Proc. Estonian Acad. Sci. Phys. Math., 2002, **51**, 1, 3–17

Dual pairs of sequence spaces. II

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Received 8 August 2001, in revised form 12 November 2001

Abstract. The authors proceed their investigation of dual pairs (E, E^S), where E is a sequence space, S is a K-space on which a sum s is defined in the sense of Ruckle, and E^S is the space of all corresponding factor sequences. Here, the particular case is considered that the sum s has the representation $s(z) = \lim_\gamma \sum_k v_{\gamma k} z_k$ $(z \in S)$, where Γ is a directed set of indices γ and $(v_{\gamma k})_k$ is a finite sequence for each $\gamma \in \Gamma$. On the basis of this representation the S-sections of any sequence $x = (x_k)$ and both, their convergence $(AK(S))$ and boundedness $(AB(S))$ in K-spaces E are studied. Further, inclusion theorems due to Bennett and Kalton are proved in this more general situation. Following an idea of Schaefer to consider "section convergence barrels", the notion of $AK(S)$-barrelled K-spaces is introduced which leads to the result that a Mackey K-space E containing all finite sequences is $AK(S)$-barrelled if and only if $E^S \subset E'$. The paper covers some results concerning the Köthe–Toeplitz duals and related section properties, for example, the $\beta(T)$-dual and the STK-property (considered by Buntinas and Meyers).

Key words: topological sequence spaces, Köthe–Toeplitz duals, section convergence, sum space, solid (normal) topology, inclusion theorems.

1. INTRODUCTION

In [1] the authors defined and investigated dual pairs (E, E^S), where E is a sequence space, S is a K-space on which a sum s is defined in the sense of Ruckle [2], and E^S is the space of all corresponding factor sequences. Moreover, in generalization of the SAK-property in the case of the dual pair (E, E^β) and matrix maps, the SK-property and the quasi-matrix maps were introduced and studied. In that general situation well-known inclusion theorems due to Bennett and Kalton [3] and Grosse-Erdmann [4] were proved. The authors justified these generalizations by several applications to different kinds of Köthe–Toeplitz duals and related section properties, for example, the $\beta(T)$-dual and the STK-property (cf. Buntinas [5]).

In this note we consider dual pairs (E, E^S), where the sum s has the representation

$$s(z) = \lim_{\gamma} \sum_{k} v_{\gamma k} z_k \quad (z \in S)$$

and Γ is a directed set of indices γ, $(v_{\gamma k})_k$ is a finite sequence for each $\gamma \in \Gamma$. This representation enables us to define in Section 2 the S-sections of any sequence $x = (x_k)$ and study in K-spaces E the properties $AK(S)$ and $AB(S)$. In Section 3 we complete an inclusion theorem which is proved in [1] and generalize a further inclusion theorem due to Bennett and Kalton [3]. These results are applied to certain dual pairs (E, E^S), where S is c_T, bv_T, and fs, respectively. In Section 4, following Schaefer [6], we introduce the $AK(S)$-barrelledness of K-spaces and show that a Mackey K-space E containing φ is $AK(S)$-barrelled if and only if $E^S \subset E'$, i.e., the functional, defined by $x \mapsto s((u_k x_k))$, is continuous on E for each $u \in E^S$. The last result is verified for the particular cases $S = cs$ and $S = \ell$.

The terminology from the theory of locally convex spaces and summability is standard; we refer to Wilansky [7,8] and Boos [9].

Let ω be the space of all complex (or real) sequences and φ the subspace of all finitely nonzero sequences. Obviously, $\varphi = \operatorname{span}\{e^k \mid k \in \mathbb{N}\}$, where $e^k := (0, \ldots, 0, 1, 0 \ldots)$ with 1 in the kth position, and φ contains the *sections* $x^{[n]} := \sum_{k=1}^{n} x_k e^k$ $(n \in \mathbb{N})$ of all sequences $x \in \omega$.

A sequence space is a subspace of ω. If a sequence space E carries a locally convex topology such that the coordinate functionals π_n $(n \in \mathbb{N})$ defined by $\pi_n(x) := x_n$ $(x \in \omega)$ are continuous, then E is called a *K-space*. Note that φ is $\sigma(E', E)$-dense in the topological dual E' for each K-space E, where we identify φ with $\operatorname{span}\{\pi_n \mid n \in \mathbb{N}\}$. For any K-space E containing φ, the *f-dual* E^f is defined by

$$E^f \;:=\; \left\{ u_f := (f(e^k)) \;\middle|\; f \in E' \right\}.$$

A Fréchet (Banach) K-space is said to be an *FK-(BK-)space*. The following BK-spaces will be important in the sequel:

$$m := \left\{ x \in \omega \;\middle|\; \sup_k |x_k| < \infty \right\}, \qquad c := \left\{ x \in \omega \;\middle|\; \lim x := \lim_k x_k \text{ exists} \right\},$$
$$c_0 := \left\{ x \in c \;\middle|\; \lim x = 0 \right\}, \qquad bv := \left\{ x \in \omega \;\middle|\; \sum_k |x_k - x_{k+1}| < \infty \right\},$$
$$bs := \left\{ x \in \omega \;\middle|\; \sup_n \left| \sum_{k=1}^{n} x_k \right| < \infty \right\}, \quad cs := \left\{ x \in \omega \;\middle|\; \sum_k x_k \text{ converges} \right\},$$
$$\ell := \left\{ x \in \omega \;\middle|\; \sum_k |x_k| < \infty \right\}.$$

Furthermore, ω is an FK-space under its product topology.

For sequence spaces E and F we define

$$E{\cdot}F := \Big\{ux := (u_k x_k) \ \Big| \ u \in E, x \in F\Big\}, \ E^F := \Big\{u \in \omega \ \Big| \ \forall\, x \in E : ux \in F\Big\}.$$

The α-*dual* and β-*dual* of E are defined as $E^\alpha := E^\ell$ and $E^\beta := E^{cs}$. If $A = (a_{nk})$ is an infinite matrix such that $Ax := \left(\sum_k a_{nk} x_k\right)_n$ exists and $Ax \in F$ for each $x \in E$, then the linear map

$$A : E \to F\,, \quad x \mapsto Ax \tag{1.1}$$

is called a *matrix map*.

Let S be a K-space with $\varphi \subset S$ and let $s \in S'$ be a sum on S, that is,

$$s(z) = \sum_k z_k \ \text{ for each } \ z \in \varphi$$

(cf. Ruckle [2]). If E is a sequence space containing φ, then (E, E^S) is a dual pair under the bilinear functional

$$\langle\ ,\ \rangle : E \times E^S \to \mathbb{K} \ : \ (x, u) \mapsto \langle x, u\rangle := s(ux).$$

Since $\varphi \subset E^S$, $(E, \sigma(E, E^S))$ is a K-space. E is called an SK-*space* if $E = E_{SK}$, where

$$E_{SK} := \{x \in E \mid \forall\, f \in E' \ : \ u_f x \in S \ \text{ and } \ f(x) = s(u_f x)\}.$$

For example, $(E, \tau(E, E^S))$ is an SK-space. If we put

$$S \ := \ cs \quad \text{and} \quad s(z) \ := \ \sum_k z_k \quad (z \in cs), \tag{1.2}$$

then E_{SK} is the subspace of all elements of E which are the weak limits of their sections.

Let $A = (a_{nk})$ be an infinite matrix. We put $a^{(n)} := (a_{nk})_k$ $(n \in \mathbb{N})$ and

$$\omega_{\mathfrak{A}} := \bigcap \Big\{\{a^{(n)}\}^S \ \Big| \ n \in \mathbb{N}\Big\}.$$

For a sequence space F we define

$$F_{\mathfrak{A}} := \Big\{x \in \omega_{\mathfrak{A}} \ \Big| \ \mathfrak{A}x := \big(s\big(a^{(n)}x\big)\big) \in F\Big\}.$$

If E is a sequence space with $E \subset F_{\mathfrak{A}}$, then the linear map

$$\mathfrak{A} : E \to F, \ x \mapsto \mathfrak{A}x$$

is called a *quasi-matrix map*.

Proposition 1.1 (cf. Proposition 4.2 in [1]). *Let E and F be K-spaces. Each of the following statements implies the continuity of any quasi-matrix map $\mathfrak{A} : E \to F$:*

(a) *S and F are separable FK-spaces, E is a Mackey space and $(E', \sigma(E', E))$ is sequentially complete.*

(b) *S and F are FK-spaces and E is barrelled.*

Remark 1.2. In Proposition 4.2 of [1] the statements in Proposition 1.1 are proved in a more general situation where S and F are assumed to be L_φ- and A_φ-spaces, respectively. Note that any FK-space is an A_φ-space and each separable FK-space is an L_φ-space. For the notion of L_φ- and A_φ-spaces we refer to [10].

If S and E are (separable) FK-spaces, topologized, respectively, by families $\mathcal{Q}$ and $\mathcal{P}$ of seminorms, then $E_\mathfrak{A}$ is a (separable) FK-space with the family of seminorms (cf. [1], Proposition 4.4 and Remark 4.5)

$$\begin{aligned} r_k : E_\mathfrak{A} \to \mathbb{R}\,,\ x \mapsto r_k(x) := |x_k| \qquad & (k \in \mathbb{N}), \\ q \circ \operatorname{diag}_{a^{(n)}} : E_\mathfrak{A} \to \mathbb{R}\,,\ x \mapsto q(a^{(n)}x) \qquad & (q \in \mathcal{Q}, n \in \mathbb{N}), \\ p \circ \mathfrak{A} : E_\mathfrak{A} \to \mathbb{R}\,,\ x \mapsto p(\mathfrak{A}x) \qquad & (p \in \mathcal{P}). \end{aligned}$$

Here, $\operatorname{diag}_{a^{(n)}}$ denotes the diagonal matrix with the diagonal $a^{(n)}$.

Obviously, $\varphi \subset c_\mathfrak{A}$ holds if and only if $a_k := \lim_n a_{nk}$ exists for each $k \in \mathbb{N}$. Further, A is said to be an Sp_1-*matrix* if $a := (a_k) = e := (1, 1, \ldots)$, and it is called an Sp_1^*-*matrix* if, in addition, each column of A belongs to bv. For example, the *summation matrix* $\Sigma = (\sigma_{nk})$, with $\sigma_{nk} = 1$ for $k \leq n$ and $\sigma_{nk} = 0$ otherwise, is an Sp_1^*-matrix. Note that in the case of (1.2) a quasi-matrix map $\mathfrak{A}$ is the matrix map A (cf. (1.1)) and we write E_A instead of $E_\mathfrak{A}$. It is well known that the convergence domain c_A of any matrix A is a separable FK-space.

2. PROPERTIES $AK(S)$ AND $AB(S)$

A typical sum s on a K-space S has often the representation

$$s(z) = \lim_{\gamma \in \Gamma} s_\gamma(z) \ \text{ with } \ s_\gamma(z) := \sum_k v_{\gamma k} z_k \quad (z \in S), \tag{2.1}$$

where Γ is a directed index set and $v_\gamma := (v_{\gamma k})_k \in \varphi$ for each $\gamma \in \Gamma$. (Note, since s is a sum, we have $1 = s(e^k) = \lim_\gamma v_{\gamma k}$ $(k \in \mathbb{N})$.) In particular, s is of that type if Δ is any directed set and R is a further non-empty set, and if s satisfies for all $z \in S$ the condition

$$s(z) := \lim_{\delta \in \Delta} \sum_k v_{\delta k}^\rho z_k \ \text{ uniformly in } \rho \in R, \tag{2.2}$$

where $(v_{\delta k}^\rho)_k \in \varphi$ for $\delta \in \Delta$ and $\rho \in R$. (To see this, consider in (2.1) the set $\Gamma := \Delta \times R$ with the natural partial order defined by that of Δ.)

In the sequel we assume that the sum $s \in S'$ is defined by (2.1) *and, in addition, that*

$$S = \left\{ z \in \omega \;\middle|\; \text{ the net } \left(\sum_k v_{\gamma k} z_k\right)_\gamma \text{ is bounded and convergent} \right\}. \qquad (2.3)$$

We observe that $(S, \| \|_S)$ is a BK-space, where

$$\|z\|_S \;:=\; \sup_{\gamma\in\Gamma} |s_\gamma(z)| \quad (z \in S).$$

Let ω_Γ denote the Hausdorff locally convex space of all scalar nets $(u_\gamma)_{\gamma\in\Gamma}$ equipped with the product topology. The vector subspace c_Γ of all bounded convergent nets is a Banach space with the supremum norm $\| \|_\infty$ defined by $\|(u_\gamma)\|_\infty := \sup_{\gamma\in\Gamma} |u_\gamma|$ $((u_\gamma) \in c_\Gamma)$. Obviously, $S = V^{-1}(c_\Gamma)$ and $\| \|_S = \| \|_\infty \circ V$, where V is the continuous linear map

$$V \;:\; \omega \longrightarrow \omega_\Gamma\,, \quad z \longmapsto (s_\gamma(z)).$$

By Theorem 5 of [11] S, together with the family of seminorms $\{\| \|_S\} \cup \{r_k \mid k \in \mathbb{N}\}$, is an FK-space. Since $\sup_{\gamma\in\Gamma} |v_{\gamma k}| \neq 0$ for each $k \in \mathbb{N}$, we get that $(S, \| \|_S)$ is even a BK-space.

We illustrate this situation with the following examples.

Example 2.1. If $\Gamma := \mathbb{N}$, then the sum (2.1) has the form

$$s(z) = \lim_n \sum_k v_{nk} z_k \quad (z \in S),$$

and from (2.3) we get $S = c_V$. Thereby $V = (v_{nk})$ is a row-finite Sp_1-matrix (cf. Case 1 in [1]).

Example 2.2. We consider another important example of this situation (cf. Case 2 in [1]). Let $\Gamma := \Phi$, the collection of all finite subsets of $\mathbb{N}$ directed by the set inclusion, and let $T = (t_{nk})$ be a row-finite Sp_1^*-matrix. Put

$$S := bv_T \;\text{ and }\; s(z) := \lim_{F\in\Phi} \sum_{i\in F} (t_{ik} - t_{i-1,k}) z_k \quad (z \in bv_T). \qquad (2.4)$$

Then the sum s has the representation (2.1) with $v_{Fk} := \sum_{i\in F}(t_{ik} - t_{i-1,k})$ $(F \in \Phi\,,\ k \in \mathbb{N})$, and the condition (2.3) is satisfied. Note that $\sum_k |(Tz)_i - (Tz)_{i-1}| = \sum_i |\sum_k (t_{ik} - t_{i-1,k}) z_k| < \infty$, which implies $s(z) = \sum_i ((Tz)_i - (Tz)_{i-1}) = \lim_i (Tz)_i = \lim_T z$ for each $z \in bv_T$.

In the particular case of $T = \Sigma$ we obviously have

$$S \;=\; \ell \;\text{ and }\; s(z) \;=\; \lim_{F\in\Phi} \sum_{i\in F} z_i \;=\; \sum_i z_i \quad (z \in \ell). \qquad (2.5)$$

Example 2.3. We use the notation (cf. [12])

$$\mathcal{M}_u := \left\{ y = (y_{nr})_{n,r\in\mathbb{N}} \;\middle|\; \sup_{n,r} |y_{nr}| < \infty \right\}$$

and

$$\mathcal{F} := \left\{ y \in \mathcal{M}_u \;\middle|\; \exists\, b_y \in \mathbb{K} \;:\; \lim_n y_{nr} = b_y \;\text{ uniformly in } r \in \mathbb{N} \right\}.$$

Let $\mathcal{V} = \left(V^{(r)}\right)$ be a sequence of row-finite matrices with the property

$$\lim_n v_{nk}^{(r)} = 1 \;\text{ uniformly in }\; r \in \mathbb{N} \;\text{ for each }\; k \in \mathbb{N}.$$

For any $z = (z_k) \in \omega$ we put $\mathcal{V}z := \left(\sum_k v_{nk}^{(r)} z_k\right)_{nr}$ and use the notation

$$\begin{aligned} S &:= \mathcal{F}_{\mathcal{V}} := \left\{ z \in \omega \;\middle|\; \mathcal{V}z \in \mathcal{F} \right\}, \\ s(z) &:= \mathcal{F}\text{–}\lim \mathcal{V}z := \lim_n \sum_k v_{nk}^{(r)} z_k \;\text{ uniformly in } r \in \mathbb{N} \quad (z \in S). \end{aligned} \tag{2.6}$$

Then the sum s is defined in the sense of (2.2) and the condition (2.3) is satisfied.

Let a K-space S be equipped with a sum (2.1). Then, for each $x = (x_k)$ and $\gamma \in \Gamma$, the sequence

$$P_\gamma(x) := \sum_k v_{\gamma k} x_k e^k \quad (\gamma \in \Gamma)$$

is called the γth S-*section* of x. If E is a K-space containing φ, we define

$$\begin{aligned} E_{AB(S)} &:= \left\{ x \in \omega \;\middle|\; (P_\gamma(x))_{\gamma\in\Gamma} \text{ is a bounded net in } E \right\}, \\ E_{AK(S)} &:= \left\{ x \in E_{AB(S)} \cap E \;\middle|\; \lim_\gamma P_\gamma(x) \;\text{ exists in } E \right\}. \end{aligned}$$

E is said to be an $AB(S)$-space if $E \subset E_{AB(S)}$ and an $AK(S)$-space if $E = E_{AK(S)}$. Obviously, $\lim_\gamma P_\gamma(x) = x$ in E for every $x \in E_{AK(S)}$. This implies that

$$f(x) = \lim_\gamma \sum_k v_{\gamma k} x_k f(e^k) \quad (x \in E_{AK(S)})$$

for every $f \in E'$. From (2.3) we get $E_{AK(S)} \cdot E^f \subset S$ and $f(x) = s(u_f x)$ $(x \in E_{AK(S)}, f \in E')$. Consequently, $(E, \sigma(E, E'))_{AK(S)} = (E, \sigma(E, E'))_{SK} = (E, \tau_E)_{SK}$ and $(E, \sigma(E, E^S))$ is an $AK(S)$-space. On account of (2.3) we have $E_{SK} \subset E_{AB(S)}$. Therefore

$$\varphi \subset E_{AK(S)} \subset E_{SK} \subset E_{AB(S)}.$$

We remark that $E_{AB(S)} = (E^f)^{\widetilde{S}}$, where

$$\widetilde{S} := \left\{ z \in \omega \;\Big|\; \sup_{\gamma} \left| \sum_k v_{\gamma k} z_k \right| < \infty \right\}.$$

The following proposition is a direct generalization of the corresponding result in the "classical" case (1.2) (cf. [13], Corollary 1 of Proposition 5). Therefore we omit the proof.

Proposition 2.4. *For a barrelled K-space E containing φ the following statements are equivalent*:

(a) *E is an $AK(S)$-space.*

(b) *E is an SK-space.*

(c) *E is an AD- and $AB(S)$-space.*

3. INCLUSION THEOREMS

From Theorems 5.1 and 5.2 of [1] we verify the following inclusion theorems of Bennett–Kalton type.

Theorem 3.1 (cf. [1], Theorem 5.1). *Let S be a separable FK-space with a sum $s \in S'$. For a sequence space E containing φ the following statements are equivalent*:

(a) *$(E^S, \sigma(E^S, E))$ is sequentially complete.*

(b) *Any quasi-matrix map $\mathfrak{A} : (E, \tau(E, E^S)) \to F$ is continuous whenever F is a separable FK-space.*

(c) *The implication $E \subset F \Rightarrow E \subset F_{SK}$ holds whenever F is a separable FK-space.*

Theorem 3.2 (cf. [1], Theorem 5.2). *Let S be an FK-space with a sum $s \in S'$. For a sequence space E containing φ the following statements are equivalent*:

(a) *$(E, \tau(E, E^S))$ is barrelled.*

(b) *Any quasi-matrix map $\mathfrak{A} : (E, \tau(E, E^S)) \to F$ is continuous whenever F is an FK-space.*

(c) *The implication $E \subset F \Rightarrow E \subset F_{SK}$ holds whenever F is an FK-space.*

Now we assume that S and the sum s are given by (2.1) and (2.3). This enables us to complete Theorem 3.2.

Theorem 3.3. *For any sequence space E containing φ each of the statements* (a), (b), *and* (c) *of Theorem* 3.2 *is equivalent to*

(d) *The implication $E \subset F \Rightarrow E \subset F_{AK(S)}$ holds whenever F is an FK-space.*

Proof. Clearly, (d) $\Rightarrow$ (c). Conversely, if (c) holds, then $(E, \tau(E, E^S))$ is barrelled. Thus, since $(E, \tau(E, E^S))$ is an SK-space, it has the

$AK(S)$-property by Proposition 2.4. Therefore $E \subset F_{AK(S)}$ since the inclusion map $i : (E, \tau(E, E^S)) \to F$ is continuous. $\square$

The next theorem extends a further inclusion theorem due to Bennett and Kalton (cf. [3], Theorem 6, and also [14,15]).

Theorem 3.4. *Suppose that a BK-space S and a sum $s \in S'$ is defined by* (2.1) *and* (2.3), *where the index set Γ contains a cofinal sequence (γ_n). Let S be separable. For a sequence space E containing φ the following statements are equivalent*:

(a) $(E, \tau(E, E^S))$ *is an $AK(S)$-space and $(E^S, \sigma(E^S, E))$ is sequentially complete.*

(b) *The implication $E \subset F \Rightarrow E \subset F_{AK(S)}$ holds whenever F is a separable FK-space.*

(c) *The implication $E \subset c_{\mathfrak{A}} \Rightarrow E \subset (c_{\mathfrak{A}})_{AK(S)}$ holds for every quasi-matrix map $\mathfrak{A}$.*

Proof.

(a) $\Rightarrow$ (b): By Theorem 3.1, the inclusion map $i : (E, \tau(E, E^S)) \to F$ is continuous for each separable FK-space F. Since $(E, \tau(E, E^S))$ has the $AK(S)$-property, we get $E \subset F_{AK(S)}$.

(b) $\Rightarrow$ (c) is valid, since $c_{\mathfrak{A}}$ is a separable FK-space.

(c) $\Rightarrow$ (a): We first remark that (c) implies the sequential completeness of $(E^S, \sigma(E^S, E))$ on account of Theorem 3.1. Assume that $(E, \tau(E, E^S))$ is not an $AK(S)$-space. Then there exist an $x \in E$ and an absolutely convex $\sigma(E^S, E)$-compact subset $K \subset E^S$ such that

$$\sup_{a \in K} |s(a(x - P_{\gamma_n}(x)))| \not\to 0 \ (n \to \infty).$$

Therefore we may choose an index sequence (n_ν) and a sequence $(a^{(\nu)})$ in K such that

$$\left|s\big(a^{(\nu)}\big(x - P_{\gamma_{n_\nu}}(x)\big)\big)\right| \geq \varepsilon > 0 \quad (\nu \in \mathbb{N}). \tag{3.1}$$

Since K is $\sigma(E^S, E)$-compact, it is $\sigma(E^S, E)$-sequentially compact (cf. [16], Theorem 3.10). Thus we may assume without loss of generality that $(a^{(\nu)})$ is $\sigma(E^S, E)$-convergent. (Otherwise we switch over to a subsequence of $(a^{(\nu)})$.) Now, if A denotes the matrix given by $a_{ik} := a_k^{(i)}$ $(i, k \in \mathbb{N})$, then the last assumption gives us $E \subset c_{\mathfrak{A}}$. From (c) we get $E \subset (c_{\mathfrak{A}})_{AK(S)}$ contradicting (3.1). So $(c) \Rightarrow (a)$. $\square$

We now examine the $AK(S)$-property of K-spaces for certain K-spaces S. For that, throughout this section, let E be a sequence space containing φ.

Example 3.5. Let $S = c_T$ and $s(z) = \lim_T z$ $(z \in c_T)$, where $T = (t_{nk})$ is a row-finite Sp_1-matrix. The S-sections introduced above are the Toeplitz sections. Then the $AK(c_T)$-property is just the TK-property, that is the T-sectional convergence in the sense of Buntinas [5] and Meyers [17]. Recall that

$$F_{TK} := \left\{ x \in F \;\Big|\; \sum_k t_{nk} x_k e^k \longrightarrow x \;\text{ in } (F, \tau_F) \right\},$$

where (F, τ_F) is a K-space containing φ. Furthermore, we have

$$E^{c_T} = E^{\beta(T)} := \left\{ u \in \omega \;\Big|\; \forall\, x \in E \;:\; \lim_n \sum_k t_{nk} u_k x_k \;\text{ exists} \right\}.$$

As a consequence of Theorem 3.3 we get:
$(E, \tau(E, E^{\beta(T)}))$ is barrelled if and only if the implication $E \subset F \Rightarrow E \subset F_{TK}$ holds for every FK-space F.

From Theorem 3.4 we conclude:
$(E, \tau(E, E^{\beta(T)}))$ enjoys the TK-property and $(E^{\beta(T)}, \sigma(E^{\beta(T)}, E))$ is sequentially complete if and only if the implication $E \subset F \Rightarrow E \subset F_{TK}$ holds for each separable FK-space F.

Example 3.6. In the situation of (2.4) we have

$$E^{bv_T} = E^{\alpha(T)} := \left\{ u \in \omega \;\Big|\; \forall\, x \in E : \sum_i \left| \sum_k (t_{ik} - t_{i-1,k}) u_k x_k \right| < \infty \right\}.$$

Moreover, the $AK(bv_T)$-property is just the UTK-property, which is, in turn, the unconditional T-sectional convergence (cf. Fleming [18], DeFranza and Fleming [19]). It is easy to establish the Inclusion Theorems 3.3 and 3.4 in this context. We consider here the important special case of (2.5). Then $E^\ell = E^\alpha$ and the $AK(\ell)$-property is the UAK-property (cf. Sember [20], Sember and Raphael [21]). Remember, for any K-space (F, τ_F) containing φ, the notation

$$E_{UAK} := \left\{ x \in E \;\Big|\; \sum_{k \in \mathcal{F}} x_k e^k \underset{\mathcal{F}}{\rightarrow} x(\tau_E) \right\},$$

where $\mathcal{F}$ is the collection of all finite subsets of $\mathbb{N}$ directed by the set inclusion.

From Theorem 3.3 we derive:
$(E, \tau(E, E^\alpha))$ is barrelled if and only if the implication $E \subset F \Rightarrow E \subset F_{UAK}$ holds for every FK-space F.

The following corollary is an immediate consequence of Theorem 3.4.

Corollary 3.7. *The implication $E \subset F \Rightarrow E \subset F_{UAK}$ holds for every separable FK-space F if and only if $(E, \tau(E, E^\alpha))$ has the UAK-property and $(E^\alpha, \sigma(E^\alpha, E))$ is sequentially complete.*

Example 3.8. Let $\mathcal{A}$ be the sequence of the matrices $A^{(r)} = (a_{nk}^{(r)})_{n,k}$ with

$$a_{nk}^{(r)} := \begin{cases} \frac{1}{n} & \text{if } r \le k \le r+n-1, \\ 0 & \text{otherwise,} \end{cases} \qquad (n, k \in \mathbb{N}).$$

We define

$$S := fs := \left\{ z \in \omega \;\Big|\; \exists\, b_z \in \mathbb{K} : \lim_n \sum_k a_{nk}^{(r)} \sum_{i=1}^{k} z_i = b_z \text{ uniformly in } r \in \mathbb{N} \right\}$$

and put $f\text{–}\sum_i z_i = b_z$ when $z \in fs$. Note that $S = \Sigma^{-1} f$, where f stands for the BK-space of all almost convergent sequences. We have

$$E^S = E^{fs} = \left\{ u \in \omega \;\Big|\; \forall\, x \in E \;:\; f\text{–}\sum_k u_k x_k \text{ exists} \right\},$$

and particularly $bv^{fs} = fs$ in the case $E = bv$. Now, if we put $s(z) := f\text{–}\sum_i z_i$ $(z \in fs)$, then s has the representation (2.6) with $\mathcal{V} = (A^{(r)}\Sigma)$.

If F is a K-space containing φ, then

$$\begin{aligned} F_{AB(fs)} &= \left\{ x \in \omega \;\Big|\; \left(\frac{1}{n} \sum_{k=r}^{r+n-1} \sum_{i=1}^{k} x_i e^i \right)_{n,r} \text{ is bounded in } F \right\} \\ &= \left\{ x \in \omega \;\Big|\; \forall\, f \in F' \;:\; \sup_{n,r} \left| \frac{1}{n} \sum_{k=r}^{r+n-1} \sum_{i=1}^{k} x_i f(e^i) \right| < \infty \right\} \\ &= \left\{ x \in \omega \;\Big|\; \forall\, f \in F' \;:\; \sup_k \left| \sum_{i=1}^{k} x_i f(e^i) \right| < \infty \right\} \;=\; F_{AB}. \end{aligned}$$

We introduce the properties $fSAK$ and fAK with respect to F as follows:

$$\begin{aligned} F_{fSAK} &:= F_{fsK} \\ &= \left\{ x \in F \;\Big|\; \forall\, f \in F' : \; u_f x \in fs \text{ and } f(x) = f - \sum_k x_k f(e^k) \right\}, \\ F_{fAK} &:= F_{AK(fs)} \\ &= \left\{ x \in F \;\Big|\; \frac{1}{n} \sum_{k=r}^{r+n-1} \sum_{i=1}^{k} x_i e^{(i)} \to x(\tau_E) \text{ uniformly in } r \in \mathbb{N} \right\}. \end{aligned}$$

Note that the fAK-property, that is, *the almost sectional convergence*, differs from the AK-property: If, for instance, $E := c_B$ with $B := \Sigma^{-1}$, then $e = (1, 1, 1, \ldots)$ belongs to $(c_B)_{fAK}$ but not to $(c_B)_{AK} = c_0$. For a matrix $A = (a_{nk})$ the corresponding quasi-matrix map $\mathfrak{A}$ is defined by $\mathfrak{A}x = (f\text{-}\sum_k a_{nk}x_k)_n$. From Theorems 3.2 and 3.3 we conclude:

Theorem 3.9. *For a sequence space E containing φ the following statements are equivalent*:

(a) $(E, \tau(E, E^{fs}))$ *is barrelled.*

(b) *Any quasi-matrix map* $\mathfrak{A} : (E, \tau(E, E^{fs})) \to F,\ x \mapsto (f\text{-}\sum_k a_{nk}x_k)_n$ *is continuous whenever F is an FK-space.*

(c) *The implication $E \subset F \Rightarrow E \subset F_{fSAK}$ holds whenever F is an FK-space.*

(d) *The implication $E \subset F \Rightarrow E \subset F_{fAK}$ holds whenever F is an FK-space.*

Remark 3.10. Theorems 3.2 and 3.3 and the corresponding assertions in Examples 3.5, 3.6, and 3.8 (including Theorem 3.9) remain true if we replace "FK-space" by "A_φ-space". Analogously, in Theorems 3.1 and 3.4 and in the corresponding statements in 3.5 and 3.6 we may replace "separable FK-space" by "L_φ-space".

4. $AK(S)$-BARRELLED SPACES

If E is a K-space, then we have $\varphi \subset E'$, that is, more precisely, span $\{\pi_n \mid n \in \mathbb{N}\} \subset E'$. Moreover, if $(E', \sigma(E', E))$ is sequentially complete (in particular, if E is barrelled), then $E^\beta \subset E'$, that is, for each $u \in E^\beta$, the linear functional, defined by $x \mapsto \sum_k u_k x_k$, is continuous on E. Obviously, the sequential completeness of $(E', \sigma(E', E))$ is not a necessary condition for that inclusion. The aim of this section is to give a topological characterization of the inclusion $E^S \subset E'$, where S is a BK-space equipped with a sum $s \in S'$ given by (2.1) and (2.3). *In addition, we assume that S is an $AK(S)$-space.*

Let E be a K-space containing φ. Let U be a barrel in E and let E_U denote the seminormed space (E, p_U), where p_U is the Minkowski functional with respect to U.

Definition 4.1. *A barrel U in a K-space E is called an $AK(S)$*-barrel *if E_U is an $AK(S)$-space.*

Let $U^\oplus$ denote the polar of U with respect to the dual pair (E, E^*), where E^* is the algebraic dual of E. Then we have $E'_U = \mathrm{span}\, U^\oplus$ for each barrel U in E. Now, if U is an $AK(S)$-barrel, then

$$\forall\, x \in E_U\ \forall\, f \in E'_U\ :\ u_f x \in S\ \text{ and }\ f(x) = s(u_f x).$$

Thus, $U^\oplus \subset E^S$ and

$$\lim_\gamma \sup_{u \in U^\oplus} \left|\lim_{\rho \in \Gamma} \sum_k v_{\rho k} u_k (x - P_\gamma(x))_k\right| \;=\; \lim_\gamma p_U(x - P_\gamma(x)) \;=\; 0$$

for each $x \in E$.

Let now $u \in E^S$ be fixed. We put $B := \{P_\gamma(u) \mid \gamma \in \Gamma\} \subset \varphi$ and $U := B^\circ$, where $^\circ$ stands for the polar with respect to the dual pair (E, E'). Since

S is an $AK(S)$-space, $\{s \circ P_\gamma \mid \gamma \in \Gamma\}$ is bounded in $(S', \sigma(S', S))$. Hence $\sup_\gamma |\langle x, P_\gamma(u)\rangle| = \sup_\gamma |s \circ P_\gamma(ux)| < \infty$ for each $x \in E$. Then B is $\sigma(E', E)$-bounded; thus U is a barrel in E. Moreover,

$$\begin{aligned} p_U(x - P_\rho(x)) &= \sup_{y \in B^{\circ\circ}} |s(y(x - P_\rho(x)))| = \sup_\gamma |s(P_\gamma(u)(x - P_\rho(x)))| \\ &= \sup_\gamma \left| \lim_{\eta \in \Gamma} \sum_k v_{\eta k} v_{\gamma k} u_k [x - P_\rho(x)]_k \right| \\ &= \sup_\gamma \left| \sum_k v_{\gamma k} [ux - P_\rho(ux)]_k \right| \\ &= \|ux - P_\rho(ux)\|_S \underset{\rho}{\longrightarrow} 0 \end{aligned}$$

for each $x \in E$. Therefore, U is an $AK(S)$-barrel. We summarize our observations in the following proposition.

Proposition 4.2. *Let E be a K-space with $\varphi \subset E$. Then for every $u \in E^S$, the polar $\{P_\gamma(u) \mid \gamma \in \Gamma\}^\circ$ is an $AK(S)$-barrel in E.*

Definition 4.3. *A K-space E containing φ is said to be $AK(S)$-*barrelled *if each $AK(S)$-barrel in E is a τ_E-neighbourhood of 0.*

The following theorem answers the question stated above.

Theorem 4.4. *For any K-space E with $\varphi \subset E$ the following statements are equivalent*:

(a) $E^S \subset E'$, *i.e., the functional, defined by $x \mapsto s(ux)$, is continuous on E for each $u \in E^S$.*

(b) $(E, \tau(E, E'))$ *is $AK(S)$-barrelled.*

(c) $\{P_\gamma(u) \mid \gamma \in \Gamma\}$ *is $\tau(E, E')$-equicontinuous for each $u \in E^S$.*

Proof.
(a) $\Rightarrow$ (b): Let $E^S \subset E'$ and let U be an $AK(S)$-barrel in E. Then $U^\circ \subset U^\oplus \subset E^S$. We have to prove the $\sigma(E^S, E)$-compactness of U°. Since in K-spaces compactness and sequential compactness coincide (cf. [16], Theorem 3.10), it is sufficient to show that U° is $\sigma(E^S, E)$-sequentially compact in E^S.

To that end, let $(a^{(n)})$ with $a^{(n)} = (a_k^{(n)})_k$ be a sequence in U°. It is $\sigma(E^S, E)$-bounded and therefore coordinatewise bounded. Thus, without loss of generality, we may assume that $(a^{(n)})$ converges coordinatewise to an $a \in \omega$. We have $\sup_n |s(a^{(n)}x)| < \infty$ $(x \in E)$, therefore $E \subset m_{\mathfrak{A}}$, where $\mathfrak{A}$ is the quasi-matrix map defined by the matrix $A := (a_k^{(n)})$. We show that $E \subset (m_{\mathfrak{A}})_{AK(S)}$. Obviously, for every $x \in m_{\mathfrak{A}}$ we have

$$x \in (m_{\mathfrak{A}})_{AK(S)} \iff \begin{array}{ll} \text{(i)} & \sup_n |s(a^{(n)}(x - P_\gamma(x)))| \underset{\gamma}{\longrightarrow} 0, \\ \text{(ii)} & \left\| a^{(n)}x - P_\gamma(a^{(n)}x) \right\|_S \underset{\gamma}{\longrightarrow} 0 \quad (n \in \mathbb{N}), \\ \text{(iii)} & r_k(x - P_\gamma(x)) \underset{\gamma}{\longrightarrow} 0 \quad (k \in \mathbb{N}). \end{array}$$

Thereby, condition (iii) is clearly satisfied, and (ii) holds by the $AK(S)$-property of S. Since U is an $AK(S)$-barrel, condition (i) is satisfied. Namely,

$$\begin{aligned} \sup_n \left| s\big(a^{(n)}(x - P_\gamma(x))\big) \right| &= \sup_n \left| \lim_{\rho \in \Gamma} \sum_k v_{\rho k} a_k^{(n)} [x - P_\gamma(x)]_k \right| \\ &\leq \sup_{v \in U^\oplus} \left| \lim_\rho \sum_k v_{\rho k} v_k [x - P_\gamma(x)]_k \right| \\ &= p_U(x - P_\gamma(x)) \underset{\gamma}{\longrightarrow} 0. \end{aligned}$$

Altogether, $E \subset (m_{\mathfrak{A}})_{AK(S)} = (c_{\mathfrak{A}})_{AK(S)} \subset (c_{\mathfrak{A}})_{SK}$. Then $ax \in S$ and $\lim_{\mathfrak{A}} x = s(ax)$ for each $x \in E$. This implies $a \in E^S$ and $a^{(n)} \to a(\sigma(E^S, E))$. Hence U° is $\sigma(E^S, E)$-sequentially compact.

(b) $\Rightarrow$ (c): By Proposition 4.2, $\{P_\gamma(u) \mid \gamma \in \Gamma\}^\circ$ is an $AK(S)$-barrel for each $u \in E^S$. On account of (b) it is a $\tau(E, E')$-neighbourhood of 0; thus $\{P_\gamma(u) \mid \gamma \in \Gamma\}$ is $\tau(E, E')$-equicontinuous.

(c) $\Rightarrow$ (a): Let $u \in E^S$, then $\{P_\gamma(u) \mid \gamma \in \Gamma\}$ is $\tau(E, E')$-equicontinuous. For each $\epsilon > 0$ there exists a $\tau(E, E')$-neighbourhood V of 0 in E with $|\sum_k v_{\gamma k} u_k x_k| \leq \epsilon$ $(x \in V,\ \gamma \in \Gamma)$. That yields $|s(ux)| = |\lim_\gamma \sum_k v_{\gamma k} u_k x_k| \leq \epsilon$ $(x \in V)$, which proves that the linear functional, defined by $x \mapsto s(ux)$, is continuous on $(E, \tau(E, E'))$. $\square$

It is an easy task to verify Theorem 4.4 for the particular spaces $S = c_T$ and $S = bv_T$ discussed above. We consider here the two most important special cases of (1.2) and (2.5).

If $S = cs$, $AK(S)$-barrels are said to be AK-barrels (cf. Schaefer [[6]]). A barrel U in a K-space E is an AK-barrel if and only if $U^\oplus \subset E^\beta$ and the series $\sum_k u_k x_k$ converges uniformly in $u \in U^\oplus$ for each $x \in E$. The AK-barrels differ from the other barrels by their "toleration" of the sectional convergence. If $\mathcal{B}$ is a neighbourhood basis of 0 in E consisting of barrels, then E is an AK-space if and only if each $U \in \mathcal{B}$ is an AK-barrel. This yields that the strongest AK-topology on a sequence space E is defined by the neighbourhood basis $\mathcal{B}_0$ of 0, where $\mathcal{B}_0 := \big\{U \subset E \mid U \text{ is a } \sigma(E, E^\beta)\text{-}AK\text{-barrel}\big\}$ (cf. [[6]]). So we can formulate Theorem 4.4 in the special case $S = cs$.

Theorem 4.5. *For any K-space E containing φ the following statements are equivalent*:

(a) $E^\beta \subset E'$.

(b) $(E, \tau(E, E'))$ *is AK-barrelled.*

(c) $\{u^{[n]} \mid n \in \mathbb{N}\}$ *is $\tau(E, E')$-equicontinuous for each $u \in E^\beta$.*

In the case $S = \ell$, $AK(S)$-barrels are called UAK-barrels. A barrel U in a K-space E is a UAK-barrel if and only if $U^\oplus \subset E^\alpha$ and the series $\sum_k u_k x_k$ converges unconditionally and uniformly in $u \in U^\oplus$ for each $x \in E$. Obviously, E is a UAK-space if and only if there exists a neighbourhood basis of 0 consisting of UAK-barrels. The strongest UAK-topology on a sequence space E is defined by the neighbourhood basis of 0 consisting of all $\sigma(E, E^\alpha)$-UAK-barrels. By that, Theorem 4.4 has the following form:

Theorem 4.6. *For any K-space E containing φ the following statements are equivalent*:

(a) $E^\alpha \subset E'$.

(b) $(E, \tau(E, E'))$ *is UAK-barrelled.*

(c) $\left\{\sum_{k \in F} u_k e^k \mid F \in \Phi\right\}$ *is $\tau(E, E')$-equicontinuous for each $u \in E^\alpha$.*

(d) $\{u^{[n]} \mid n \in \mathbb{N}\}$ *is $\tau(E, E')$-equicontinuous for each $u \in E^\alpha$.*

ACKNOWLEDGEMENTS

The present work was supported by Deutscher Akademischer Austauschdienst (DAAD) and the Estonian Science Foundation (grant No. 3991).

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Jadaruumide duaalsed paarid. II

Johann Boos ja Toivo Leiger

Autorid jätkavad varasemas töös [1] alustatud duaalsete paaride $\left(E, E^S\right)$ uurimist. E tähistab jadaruumi, S on selline K-ruum, milles on defineeritud Ruckle'i üldistatud summa s, ja E^S on vastavate faktorjadade ruum. Siinses artiklis on vaadeldud kõige sagedamini esinevat erijuhtu, kus summa on esitatud kujul $s\left(z\right) = \lim_\gamma \sum_k v_{\gamma k} z_k \ (z \in S)$, indeksite γ hulk Γ on suunatud ja $\left(v_{\gamma k}\right)_k$ on iga $\gamma \in \Gamma$ puhul lõplik jada. Niisuguse esituse abil on defineeritud jadade S-lõiked ning uuritud nende koonduvust $(AK(S))$ ja tõkestatust $(AB(S))$ mingis K-ruumis E. Selles kontekstis on tõestatud kõigepealt tuntud Bennetti–Kaltoni sisalduvusteoreemid. Teiseks, lähtudes Schaeferi “lõikekoonduvustünni” (*section convergence barrel*) mõistest, on defineeritud $AK(S)$-tünniruumid ja tõestatud, et Mackey K-ruum E on $AK(S)$-tünniruum parajasti siis, kui kehtib sisalduvus $E^S \subset E'$. Artikli põhitulemuste rakendamisel saadakse rida väiteid Köthe–Toeplitzi kaasruumide ja lõigetega seotud omaduste kohta konkreetsetes situatsioonides, näiteks $\beta\left(T\right)$-kaasruumi ja STK-omaduse seostest (need mõisted on tuntud Buntinase ja Meyersi töödest).