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Convergence and λ-boundedness of functional series with respect to multiplicative systems

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Abstract. The series $\sum c_k g_k(t)$, where $\{g_k\}$ is a product system defined by a multiplicative system, is studied. Some sufficient conditions for p-maximal convergence with speed of this series are found. Also the series $\sum < f, w_k > g_k(t)$ with $f \in L^p_{[0,1]}$, and $\{w_k\}$ being a Walsh system is considered. It is proved that this series converges almost everywhere for various product systems. In the last section the λ-boundedness of this series is discussed.

Key words: multiplicative systems, Walsh functions, convergence with speed, convergence almost everywhere, λ-boundedness.

1. INTRODUCTION

Let $f = \{f_k\}_{k=0}^{\infty}$ be a system of integrable functions on $[a, b]$ satisfying

$$|f_k(t)| \le 1 \quad \text{almost everywhere (a.e.) on } [a, b].$$

The *product system* $\{g_n\}$ *of* $\{f_k\}$ is then given by

$$g_0(t) = 1 \quad \text{and} \quad g_n(t) = f_{n_0+1}(t) f_{n_1+1}(t) \ldots f_{n_k+1}(t) \qquad (t \in [a, b]),$$

where $n = 2^{n_0} + 2^{n_1} + \ldots + 2^{n_k}$ $(n_0 < n_1 < \ldots < n_k)$ is the dyadic representation of n. If $\{g_n\}$ is orthogonal, then $\{f_k\}$ is called *orthogonal multiplicative*. If

$$\int_a^b g_n(t)dt = 0 \quad \text{for } n = 1, 2 \ldots ,$$

then it is said that $\{f_k\}$ is a *strongly multiplicative system* (see [1]). For example, the Rademacher system is orthogonal multiplicative and the Walsh system $\{w_n\}_{n=0}^{\infty}$ is its product system. If

$$\sum_{n=0}^{\infty}\left|\int_a^b g_n(t)dt\right| < \infty,$$

then the system $\{f_k\}$ is called *weakly multiplicative* (see [2], p. 292). If

$$\int_0^1 \left|\sum_{n=0}^{2^m-1}\Big(\int_a^b g_n(\tau)d\tau\Big)w_n(t)\right|^p dt = O(1),$$

then $\{f_k\}$ is called p-*weakly multiplicative* $(1 \le p \le \infty)$ (see [2], p. 330). Particularly, the system $\{f_k\}$ with

$$\sum_{n=0}^{\infty}\Big(\int_a^b g_n(t)dt\Big)^2 < \infty$$

is 2-weakly multiplicative (see [3]).

Clearly, every orthogonal multiplicative system, strongly multiplicative system, and weakly multiplicative system is p-weakly multiplicative.

We first consider the series

$$\sum_{k=0}^{\infty} c_k f_k(t) \tag{1}$$

and

$$\sum_{k=0}^{\infty} c_k g_k(t). \tag{2}$$

Notice that if the series (2) converges a.e. on $[a,b]$ for all $(c_k) \in \ell^2$, then the same statement is true for the series (1).

In [4] it is proved that the series (1) converges a.e. on [a,b] for all rearrangements of $\{c_k f_k\}$ if $(c_k) \in \ell^2$ and $\{f_k\}$ is a p-weakly multiplicative system for a number p with $1 < p < \infty$.

The series (2) is called p-*maximally convergent a.e. on* $[a,b]$ if it is convergent a.e. on $[a,b]$ and

$$\int_a^b \sup_n \left|\sum_{k=0}^{n} c_k g_k(t)\right|^p dt < \infty.$$

Theorem A ([4]). *A series* (2) *is* 1-*maximally convergent a.e. on* $[a,b]$ *if* $(c_k) \in \ell^2$ *and* $\{g_k\}$ *is the product system of a* p-*weakly multiplicative system for* $2 \le p < \infty$.

On the other hand, Schipp in [5] proved

Theorem B ([5]). *A series* (2) *is* 2*-maximally convergent a.e. on* $[a,b]$ *if* $(c_k) \in \ell^2$ *and* $\{g_k\}$ *is the product system of a weakly multiplicative system.*

In this paper we study p-maximal convergence a.e. of the series

$$\sum_{k=0}^{\infty} c_k g_k(t)$$

in the sense of the convergence with speed. Let $\lambda = (\lambda_k)$ be a sequence such that $0 < \lambda_k \nearrow \infty$. The series (2), which is convergent a.e. on $[a,b]$, is called

1) λ-convergent (or convergent with speed λ) a.e. on $[a,b]$ if the limit

$$\lim_n \lambda_n \sum_{k=n+1}^{\infty} c_k g_k(t)$$

exists a.e. on $[a,b]$;

2) λ-bounded a.e. on $[a,b]$ if

$$\sup_n \lambda_n \left| \sum_{k=n+1}^{\infty} c_k g_k(t) \right| < \infty \quad \text{a.e. on } [a,b].$$

Clearly, the λ-convergence implies the λ-boundedness.

Definition 1. *If a series* (2) *is* λ *-convergent a.e. on* $[a,b]$ *and*

$$\int_a^b \sup_n \lambda_n^p \left| \sum_{k=n+1}^{\infty} c_k g_k(t) \right|^p dt < \infty, \tag{3}$$

then it is said that the series (2) *is* p*-maximally* λ*-convergent a.e. on* $[a,b]$.

Definition 2. *If the series* (2) *is* λ*-bounded and* (3) *is valid, then it is said that the series* (2) *is* p*-maximally* λ*-bounded.*

In Section 2 we characterize p-maximal λ-convergence a.e. of the series (2) for $p = 1$ and $p = 2$. For this, we consider the sequence space

$$\ell^2_\lambda := \left\{ c = (c_k) | \sum_{k=0}^{\infty} \lambda_k^2 c_k^2 < \infty \right\}.$$

Obviously, ℓ^2_λ endowed with the norm

$$\| c \| = \Big(\sum_{k=0}^{\infty} c_k^2 \lambda_k^2 \Big)^{1/2}$$

is a Banach space and the sequences $e_i := (\delta_{ki})_{k=0}^{\infty}$ $(i = 0, 1, \ldots)$ form a total set in $(\ell^2_\lambda, \| \, \|)$ (cf. [6], p. 138).

In Section 3 we consider the series (2) where

$$c_k =< f, w_k >:= \int_0^1 f(t) w_k(t) dt \qquad (f \in L^p_{[0,1]})$$

or

$$c_k =< f, g_k >:= \int_a^b f(t) g_k(t) dt \qquad (f \in L^p_{[a,b]})$$

and find some sufficient conditions for p-maximal convergence a.e. ($1 \le p < \infty$) of these series.

In Section 4 we characterize p-maximal λ-boundedness a.e. of the series $\sum_{k=0}^{\infty} < f, g_k > g_k(t)$, where $f \in L^p_{[a,b]}$.

2. p-MAXIMAL λ-CONVERGENCE

We shall prove the following theorem.

Theorem 1. *If $(c_k) \in \ell^2_\lambda$ and $\{g_k\}$ is the product system of a weakly multiplicative system, then the series (2) is 2-maximally λ-convergent a.e. on $[a, b]$.*

To prove Theorem 1 we need the following corollary of the Banach–Steinhaus theorem.

Lemma ([7], p. 361). *Let D_n $(n = 0, 1, \ldots)$ be continuous sublinear operators from a Banach space X to the Frechet space $M_{[a,b]}$ of all functions totally measurable on $[a, b]$. Suppose that the following conditions hold*:

1. $\sup\limits_n |D_n(x, t)| < \infty$ *a.e. on $[a, b]$ for every $x \in X$,*

2. *the limit $\lim\limits_n D_n(\bar{x}, t)$ exists a.e. on $[a, b]$ for every $\bar{x}$ from a total set in X.*

Then the limit $\lim\limits_n D_n(x, t)$ exists a.e. on $[a, b]$ for all $x \in X$.

Proof of Theorem 1. Let $\{g_k\}$ be the product system of a weakly multiplicative system. Because

$$K_m(t, u) := \sum_{j=0}^{2^m - 1} g_j(t) w_j(u) \ge 0 \qquad (t \in [a, b], \; u \in [0, 1], \; m = 0, 1, \ldots)$$

(see [2], p. 293) and the Walsh system is orthogonal, by the Cauchy–Schwartz inequality we get

$$\begin{aligned}\Big\{\int_a^b\Big(\sum_{k=0}^m c_k g_k(t)\Big)^2 dt\Big\}^{1/2} &= \Big\{\int_a^b\Big(\int_0^1 \sum_{k=0}^m c_k w_k(\tau)K_m(t,\tau)d\tau\Big)^2 dt\Big\}^{1/2} \\ &\le \Big\{\int_a^b\bigg(\int_0^1\Big(\sum_{k=0}^m c_k w_k(\tau)\Big)^2 K_m(t,\tau)d\tau\bigg)\Big(\int_0^1 K_m(t,u)du\Big)dt\Big\}^{1/2} \\ &= \Big\{\int_0^1\Big(\sum_{k=0}^m c_k w_k(\tau)\Big)^2\Big(\int_a^b K_m(t,\tau)dt\Big)d\tau\Big\}^{1/2} \\ &\le \Big\{\sum_{k=0}^m c_k^2\Big\}^{1/2}\Big\{\sum_{\nu=0}^{2^m-1}\Big|\int_a^b g_\nu(t)dt\Big|\Big\}^{1/2} = O(1)\Big\{\sum_{k=0}^m c_k^2\Big\}^{1/2}.\end{aligned}$$

Thus the sequence (A_m) of the continuous linear operators

$$A_m : \ell_\lambda^2 \to L_{[a,b]}^2, \qquad (c_k) \mapsto \sum_{k=0}^m c_k g_k(t)$$

is pointwise bounded. Since

$$\lim_m \| A_m(e_k) \| = \lim_m \Big\{\int_a^b\Big(\sum_{k=0}^m \delta_{ki} g_k(t)\Big)^2 dt\Big\}^{1/2} = \Big\{\int_a^b g_i^2(t)dt\Big\}^{1/2}$$

for each $k = 0, 1, \ldots$, by the Banach–Steinhaus theorem we have that (A_m) is pointwise convergent to a linear operator

$$A : \ell_\lambda^2 \to L_{[a,b]}^2, \qquad (c_k) \mapsto \sum_{k=0}^\infty c_k g_k(t)$$

which is continuous. Consequently,

$$\lim_m \Big\{\int_a^b\Big(\sum_{k=m+1}^\infty c_k g_k(t)\Big)^2 dt\Big\}^{1/2} = 0 \quad \text{for each } (c_k) \in \ell_\lambda^2.$$

Therefore

$$\Big\{\int_a^b\Big(\sum_{k=m+1}^\infty c_k g_k(t)\Big)^2 dt\Big\}^{1/2} = O(1)\Big\{\sum_{k=m+1}^\infty c_k^2\Big\}^{1/2} \quad \Big((c_k) \in \ell_\lambda^2\Big),$$

and using the Minkowski inequality, we have

$$\Big\{\int_a^b \max_{n\le m}\lambda_n^2\Big(\sum_{k=n+1}^{\infty} c_k g_k(t)\Big)^2 dt\Big\}^{1/2}$$
$$\le \Big\{\int_a^b \max_{n\le m}\lambda_n^2\Big(\sum_{k=n+1}^{m} c_k g_k(t)\Big)^2 dt\Big\}^{1/2} + \Big\{\int_a^b \lambda_m^2\Big(\sum_{k=m+1}^{\infty} c_k g_k(t)\Big)^2 dt\Big\}^{1/2}$$
$$\le \Big\{\int_a^b \max_{n\le m}\lambda_n^2\Big(\sum_{k=n+1}^{m} c_k g_k(t)\Big)^2 dt\Big\}^{1/2} + O(1)\Big\{\sum_{k=m+1}^{\infty} c_k^2\lambda_k^2\Big\}^{1/2}.$$

By Abel's transformation in view of

$$\sum_{k=n+1}^{m} a_k u_k = \sum_{k=n+1}^{m-1}(a_k - a_{k+1})\sum_{\nu=0}^{k} u_\nu - a_{n+1}\sum_{k=0}^{n} u_k + a_m\sum_{k=0}^{m} u_k \qquad (4)$$

we obtain

$$\Big\{\int_a^b \max_{n\le m}\lambda_n^2\Big(\sum_{k=n+1}^{m} c_k g_k(t)\Big)^2 dt\Big\}^{1/2}$$
$$\le O(1)\Big\{\int_a^b \max_{k\le m}\Big(\sum_{\nu=0}^{k} c_\nu\lambda_\nu g_\nu(t)\Big)^2 dt\Big\}^{1/2} \max_{n\le m}\lambda_n \sum_{k=n+1}^{m-1}\Big(\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}\Big)$$
$$+ \Big\{\int_a^b \max_{n\le m}\Big(\sum_{\nu=0}^{n} c_\nu\lambda_\nu g_\nu(t)\Big)^2 dt\Big\}^{1/2} + \Big\{\int_a^b \Big(\sum_{\nu=0}^{m} c_\nu\lambda_\nu g_\nu(t)\Big)^2 dt\Big\}^{1/2}$$
$$= O(1)\Big\{\int_a^b \max_{k\le m}\Big(\sum_{\nu=0}^{k} c_\nu\lambda_\nu g_\nu(t)\Big)^2 dt\Big\}^{1/2}.$$

Then by Theorem B

$$\Big\{\int_a^b \max_{n\le m}\lambda_n^2\Big(\sum_{k=n+1}^{\infty} c_k g_k(t)\Big)^2 dt\Big\}^{1/2} = O(1) \qquad \Big((c_k)\in \ell_\lambda^2\Big), \qquad (5)$$

which gives

$$\sup_n \lambda_n\Big|\sum_{k=n+1}^{\infty} c_k g_k(t)\Big| < \infty \quad \text{a.e. on } [a,b] \text{ for each } (c_k)\in\ell_\lambda^2. \qquad (6)$$

Therefore the linear operators

$$D_n : \ell_\lambda^2 \longrightarrow M_{[a,b]}, \quad (c_k)\mapsto \lambda_n\sum_{k=n+1}^{\infty} c_k g_k(t) \quad (n = 0, 1, \dots)$$

are continuous and the statements 1 (cf. (6)) and 2 of Lemma are fulfilled. By Lemma, the limit

$$\lim_n \lambda_n \sum_{k=n+1}^{\infty} c_k g_k(t)$$

exists a.e. on $[a,b]$ for every $(c_k) \in \ell^2_\lambda$. Hence the series (2) is λ-convergent a.e. on $[a,b]$ and 2-maximally λ-convergent by (5). The proof of the theorem is now complete.

Analogously, if $\{g_k\}$ is the product system of a 2-weakly multiplicative system, then by orthogonality of the Walsh system we have

$$\begin{aligned}\int_a^b \Big|\sum_{k=0}^{m} c_k g_k(t)\Big|\, dt &\le \int_0^1 \Big|\sum_{k=0}^{m} c_k w_k(\tau)\Big| \Big(\int_a^b K_m(t,\tau)dt\Big) d\tau \\ &\le \Big\{\int_0^1 \Big(\sum_{k=0}^{m} c_k w_k(\tau)\Big)^2 d\tau\Big\}^{1/2} \Big\{\int_0^1 \Big(\int_a^b K_m(t,\tau)\Big)^2 d\tau\Big\}^{1/2} \\ &= \Big\{\sum_{k=0}^{m} c_k^2\Big\}^{1/2} \Big\{\sum_{\nu=0}^{2^m-1} \Big(\int_a^b g_\nu(t)dt\Big)^2\Big\}^{1/2} = O(1)\Big\{\sum_{k=0}^{m} c_k^2\Big\}^{1/2}.\end{aligned}$$

Applying the Banach–Steinhaus theorem, we get that for every $c \in \ell^2_\lambda$

$$\int_a^b \Big|\sum_{k=n+1}^{\infty} c_k g_k(t)\Big|\, dt = O(1)\Big\{\sum_{k=n+1}^{\infty} c_k^2\Big\}^{1/2}.$$

By Abel's transformation (4) and Theorem A we obtain

$$\begin{aligned}&\int_a^b \max_{n\le m} \lambda_n \Big|\sum_{k=n+1}^{\infty} c_k g_k(t)\Big|\, dt \\ &\qquad = O(1)\int_a^b \max_{n\le m} \Big|\sum_{k=0}^{n} c_k \lambda_k g_k(t)\Big|\, dt + \int_a^b \lambda_m \Big|\sum_{k=m+1}^{\infty} c_k g_k(t)\Big|\, dt \\ &\qquad = O(1)\int_a^b \max_{n\le m} \Big|\sum_{k=0}^{n} c_k \lambda_k g_k(t)\Big|\, dt + O(1) \parallel c \parallel_{l^2_\lambda} = O(1) \parallel c \parallel_{l^2_\lambda}.\end{aligned}$$

Using Lemma we get the following result.

Theorem 2. *If $(c_k) \in \ell^2_\lambda$ and $\{g_k\}$ is the product system of a 2-weakly multiplicative system, then the series (2) is 1-maximally λ-convergent a.e. on $[a,b]$.*

3. p-MAXIMAL CONVERGENCE OF THE SERIES $\sum < f, w_k > g_k(t)$ AND $\sum < f, g_k > g_k(t)$

We shall prove the following theorem.

Theorem 3. *Let $1 < p, q < \infty$ be conjugate exponents ($\frac{1}{p} + \frac{1}{q} = 1$) and let f be a function in $L^p_{[0,1]}$. If $\{g_k\}$ is the product system of a q-weakly multiplicative system, then the series*

$$\sum_{k=0}^{\infty} < f, w_k > g_k(t) \tag{7}$$

is 1-maximally convergent a.e. on $[a, b]$.

Proof. On the one hand,

$$\begin{aligned}
&\int_a^b \max_{n \le m} \left| \sum_{k=0}^{n} < f, w_k > g_k(t) \right| dt \\
&\qquad = \int_a^b \max_{n \le m} \left| \int_0^1 \sum_{k=0}^{n} < f, w_k > w_k(\tau) K_m(t, \tau) d\tau \right| dt \\
&\qquad \le \int_0^1 \max_{n \le m} \left| \sum_{k=0}^{n} < f, w_k > w_k(\tau) \right| \left| \int_a^b K_m(t, \tau) dt \right| d\tau.
\end{aligned}$$

On the other hand, from [2], p. 103, it follows that

$$\sup_n \left| \sum_{k=0}^{n} < f, w_k > w_k(\tau) \right| \in L^p_{[0,1]}. \tag{8}$$

Therefore by the Hölder inequality

$$\int_a^b \max_{n \le m} \left| \sum_{k=0}^{n} < f, w_k > g_k(t) \right| dt = O(1) \Big\{ \int_0^1 \left| \int_a^b K_m(t, \tau) dt \right|^q d\tau \Big\}^{1/q} = O(1).$$

The assertion now follows from Lemma.

Since by the Hölder inequality

$$\Big\{\int_a^b \max_{n\le m}\Big|\sum_{k=0}^n <f,w_k> g_k(t)\Big|^p dt\Big\}^{1/p}$$
$$=\Big\{\int_a^b \max_{n\le m}\Big|\int_0^1 \sum_{k=0}^n <f,w_k> w_k(\tau)K_m(t,\tau)d\tau\Big|^p dt\Big\}^{1/p}$$
$$\le\Big\{\int_a^b \int_0^1 \max_{n\le m}\Big|\sum_{k=0}^n <f,w_k> w_k(\tau)\Big|^p K_m(t,\tau)d\tau\Big[\int_0^1 K_m(t,\tau)d\tau\Big]^{p/q}dt\Big\}^{1/p}$$
$$=\Big\{\int_0^1 \max_{n\le m}\Big|\sum_{k=0}^n <f,w_k> w_k(\tau)\Big|^p \Big(\int_a^b K_m(t,\tau)dt\Big)d\tau\Big\}^{1/p}$$
$$=O(1)\Big\{\int_0^1 \max_{n\le m}\Big|\sum_{k=0}^n <f,w_k> w_k(\tau)\Big|^p d\tau\Big\}^{1/p}\Big\{\sum_{\nu=0}^\infty\Big|\int_a^b g_\nu(t)dt\Big|\Big\}^{1/p},$$

by (8) we get

$$\Big\{\int_a^b \max_{n\le m}\Big|\sum_{k=0}^n <f,w_k> g_k(t)\Big|^p dt\Big\}^{1/p}=O(1).$$

Now Lemma leads to the following theorem.

Theorem 4. *If $\{g_k\}$ is the product system of a weakly multiplicative system, then the series (7) with $f\in L^p_{[0,1]}$ $(1<p<\infty)$ is p-maximally convergent a.e. on $[a,b]$.*

Set

$$h_n(t):=\sum_{k=0}^n\Big(1-\frac{k}{n+1}\Big)<f,g_k> w_k(t),$$

where $f\in L^p_{[a,b]}$ and $\{g_k\}$ is the product system of a weakly multiplicative system. We shall prove that $h_n\in L^p_{[0,1]}$. Indeed, since (see [8])

$$\operatorname{vrai}\sup_n\int_0^1\Big|\sum_{k=0}^n\Big(1-\frac{k}{n+1}\Big)w_k(\tau)w_k(t)\Big|d\tau=O(1),$$

using the Hölder inequality, we get

$$
\begin{aligned}
&\Big\{\int_0^1 |h_n(t)|^p\,dt\Big\}^{1/p}\\
&=\Big\{\int_0^1 \Big|\int_0^1 \sum_{k=0}^{n}\Big(1-\frac{k}{n+1}\Big)w_k(t)w_k(\tau)\sum_{\nu=0}^{2^n-1} w_\nu(\tau) < f, g_\nu > d\tau\Big|^p\,dt\Big\}^{1/p}\\
&\le \Big\{\int_0^1\int_0^1 \Big|\sum_{k=0}^{n}\Big(1-\frac{k}{n+1}\Big)w_k(t)w_k(\tau)\Big|\,\Big|\sum_{\nu=0}^{2^n-1} w_\nu(\tau) < f, g_\nu >\Big|^p\,d\tau\\
&\qquad\times\Big[\int_0^1 \Big|\sum_{k=0}^{n}\Big(1-\frac{k}{n+1}\Big)w_k(t)w_k(\tau)\Big|\,d\tau\Big]^{p/q}dt\Big\}^{1/p}\\
&=O(1)\Big\{\int_0^1\int_0^1 \Big|\sum_{k=0}^{n}\Big(1-\frac{k}{n+1}\Big)w_k(t)w_k(\tau)\Big|\,dt\,\Big|\sum_{\nu=0}^{2^n-1} w_\nu(\tau) < f, g_\nu >\Big|^p\,d\tau\Big\}^{1/p}\\
&=O(1)\Big\{\int_0^1 \Big|\int_a^b f(u)K_n(u,\tau)du\Big|^p\,d\tau\Big\}^{1/p}
\end{aligned}
$$

and using the Hölder inequality once again, we have

$$
\begin{aligned}
&\Big\{\int_0^1 |h_n(t)|^p\,dt\Big\}^{1/p}\\
&\qquad=O(1)\Big\{\int_0^1\int_a^b |f(u)|^p\,K_n(u,\tau)du\Big[\int_a^b K_n(u,\tau)du\Big]^{p/q}d\tau\Big\}^{1/p}\\
&\qquad=O(1)\Big\{\int_a^b |f(u)|^p\int_0^1 K_n(u,\tau)d\tau du\Big\}^{1/p}\Big\{\sum_{\nu=0}^{2^n-1}\Big|\int_a^b g_\nu(u)du\Big|\Big\}^{1/q}\\
&\qquad=O(1)\Big\{\int_a^b |f(u)|^p\,du\Big\}^{1/p}.
\end{aligned}
$$

Therefore $h(t) := \lim_n h_n(t) \in L^p_{[0,1]}$ and $< f, g_\nu >$ are the Walsh–Fourier coefficients of h for every $k = 0, 1, 2, \ldots$:

$$
\begin{aligned}
< h, w_\nu > &= \int_0^1 w_\nu(t)\lim_n \sum_{k=0}^{n}\Big(1-\frac{k}{n+1}\Big) < f, g_k > w_k(t)dt\\
&= \lim_n \sum_{k=0}^{n}\Big(1-\frac{k}{n+1}\Big) < f, g_k > \int_0^1 w_k(t)w_\nu(t)dt\\
&= \lim_n\Big(1-\frac{\nu}{n+1}\Big) < f, g_\nu > = < f, g_\nu > .
\end{aligned}
$$

This yields the following result.

Theorem 5. *If $\{g_k\}$ is a product system of a weakly multiplicative system, then the series*

$$\sum_{k=0}^{\infty} < f, g_k > g_k(t),$$

where $f \in L^p_{[a,b]}$, is p-maximally convergent a.e. on $[a,b]$.

4. p-MAXIMAL λ-BOUNDEDNESS

Let $\{g_k\}$ be the product system of a weakly multiplicative system. From Theorem 4 it follows that the series (7) is for every $f \in L^p_{[a,b]}$ $(1 < p < \infty)$ p-maximally convergent a.e. on $[a,b]$ (and in $L^p_{[a,b]}$) to some function $g \in L^p_{[a,b]}$.

We will prove the following theorem.

Theorem 6. *Let $\{g_k\}$ be the product system of a weakly multiplicative system and let $f \in L^p_{[0,1]}$. If the series*

$$\sum_{k=0}^{\infty} < f, w_k > w_k(t) \tag{9}$$

is p-maximally λ-bounded a.e. on $[0,1]$, then the series (7) for the same f is p-maximally λ-bounded a.e. on $[a,b]$.

Proof. Let (s_m) be a sequence of natural numbers. Because the Walsh system is orthogonal, by the Minkowsky inequality we obtain

$$\begin{aligned} C_m &:= \Big\{ \int_a^b \max_{n\le m} \lambda_n^p \Big| \sum_{k=0}^{n} < f, w_k > g_k(t) - g(t) \Big|^p dt \Big\}^{1/p} \\ &\le \Big\{ \int_a^b \max_{n\le m} \lambda_n^p \Big[\int_0^1 \Big| \sum_{k=0}^{n} < f, w_k > w_k(\tau) - f(\tau) \Big| K_{s_m}(t,\tau) d\tau \Big]^p dt \Big\}^{1/p} \\ &\quad + \Big\{ \int_a^b \max_{n\le m} \lambda_n^p \Big| \int_0^1 f(\tau) K_{s_m}(t,\tau) d\tau - g(t) \Big|^p dt \Big\}^{1/p}. \end{aligned}$$

By the Hölder inequality it follows that

$$\begin{aligned} C_m \le \Big\{ \int_a^b \max_{n\le m} \lambda_n^p \int_0^1 \Big| \sum_{k=0}^{n} < f, w_k > w_k(\tau) - f(\tau) \Big|^p K_{s_m}(t,\tau) d\tau \\ \times \Big[\int_0^1 K_{s_m}(t,\tau) d\tau \Big]^{p/q} dt \Big\}^{1/p} \\ + \Big\{ \int_a^b \lambda_m^p \Big| \sum_{\nu=0}^{2^{s_m}-1} < f, w_\nu > g_\nu(t) - g(t) \Big|^p dt \Big\}^{1/p}. \end{aligned}$$

Thus

$$C_m \leq \Big\{\int_0^1 \max_{n\leq m} \lambda_n^p \Big|\sum_{k=0}^{n} < f, w_k > w_k(\tau) - f(\tau)\Big|^p \Big(\int_a^b K_{s_m}(t,\tau)dt\Big)d\tau\Big\}^{1/p}$$
$$+ \Big\{\int_a^b \lambda_m^p \Big|\sum_{\nu=0}^{2^{s_m}-1} < f, w_\nu > g_\nu(t) - g(t)\Big|^p dt\Big\}^{1/p}$$
$$= O(1)\Big\{\int_0^1 \max_{n\leq m} \lambda_n^p \Big|\sum_{k=0}^{n} < f, w_k > w_k(\tau) - f(\tau)\Big|^p d\tau\Big\}^{1/p}$$
$$+ \Big\{\int_a^b \lambda_m^p \Big|\sum_{\nu=0}^{2^{s_m}-1} < f, w_\nu > g_\nu(t) - g(t)\Big|^p dt\Big\}^{1/p}.$$

From Theorem 4 it follows that there exists a subsequence (s_m) of natural numbers such that

$$\lim_m \int_a^b \lambda_m^p \Big|\sum_{\nu=0}^{2^{s_m}-1} < f, w_\nu > g_\nu(t) - g(t)\Big|^p dt = 0.$$

Therefore we have

$$C_m = O(1)\Big\{\int_0^1 \max_{n\leq m} \lambda_n^p \Big|\sum_{k=0}^{n} < f, w_k > w_k(\tau) - f(\tau)\Big|^p d\tau\Big\}^{1/p} + O(1).$$

The proof is complete.

Using Theorem 3, we can prove the following theorem.

Theorem 7. *Let $\{g_k\}$ be the product system of a q-weakly multiplicative system and let $f \in L^p_{[0,1]}$ where $\frac{1}{p} + \frac{1}{q} = 1$. If the series (9) is p-maximally λ-bounded a.e. on $[0,1]$ for f, then the series (7) is 1-maximally λ-bounded a.e. on $[a,b]$ for the same f.*

Proof. Let (s_m) be a sequence of natural numbers. As in the proof of Theorem 6, we obtain

$$D_m := \int_a^b \max_{n\leq m} \lambda_n \Big|\sum_{k=0}^{n} < f, w_k > g_k(t) - g(t)\Big| dt$$
$$\leq \int_a^b \max_{n\leq m} \lambda_n \Big|\int_0^1 \Big(\sum_{k=0}^{n} < f, w_k > w_k(\tau) - f(\tau)\Big) K_{s_m}(t,\tau)d\tau\Big| dt$$
$$\leq \int_0^1 \max_{n\leq m} \lambda_n \Big|\sum_{k=0}^{n} < f, w_k > w_k(\tau) - f(\tau)\Big| \Big|\int_a^b K_{s_m}(t,\tau)dt\Big| d\tau$$
$$+ \int_a^b \lambda_m \Big|\sum_{\nu=0}^{2^{s_m}-1} < f, w_\nu > g_\nu(t) - g(t)\Big| dt.$$

By Theorem 3, the series (7) is 1-maximally λ-convergent a.e. on $[a, b]$. Therefore the series (7) converges in $L^1_{[a,b]}$ as well. So, there exists a sequence of natural numbers s_m such that

$$\lim_m \int_a^b \lambda_m \left| \sum_{\nu=0}^{2^{s_m}-1} < f, w_\nu > g_\nu(t) - g(t) \right| dt = 0.$$

Therefore by the Hölder inequality we have

$$D_m \leq \left\{ \int_0^1 \max_{n \leq m} \lambda_n^p \left| \sum_{k=0}^{n} < f, w_k > w_k(\tau) - f(\tau) \right|^p d\tau \right\}^{1/p}$$
$$\times \left\{ \int_0^1 \left| \int_a^b K_{s_m}(t, \tau) dt \right|^q d\tau \right\}^{1/q} + O(1)$$

and the proof is complete by the hypotheses of theorem.

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Multiplikatiivsete süsteemidega määratud funktsionaalridade koonduvus ja λ-tõkestatus

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Artiklis on käsitletud rida $\sum c_k g_k(t)$, kus süsteem $\{g_k\}$ on mingi multiplikatiivse süsteemi korrutissüsteem, ja leitud piisavaid tingimusi selle rea p-maksimaalse kiirusega koonduvuse jaoks. On vaadeldud ka rida $\sum < f, w_k > g_k(t)$, kus $f \in L^p_{[0,1]}$ ja $\{w_k\}$ on Walshi süsteem, ning tõestatud, et see rida koondub peaaegu kõikjal erinevate korrutissüsteemide korral. Töö viimases osas on uuritud selle rea λ-tõkestatust peaaegu kõikjal.