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On the formation of solitons in media with higher-order dispersive effects

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Abstract. Wave propagation in microstructured materials is strongly influenced by dispersive effects. In the present paper two Korteweg–de Vries type model equations, with the third- and fifth-order dispersion, are studied. Both model equations are solved numerically, under harmonic initial and periodic boundary conditions, by making use of the pseudospectral method. The character of the solution is found to be solitonic in both cases. The number of visible and hidden solitons in the emerging train is detected. Phenomena of recurrence and super-recurrence are examined.

Key words: microstructure, dispersion, nonlinearity, Korteweg–de Vries type evolution equations, solitons, pseudospectral method.

1. INTRODUCTION

Studies of microstructured materials have led to the need to take also dispersive effects of higher order into account. Together with higher-order nonlinearity these effects can cause dramatic changes in the behaviour of the emerging waves. In the present paper two one-dimensional model equations are studied which both are based on a Korteweg–de Vries (KdV) equation

$$u_t + [P(u)]_x + du_{xxx} + bu_{xxxxx} = 0. \qquad (1)$$

Here u is the excitation, t the time coordinate, x the space coordinate, $P(u)$ the elastic potential, d the third-order dispersion parameter and b that of the fifth order.

The elastic potential $P(u)$ is expressed in the polynomial form and the order of nonlinearity is determined by the highest-order term in $P(u)$.

First, if

$$P(u) = \left(-\frac{u^2}{2} + \frac{u^4}{4}\right), \tag{2}$$

we have a KdV-type equation with quartic nonlinearity, and both the third- and fifth-order dispersion (KdV435). Here the fourth-order elastic potential (2) depicts quartic nonlinearity in the simplest (symmetrical) form that can possess two minima. The higher-order dispersion can be caused by dislocations in the crystal structure. For instance, this is the case of martensitic-austenitic shape-memory alloys [1−4].

Second, if we have the quadratic elastic potential

$$P(u) = \frac{u^2}{2}, \tag{3}$$

then Eq. (1) results in the fifth-order KdV (FKdV) equation, which is considered as the second model case. Note that the same potential corresponds to the KdV equation, only the fifth-order dispersion has been added.

Equation (1), with quartic nonlinearity (2), $d > 0$ and $b > 0$, is studied in [5−9]. In [5,6] the problem (1), (2) is studied under periodic boundary conditions and harmonic initial conditions. In these papers three different regions in the dispersion parameters plane are detected. In the first region the third-order dispersion effects dominate over the fifth-order effects and a train of negative solitons forms from the initial harmonic excitation. In the second region the fifth-order dispersive effects take over the third-order effects and a train of positive solitons forms from the initial harmonic excitation. In the third region between the previous two regions the rivalry between the third- and fifth-order dispersion creates the situation where both the trains of negative and positive solitons might start to form simultaneously.

The numerical detection of solitary wave solutions to Eq. (1) with quartic nonlinearity (2) is studied in [7−9]. In these papers it is shown that (i) both, negative and positive solitary wave solutions can be detected under asymptotic boundary conditions and (ii) interaction of two positive solitons is inelastic, whereas interaction between two negative solitons is elastic – they behave like solitons.

In the present paper the case $d > 0$ and $b < 0$ is considered. Both model equations are integrated numerically under periodic boundary conditions and harmonic initial conditions. The main goal is to analyse the time-space behaviour of solutions – dependences between dispersion parameters and the number of solitons (including hidden solitons) are detected, and recurrence and super-recurrence phenomena are discussed making use of spectral amplitudes. The problem is stated in Section 2. In Sections 3 and 4 results are presented and discussed, and in Section 5 some conclusions are drawn.

2. STATEMENT OF THE PROBLEM

Both proposed model equations are integrated numerically under periodic boundary conditions

$$u(x,t) = u(x + 2n\pi, t), \quad n = \pm 1, \pm 2, \ldots \tag{4}$$

The initial excitation is given by

$$u(x,0) = \sin x, \qquad 0 \leq x \leq 2\pi. \tag{5}$$

In the present study the dispersion parameter b is considered to be negative, which results in pure normal dispersion. The notations

$$d_l = -\log d \quad \text{and} \quad b_l = -\log(-b) \tag{6}$$

are introduced for future analysis. Note that dispersion parameters have opposite signs!

Both problems are solved numerically for the range of the dispersion parameters d_l and b_l:

$$0.8 \leq d_l \leq 2.4 \quad \text{and} \quad 0.4 \leq b_l \leq 4.8. \tag{7}$$

Considering previous experience, the pseudospectral method (PsM) [10] is used for numerical simulations of wave propagation. The results are analysed using discrete spectral characteristics [11].

The goals of the present paper are to find numerical solutions to the proposed model equations KdV435 and FKdV, to examine the behaviour and characteristics of the solutions over relatively long time intervals, and to study the recurrence and super-recurrence phenomenon.

3. TYPE OF THE SOLUTION

On the basis of numerical results over a wide range of dispersion parameters (7) one can say that the solution type of the KdV435 equation with proposed boundary (4) and initial (5) conditions is a train of negative solitons (Fig. 1), except the case of very weak dispersion. In this case (discussed below), besides the train of negative solitons, a train of positive solitons emerges. The solution type of the FKdV equation is a train of positive solitons (Fig. 2).

Visible solitons. We call a soliton visible if it can be detected from the initial train of equally spaced solitons. The number of visible solitons in the train depends on the values of the dispersion parameters and is presented in Table 1. However, in some cases more solitons become visible after a sufficiently long integration interval.

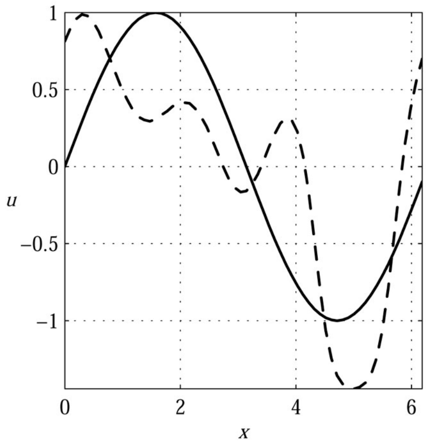

Fig. 11. Wave profiles of the solution (KdV435 equation, $d_l = 1.6$ and $b_l = 4.8$; the solid curve corresponds to the initial time moment $t = 0$, the dashed curve to the time moment $t = 4.0$).

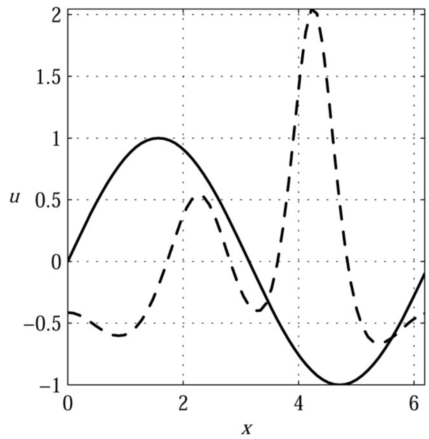

Fig. 2. Wave profiles of the solution (FKdV equation, $d_l = 1.2$ and $b_l = 4.8$; the solid curve corresponds to the initial time moment $t = 0$, the dashed curve to the time moment $t = 10.3$).

Table 1. The number of visible solitons in the train according to the values of the dispersion parameters. The left part corresponds to the KdV435 equation, the right to the FKdV equation

b_l	d_l					b_l	d_l				
	0.8	1.2	1.6	2.0	2.4		0.8	1.2	1.6	2.0	2.4
0.4	1	1	1	1	1	0.4	1	1	1	1	1
1.2	1	1	1	1	1	1.2	1	2	2	2	2
2.0	2	2	2	2	2	2.0	2	2	2	2	2
2.8	2	2	3	3	3	2.8	2	3	3	3	4
3.6	2	2	3	4	4	3.6	2	3	4	5	5
4.0	2	2	3	4	5	4.0	2	3	4	5	6
4.4	2	2	3	4	6	4.4	2	3	4	6	7
4.8	2	2	3	4	7	4.8	2	3	4	6	8

Hidden solitons. In addition to visible solitons, also hidden solitons were detected. The concept of hidden (virtual) solitons is described in detail in [11−13]. The main characteristics of hidden solitons are as follows: hidden solitons can emerge from harmonic excitation, have very small energy and amplitude, interact with visible solitons and cause distinct changes in visible soliton amplitudes and trajectories during interaction, can be detected in wave profiles for a short time interval only when several soliton interactions have taken place, if ever. Physical essence of visible and hidden solitons is the same. The influence of a hidden soliton on trajectories and amplitudes of visible solitons is presented in Fig. 3. On the basis of numerical results one can determine at least one hidden soliton in case of both

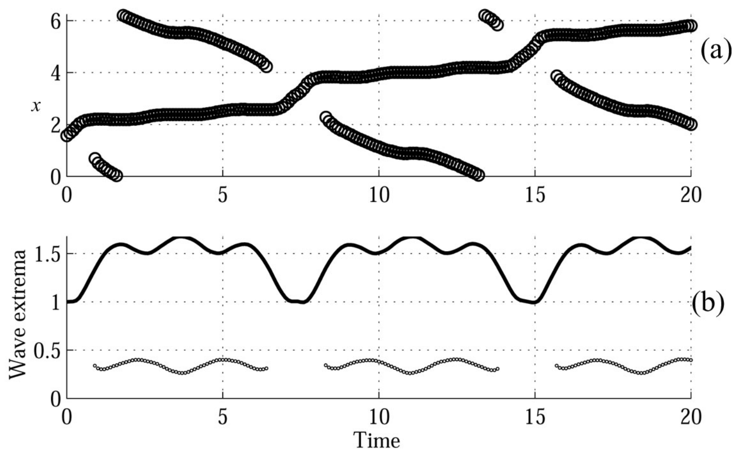

Fig. 3. Trajectories (a) and amplitude curves (b). FKdV case, $d_l = 2.4$ and $b_l = 2.0$.

model equations.

However, in the KdV435 case, with the values of dispersion parameters $d_l = 2.4$ and $b_l = \{4.0, 4.4, 4.8\}$, one cannot explain the behaviour of the solution only in terms of visible and hidden negative solitons. In those cases wave profiles are stretched in the negative as well as in the positive direction (see Fig. 4). Also, additional curves appear in the plots of trajectories and amplitudes. The explanation for such a phenomenon is that in addition to the train of negative solitons, also positive solitons start to emerge.

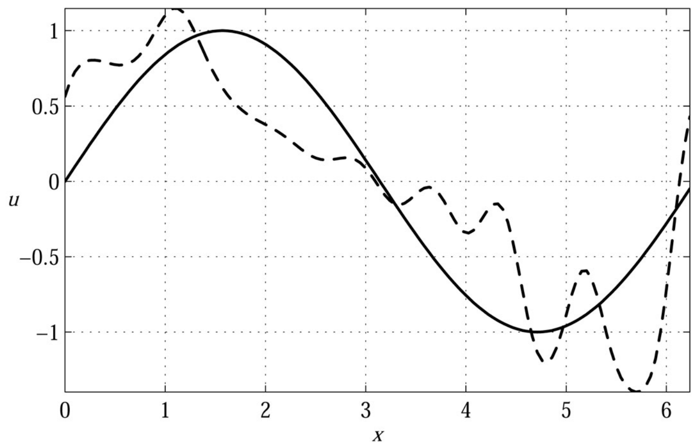

Fig. 4. Typical wave profile for the case when trains of positive and negative solitons emerge simultaneously (KdV435 equation, $d_l = 2.4$ and $b_l = 4.4$). The solid line corresponds to the time moment $t = 0$ and dashed line to $t = 2.5$.

4. RECURRENCE AND PERIODICITY

The concept of recurrence and recurrence time was first introduced by Zabusky and Kruskal [14]. The ideal recurrence time t_R can be described as an interval of time after which all solitons that have emerged from initial harmonic excitation arrive in such a phase that they reconstruct the initial state. Such a state should be repeated at $t_{R_n} \approx nt_R,\ n = 2, 3, \ldots$ However, it is shown in [15] that even the first recurrence at $t_{R_1} \equiv t_R$ is incomplete, leading to incompleteness of the following recurrences as well. One reason for such phenomena can be asymmetry of energy sharing between the modes [16].

In order to estimate the recurrence time one can use the spectral amplitudes: if $t \to t_R$, then $SA_1(t) \to 1$ and $SA_k(t) \to 0,\ k > 1$. As stated above, in many cases even the first recurrence is incomplete. However, in the same cases at certain time moments $SA_1(t_{R_S}) > SA_1(t_{R_1})$. Such a phenomenon is called super-recurrence. It is clear that if the recurrence is repeated at $t_{R_n} \approx nt_R$, then the solution can be called (quasi)periodic. For detection of the recurrence time the first three spectral amplitudes ($SA_1,\ SA_2$, and SA_3) are taken into account. We are looking for the situation where the first spectral amplitude SA_1 has maxima and other spectral amplitudes (e.g. SA_2 and SA_3) have minima at the same moment of time. If we can detect the first recurrence at $t = t_{R_1}$ (first maxima of the first spectral amplitude after initiation), then we are looking for maxima of the first spectral amplitude at the time moments $t_{R_n} \approx nt_R,\ n = 2, 3, \ldots$

We say that the solution is periodic if at least three consecutive maxima of the first spectral amplitude do not differ much from the initial value 1 (at the time moments $t = t_{R_n},\ n = 2, 3, \ldots$, and $t_{R_1} \approx t_{R_2} - t_{R_1} \approx t_{R_3} - t_{R_2} \approx \ldots$).

Recurrence. From numerical experiments one can detect that the recurrence time t_R depends on the values of the logarithmic dispersion parameters d_l and b_l (see Table 2). In case of both model equations the length of the recurrence time increases as value(s) of the logarithmic dispersion parameter(s) increases, but in the case of a fixed value of d_l (or b_l) it has a limit value.

Table 2. Recurrence time t_R against the logarithmic dispersion parameters. The left part corresponds to the KdV435 equation and right to the FKdV equation

b_l	d_l					b_l	d_l				
	0.8	1.2	1.6	2.0	2.4		0.8	1.2	1.6	2.0	2.4
0.4	0.49	0.51	0.52	0.52	0.52	0.4	0.49	0.51	0.52	0.52	0.52
1.2	2.1	2.6	2.9	3.0	3.0	1.2	2.0	2.5	2.7	2.8	2.8
2.0	4.3	6.5	8.3	9.3	9.9	2.0	3.9	5.4	6.5	7.1	7.4
2.8	5.1	8.6	12.9	*17.4*	*20.7*	2.8	4.7	7.1	**9.8**	**15.4**	**17.8**
3.6	5.3	9.2	15.7	**25.0**	**20.2**	3.6	4.8	7.8	**11.8**	**25.7**	**13.7**
4.0	5.3	9.4	16.3	**28.5**	**27.9**	4.0	4.8	7.9	*12.4*	**18.8**	**17.8**
4.4	5.3	9.4	16.7	**19.3**	**16.2**	4.4	4.8	8.0	*12.7*	**19.1**	**21.5**
4.8	5.3	9.4	16.9	**21.0**		4.8	4.8	8.0	*12.8*	**19.8**	

For both proposed model equations three different domains in the $d_l - b_l$ plane were detected, depending on the behaviour of spectral amplitudes:

(1) In the first domain (see normal font in Table 2) at least three consecutive recurrences can be detected, while time intervals between them do not differ much from each other. Values of the first spectral amplitude SA_1 at t_{R_n} are close to the initial value 1; the difference is less than 0.1 (Fig. 5).

(2) In the second domain (see bold font and empty cells in Table 2) one cannot detect the first recurrence, or time intervals between the first three consecutive recurrences do differ much from each other and values of the first spectral amplitude SA_1 at t_{R_n} differ from the initial value by more than 0.1 (see Fig. 6).

(3) The third domain (see italic font in Table 2) is characterized by the transitional behaviour between the first two domains. Here one can detect easily the first three consecutive recurrences, but time intervals between consecutive recurrences differ much from each other (see Fig. 7).

More concrete examples and time intervals $t_{R_2} - t_{R_1}$ and $t_{R_3} - t_{R_2}$ are presented in [13].

Super-recurrence. Extra long numerical integrations (numerical simulations up to the time moment $t = 1000$) were carried out in order to detect super-recurrence. The phenomenon of super-recurrence could not be detected in few cases only. In these cases the value of the first spectral amplitude SA_1 at the first recurrence is

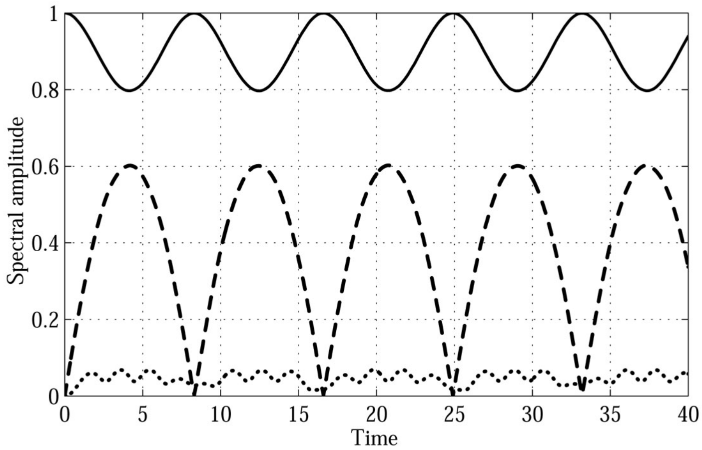

Fig. 5. The first three spectral amplitudes; the KdV435 equation, case $d_l = 1.6$ and $b_l = 2.0$ (the solid line corresponds to the first spectral amplitude SA_1, the dashed line to the second spectral amplitude SA_2, and the dotted line to the third spectral amplitude SA_3).

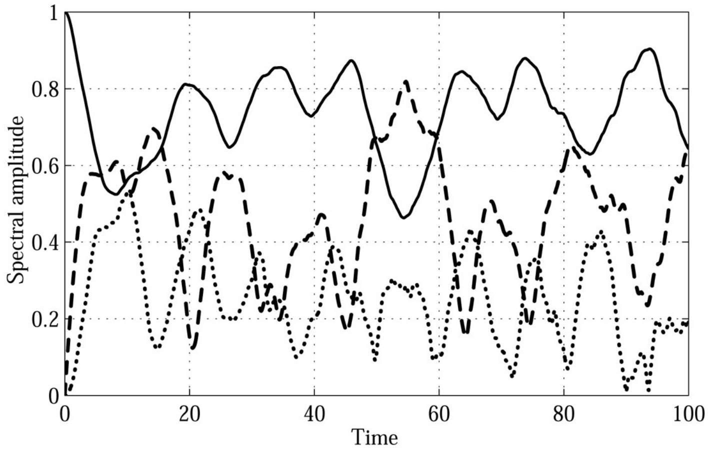

Fig. 6. The first three spectral amplitudes; the KdV435 equation, case $d_l = 2.0$ and $b_l = 4.4$ (the solid line corresponds to the first spectral amplitude SA_1, the dashed line to the second spectral amplitude SA_2, and the dotted line to the third spectral amplitude SA_3).

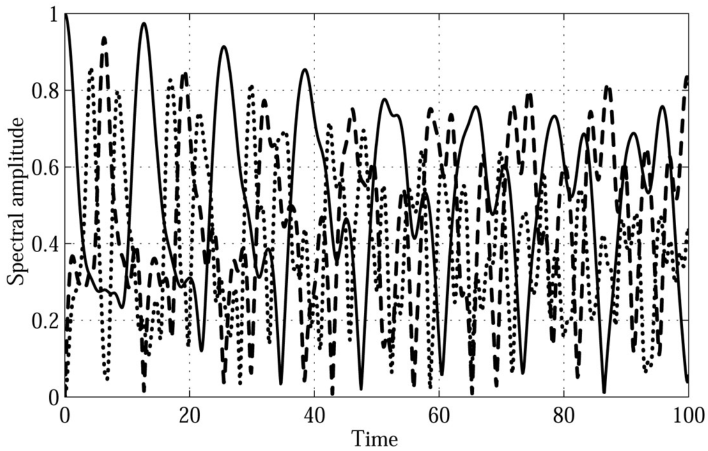

Fig. 7. The first three spectral amplitudes; the FKdV equation, case $d_l = 1.6$ and $b_l = 4.4$ (the solid line corresponds to the first spectral amplitude SA_1, the dashed line to the second spectral amplitude SA_2, and the dotted line to the third spectral amplitude SA_3).

very close to the initial value 1 and, as a rule, much higher than the values in the cases where the super-recurrence was detected (see [13]). An example of the super-recurrence phenomenon is presented in Fig. 8 where the values of the first spectral amplitude $SA_1(t_{R_n})$ are plotted against the number of recurrences n.

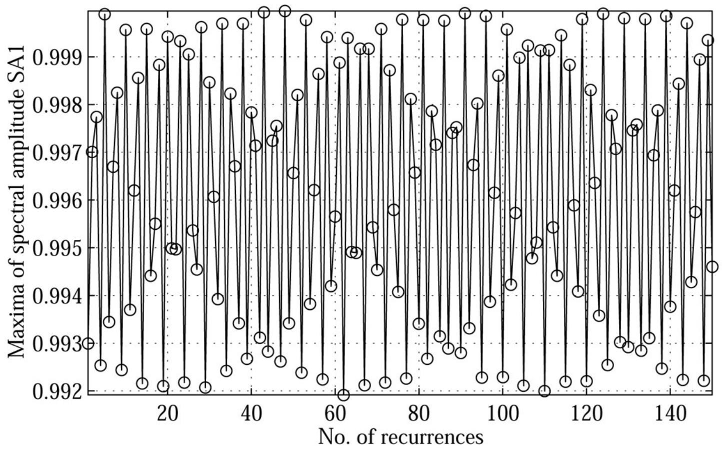

Fig. 8. Maxima of the first spectral amplitude SA_1 against the number of recurrences. FKdV equation, $d_l = 0.8$ and $b_l = 4.4$.

5. CONCLUSIONS

Numerical solutions of two KdV-type model equations (KdV435 and FKdV) are found and analysed. The initial harmonic wave results in the train of positive solitons in the case of the FKdV equation. However, in the case of the KdV435 equation, the solution type is the train of negative solitons, except the very weak dispersion case which results in simultaneous formation of trains of negative as well as positive solitons. It is demonstrated that, besides visible solitons, at least one hidden soliton can be detected in all studied cases. Recurrence and super-recurrence phenomena are analysed by making use of the first three spectral amplitudes. One can conclude that the weaker the dispersion, the worse the recurrence, like in the case of the KdV equation.

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Solitonide formeerumisest kõrgemat järku dispersiooniga keskkondades

Olari Ilison ja Andrus Salupere

Lainelevi mikrostruktuuriga keskkonnas on tugevalt mõjutatud nimetatud keskkonna dispersiivsetest omadustest. Käesolevas artiklis on vaadeldud kahte Kortewegi–de Vriesi tüüpi (KdV435 ja FKdV) mudelvõrrandit, milles dispersioon on avaldatud kolmandat ja viiendat järku liikmete abil. Mõlema mudelvõrrandi jaoks on leitud numbrilised lahendid harmoonilise alg- ja perioodilise rajatingimuse korral. Numbriliseks meetodiks on pseudospektraalmeetod. Lahenditüübiks on KdV435 korral negatiivsete solitonide jada, FKdV korral koosneb aga tekkiv solitonide jada positiivsetest solitonidest. Leitud on nähtavate ja peidetud solitonide arv jadas. Samuti on uuritud rekurentsi ja super-rekurentsi nähtust.