of addition modulo 2. If now some columns of $\mathscr{M}(n, n-\alpha)$ are formed by this operation from other columns, we will refer to them as secondary with respect to the others, referred to as primary. In the following we will consider a special class of $\mathcal{M}(n, n-\alpha)$ matrices with different rows which can be characterized as follows:

(a) there exists an $n-\alpha$ column submatrix of $\mathcal{M}(n, n-\alpha)$ with different rows;

(b) the remaining α columns of $\mathcal{M}(n, n-\alpha)$, not belonging into the above-mentioned submatrix, are secondary with respect to those in the submatrix.

Let us denote the set of vectors, corresponding to the primary columns of $\mathcal{M}(n, n-\alpha)$, by $M \underset{\mathrm{Df}}{=} \{a_1, a_2, ..., a_m\}$. These vectors build up, with respect to the operation of addition modulo 2, a commutative free group $\mathcal{P} = \langle M, \cdot, 1_M \rangle$ (see p. 14), where "$\cdot$" is the group operation and 1_M is a unit element of the group (i.e. a vector consisting only of the character "0") satisfying the condition $(\forall a_i \in M)(a_i \cdot 1_M = a_i \wedge a_i \cdot a_i = 1_M)$. To each secondary column of $\mathcal{M}(n, m)$ there corresponds an element of $\mathcal{P}(M)$. Hence, to each matrix $\mathcal{M}(n, m)$ in the above-defined class there corresponds an n-subset $\mathcal{N}_\alpha \subset \mathcal{P}(M)$ with the rank $r(\mathcal{N}_\alpha) = m$. In the following, the cases $n \le 2^m$ and $n > 2^m$ will be studied separately.

Case $n \le 2^m$. In this case the problem of finding a disjoint submatrix $\begin{bmatrix} n \\ \alpha \end{bmatrix}\left(2^{n-\alpha}, \binom{n}{\alpha}\right)$ can be reformulated into a problem of finding for every $\begin{bmatrix} n \\ \alpha \end{bmatrix}$ a matrix $\mathcal{M}(n, m)$ to which there corresponds a subset $\mathcal{N}_\alpha \subset \mathcal{P}(M)$ containing a maximum number of perfect m-subsets $\mathcal{N}' \subset \mathcal{N}_\alpha$ (in order to have, after deleting arbitrary α columns of the matrix $\mathcal{M}(n, m)$, all the rows of the resultant submatrix $\mathcal{M}(m, m)$ different again, the subset $\mathcal{N}' \subset \mathcal{N}_\alpha$, corresponding to $\mathcal{M}(m, m)$, needs to be a perfect one, for it is obvious that if a subset is perfect, then there is no information loss and therefore all rows corresponding to the given subset are different). The possible limit number for perfect subsets $\mathcal{N}'$ in $\mathcal{N}_\alpha$ is determined by Lemma 8. Let us denote by ρ the least natural number that exceeds $q\binom{n}{\alpha}$. As $n \le 2^m$, it follows from Lemma 8 that we can find for a matrix $\mathcal{M}(\mathcal{N}_\alpha)$ such a structure that the probability of finding a perfect m-subset $\mathcal{N}' \prec \mathcal{N}_\alpha$ is not less than q. It means that we have a chance to choose 2^m rows in a matrix $\begin{bmatrix} n \\ \alpha \end{bmatrix}$ such that in the corresponding matrix $\mathcal{M}(n, m)$ there exist ρ submatrices of m columns, having all their rows pairwise different. As to each submatrix of $\mathcal{M}(n, m)$, formed by deleting α of its columns, there corresponds a column in a matrix $\begin{bmatrix} n \\ \alpha \end{bmatrix}$, it is possible to separate from the matrix $\begin{bmatrix} n \\ \alpha \end{bmatrix}$ a submatrix $\begin{bmatrix} n \\ \alpha \end{bmatrix}'(2^n, \rho)$, containing a disjoint submatrix $\begin{bmatrix} n \\ \alpha \end{bmatrix}'(2^m, \rho)$. Let us consider now a submatrix $\begin{bmatrix} n \\ \alpha \end{bmatrix}''\left(2^n, \binom{n}{\alpha} - \rho\right)$ having no column with $\begin{bmatrix} n \\ \alpha \end{bmatrix}'(2^n, \rho)$. Consider again finding for the given submatrix $\begin{bmatrix} n \\ \alpha \end{bmatrix}''\left(2^n, \binom{n}{\alpha} - \rho\right)$ a $\mathcal{M}''(n, m)$ which generates a disjoint submatrix $\begin{bmatrix} n \\ \alpha \end{bmatrix}(2^m, r)$ with the maximum possible number of columns. Obviously, by a proper choice of 2^m rows in the matrix $\begin{bmatrix} n \\ \alpha \end{bmatrix}$ it is possible to realize all $\mathcal{M}(n, m)$ matrices having pairwise different rows. Hence, if we can realize a matrix $\mathcal{M}(n, m)$, then it is possible to realize any matrix $\mathcal{M}'(n, m)$ reformed from $\mathcal{M}(n, m)$ by a permutation of its columns. We can map the whole set of $\mathcal{M}(n, m)$ submatrices with m columns onto the set $\mathbb{N}_m \underset{\mathrm{Df}}{=} \{\{i_1, i_2, ..., i_\alpha, ..., i_m\} \mid (\forall i_\alpha)(i_\alpha \in N \wedge \alpha < \beta \Rightarrow i_\alpha < i_\beta)\}$, where $N = \{1, 2, ..., n\}$, and put each permutation of $\mathfrak{M}(n, m)$ columns into correspondence with a permutation from the permutation group $F_n = \{\varphi_i \mid \varphi_i : N \to N\}$. Hence, there exists a one-to-one correspondence between the columns of a matrix $\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right]$ and the elements of $\mathbb{N}_m$, which establishes the interpretation of Lemma 9 for the further use. If now $\mathfrak{M}_\rho(n, m)$ is a matrix with the corresponding subset $\mathfrak{N}_\rho \subset \mathfrak{P}(M)$ containing the maximum possible quantity of perfect m-subsets, the relative number of which with respect to all m-subsets of $\mathfrak{N}_\rho$ is not less than q, then due to the above interpretation of Lemma 9 we can choose for each submatrix $\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right]'$, formed by deleting part of $\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right]$ columns, an $\mathfrak{M}_\rho{}'(n, m)$ with the corresponding subset $\mathfrak{N}'_\rho$ having the relative number of perfect m-subsets not less than q. To these perfect m-subsets there corresponds a subset of $\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right]$ columns whose projection to $\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right]'$ is not less than $q\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right]'$. Therefore Lemma 9 yields that for any $\rho \le n$ there exists a submatrix $\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right](2^m, \rho)$ with the corresponding matrix $\mathfrak{M}_\rho(n, m)$ containing at least $r \le q\rho$ submatrices of m columns from the set of ρ columns so that in each submatrix all rows are pairwise different. Hence, we can separate from the given submatrix $\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right]''$ a disjoint submatrix $\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right]'''(2^m, r)$, where r is the least natural number greater or equal to $q\left(\left(\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right) - \rho\right)$ (due to Lemma 8 we can choose for $\mathfrak{N}_\alpha$ a structure that ensures the maximum number of perfect m-subsets for the collection of m-subsets $\mathfrak{N} \subset \mathfrak{N}_\alpha$).

In order to analyse further the process of dividing a matrix $\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right]$ into disjoint submatrices, we will use the following notations: $x_1 = q$ and $x_j \underset{\mathrm{Df}}{=} x(1 - (x_1 + x_2 + ... + x_{j-1}))x_1$ $(j = 2, 3, ...)$. From this it follows that

$$1 - \sum_{j=1}^{w} x_j = \sum_{j=0}^{w} (-1)^j \binom{w}{j} q^j = (1 - q)^w.$$

It is not hard to see that x_j $(j = 1, 2, ...)$ represents the relative column number of the jth submatrix of $\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right]$ with respect to the column number of $\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right]$. Therefore $(1-q)^w$ represents the relative column number of the residue matrix of $\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right]$ after w submatrices have been separated. It is obvious that $\lim_{n\to\infty}(1-q)^w = 0$. Therefore we can go on with the process of separating submatrices until the column number of the residue matrix is less than 4 and then divide the residue matrix to one column submatrices, which in a trivial way satisfy the separation condition. To ensure that the limit value for the actual column number of the residue matrix is 0 as well, we need the condition $\lim_{n\to\infty}\left(\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right)(1-q)^w = 0$ to be satisfied, for $\left(\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right)$ is the column number of the matrix $\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right]$ and $w = f(n)$. For a sufficiently large n with an arbitrary $\alpha \le n$, from the above it follows that

$$\binom{n}{\alpha}(1-q)^w < 1 \Rightarrow \ln\binom{n}{\alpha} + w\ln(1-q) < 0 \Rightarrow w > \ln\binom{n}{\alpha}/\ln(1-q).$$

The last inequality is satisfied if $w > \ln\left(\frac{n^n}{(n/2)^{n/2}(n/2)^{n/2}}/\ln(1/(1-q))\right)$ $\Rightarrow w > n(\ln 2/\ln(1/(1-q))) \Rightarrow w > 2.03n.$

Hence, we can take $w = 3n.$ Summing up the result in case $n \le 2^m,$ we have that the matrix $\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right]$ can be divided into no more than $3n$ submatrices, characterized by the fact that from the column sets of these submatrices 2^m rows can be chosen to form a disjoint submatrix.

Case $\boldsymbol{n > 2^m}$. We will compose now a vector, whose components are the elements of $\mathfrak{P}(M),$ substituting each column of $\mathfrak{M}(n, m)$ by an element of $\mathfrak{P}(M)$ as follows:

(a) replace arbitrary 2^m columns of $\mathfrak{M}(n, m)$ with the elements of $\mathfrak{P}(M)$;

(b) the remaining columns of $\mathfrak{M}(n, m)$ are substituted by the elements of $\mathfrak{P}(M)$ so that the numbers of different elements corresponding to columns of $\mathfrak{M}(n, m)$ do not differ by more than one.

We can apply Lemma 8 to all these subvectors with length m, having different components. The relative cardinality of this subvector set with respect to the set of all subvectors can be represented by the formula

$$\left[\binom{2^m}{m}(n/2^m)^m\right] : \binom{n}{m}.$$

If now $n \gg m,$ then

$$\binom{2^m}{m} \approx \frac{\sim 2^{m^2}}{m!}\mathrm{e}^{-\alpha(m^2/2^m - m)} \quad \text{and} \quad \binom{n}{m} \approx \frac{n^m}{m!}\mathrm{e}^{-\alpha(m^2/n - m)},$$

where $0 \le \alpha \le 1/2$ and therefore

$$\left[\binom{2^m}{m}(n/2^m)^m\right] : \binom{n}{m} \ge \mathrm{e}^{-(m^2/2^{m+1})}.$$

Thus, for sufficiently large n in the case $n > 2^m,$ we get instead of the coefficient q now $q'q,$ where $q' = \mathrm{e}^{-(m^2/2^{m+1})}.$ If $m = 3,$ then q' has its minimal value 0.56 and therefore we get for w an inequality $w > 3.86n.$ Hence, in the case $n > 2^m$ a submatrix $\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right]$ can be divided into no more than $4n$ submatrices so that it is possible to choose 2^m rows from among its rows, rendering us a disjoint submatrix as a result.

For example, we will consider the solution of the given problem for submatrices $\mathfrak{M}(n, n-1)$ and $\mathfrak{M}(n, 1)$, constructing for them the vectors

$$\mathfrak{N}_1 = \{a_1, a_2, ..., a_{n-1}, a_1 a_2 ... a_{n-1}\} \text{ and } \mathfrak{N}_{n-1} = \{\underbrace{a_1, a_1, ..., a_1}_{n \text{ times}}\},$$

whose components are elements from the set $\mathfrak{P}(M)$, and we are going to look for the perfect subsets with the power $n-1$ and 1, respectively. It is easy to see that all these subsets are perfect. But, of course, in the general case $1 < \alpha < n-1$ not all m-subsets $\mathfrak{N}'$ are perfect with respect to $\mathfrak{N}_\alpha \subset \mathfrak{P}(M)$. Hence, as a summary we get from Lemmas 8 and 9 that for any $\alpha = 0, 1, ..., n$ a matrix $\left[\begin{smallmatrix} n \\ \alpha \end{smallmatrix}\right]$ can be divided into no more than $4n$ submatrices, whose element sets represent the blocks of the partition of π_{0n}. This implies $\| \pi_{0n} \| \le 4n(n+1)$. Hence $\lim_{n\to\infty} 1/n\, H(\pi_{0n}) = 0$ and therefore

$$\begin{aligned}
&\lim_{n\to\infty} 1/n[H(\pi^{(n)} \cdot \pi_{0n}) + H((\pi_j)^n)] \\
&\qquad = \lim_{n\to\infty} 1/n[H(\pi^{(n)} \cdot \pi_{0n} \cdot (\pi_j)^n) + H((\pi^{(n)} \cdot \pi_{0n}) \oplus (\pi_j)^n)] \\
&\qquad\quad \Rightarrow \lim_{n\to\infty} 1/n\, H(\pi^{(n)} \cdot \pi_{0n}) + H(\pi_j) = H(0_X) \\
&\qquad\quad \Rightarrow \lim_{n\to\infty} 1/n\, H(\pi^{(n)}) \le \lim_{n\to\infty} 1/n\, H(\pi^{(n)} \cdot \pi_{0n}) \\
&\qquad = H(0_X) - H(\pi_j) \Rightarrow \lim_{n\to\infty} H(\pi^{(n)}) = H(0_X) - H(\pi_j).
\end{aligned}$$

This completes the proof of the lemma. □

Theorem 2. *For any simple cover* $\tau = \pi_i / \pi_j$ *we have* $H(\tau) = H(\pi_i) - H(\pi_j)$.

Proof. As in the case of Lemma 10, it is not hard to show that the theorem's statement holds for any elementary cover. Due to Lemma 3 each simple cover $\tau = \pi_i / \pi_j$ can be represented as a product of elementary covers $\tau_1, \tau_2, ..., \tau_k, ..., \tau_{mj}$, where each $\tau_k = \pi_i^{(k)} / \pi_j^{(k)}$ with

$$\pi_i^{(k)} = \left\{ B^{(\alpha)} \mid B^{(\alpha)} \in \overline{\pi}_i(B_j^{(k)}) \vee B^{(\alpha)} = \bigcup_{\substack{\beta=1 \\ \beta\neq k}}^{m_j} B_j^{(\beta)} \right\}$$

and

$$\pi_j^{(k)} = \left\{ B_j^{(k)}, \bigcup_{\substack{\beta=1 \\ \beta\neq k}}^{m_j} B_j^{(\beta)} \right\}.$$

As one can easily see, $H(\pi_i) - H(\pi_j) = \sum_{k=1}^{m_j} (H(\pi_i^{(k)}) - H(\pi_j^{(k)}))$. Due to the above result we have

$$H(\pi_i) - H(\pi_j) \le H(\tau) \le H(\tau_1 \cdot \tau_2 \cdot ...\tau_{mj})$$

$$\Rightarrow H(\pi_i) - H(\pi_j) \le H(\tau) \le \sum_{k=1}^{m_j} (H(\pi_i^{(k)}) - H(\pi_j^{(k)}))$$

$$\Rightarrow H(\tau) = H(\pi_i) - H(\pi_j).$$

The result obtained shows that the algebraic concept of extropy for partitions can be extended to simple covers as well.

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Katete informatiivsetest omadustest

Tõnu Lausmaa

Tükeldus lõplikul hulgal annab infot antud tükeldusega määratud objektist. Selline algebraline infotõlgendus on osutunud väga kasulikuks mitmesuguste struktuurse keerukuse probleemide lahendamisel. Analoogselt tükeldusega saab ka katet lõplikul hulgal interpreteerida kui antud kattega esindatud objekti struktuurset keerukust väljendavat infot. Kuid erinevalt tükeldustest puudub siiani katete kvantitatiivne infomõõt. Artiklis on alustatud katetele kvantitatiivse infomõõdu leidmist, näidates, et deduktiivselt on võimalik leida see katetele, mis väljendavad kahe tükelduse infosisalduste vahet. Saadud resultaat rajab teed sellele, et leida kõikidele praktika seisukohalt olulistele katetele kvantitatiivne infomõõt, mis võimaldab hinnata laialdase objektide klassi struktuurset keerukust.