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On the convexity theorem of M. Riesz

Veera Pavlova and Anne Tali

Department of Mathematics, Tallinn Pedagogical University, Narva mnt. 25, 10120 Tallinn, Estonia; atali@tpu.ee

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Abstract. We consider the well-known convexity theorem proved by M. Riesz in 1923, which gives certain convexity conditions for Riesz summability methods (R, α). Later on different authors have extended this theorem by modifying the convexity conditions and the definition of methods (R, α). Our aim is to extend the convexity theorem of M. Riesz to a wider class of summability methods and, afterwards, apply it to the estimation of the speed of summability.

Key words: integral summability methods, Riesz methods, integral Nörlund methods, convexity theorem of M. Riesz.

1. INTRODUCTION AND PRELIMINARIES

Let us consider the functions $x = \xi(u)$ defined for $u \geq 0$, bounded and measurable by Lebesgue on every finite interval $[0, u_0]$. Denote the set of these functions by X.

Suppose A is a linear transformation of functions $x = \xi(u) \in X$ (or, in particular, sequences $x = (\xi_n)$) into functions $Ax = \eta(u) \in X$.

If the limit $\lim_{u\to\infty} \eta(u) = \eta$ exists, we say that $x = \xi(u)$ is summable to η by the summability method A. If the function $\eta(u)$ is bounded, we say that $x = \xi(u)$ is bounded by the method A.

The most usual summability method for functions is an integral method defined with the help of the transformation

$$\eta(u) = \int_0^\infty a(u, v)\xi(v)\, dv,$$

where $a(u, v)$ is a certain function of two variables $u \geq 0$ and $v \geq 0$.

Denote by ωA the set of all these functions $x \in X$, where the transformation A is applied, and by cA the set of all functions which are summable by the method A.

The summability method A is said to be regular if

$$\lim_{u\to\infty} \xi(u) = \xi \implies \lim_{u\to\infty} \eta(u) = \xi,$$

whenever $x \in X$.

Our paper concerns the well-known convexity theorem proved by Riesz in [1]. Let us present it here.

Consider the Riesz methods $A_\alpha = (R, \alpha)$ defined for the functions $x = \xi(u) \in X$ by the transformation

$$\eta_\alpha(u) = \frac{\alpha}{u^\alpha} \int_0^u (u - v)^{\alpha-1} \xi(v)\, dv \qquad (u > 0),$$

where $\alpha > 0$ (see [2], Section 5.14). We denote

$$\xi_\alpha(u) = \int_0^u (u - v)^{\alpha-1} \xi(v)\, dv, \tag{1.1}$$

where $x = \xi(u) \in X$. So, we have $\xi_\alpha(u) = (u^\alpha/\alpha)\eta_\alpha(u)$.

Suppose everywhere in this paper that $U = U(u)$ and $V = V(u)$ are two positive, monotonically increasing functions on the interval $[0, \infty)$ and denote

$$W_{\beta\alpha}(u) = [U(u)]^{1-\alpha/\beta} [V(u)]^{\alpha/\beta} \tag{1.2}$$

for $\beta > \alpha > 0$.

Theorem A (Convexity theorem of M. Riesz). *For the family of methods* (R, α) *the implications*[1]

$$(1)\ \ \xi(u) = O(1)U(u),\ \xi_\beta(u) = O(1)V(u) \implies \xi_\alpha(u) = O(1)W_{\beta\alpha}(u),$$

$$(2)\ \ \xi(u) = O(1)U(u),\ \xi_\beta(u) = o(1)V(u) \implies \xi_\alpha(u) = o(1)W_{\beta\alpha}(u),$$

and

$$(3)\ \ \xi(u) = o(1)U(u),\ \xi_\beta(u) = O(1)V(u) \implies \xi_\alpha(u) = o(1)W_{\beta\alpha}(u)$$

are true for all $\beta > \alpha > 0$ *and* $x = \xi(u) \in X$ *as* $u \to \infty$.

The original proof of this theorem is given in [1], but its English version can be found in [3]. Later on Convexity Theorem A was generalized and modified by several authors. The modifications of this theorem with different weaker restrictions on the functions $U = U(u)$ and $V = V(u)$ can be found, for example, in [4–7].

[1] Throughout our paper the coefficients in $O(1)$ and $o(1)$ conditions may depend on values of the parameters α, β (and γ). We do not point it out with indices without special need.

In the present paper we are not going to improve the conditions on the functions $U = U(u)$ and $V = V(u)$. Also, we do not consider the modifications of Theorem A for matrix methods (see [8]). We focus ourselves on widening the class of the considered summability methods and on possible applications of this theorem. We shall extend Theorem A to a wider class of summability methods – the Riesz-type families – and apply it to the estimation of speeds of summability. Next we will introduce (see [9,10]) the notions of convergence, boundedness, and summability of functions with speed.

Suppose $\lambda = \lambda(u)$ is some positive, monotonically increasing function defined on the interval $[0, \infty[$. In the sequel we need the following notations[2]:

$$\begin{aligned} m &= \{x = \xi(u) \in X \mid \xi(u) = O(1)\}, \\ c &= \{x = \xi(u) \in X \mid \lim_{u\to\infty} \xi(u) \quad \text{exists}\}, \\ c_0 &= \{x = \xi(u) \in X \mid \lim_{u\to\infty} \xi(u) = 0\}, \\ m^\lambda &= \{x = \xi(u) \in c \mid \lambda(u)[\xi(u) - \lim_{u\to\infty} \xi(u)] \in m\}, \\ c^\lambda &= \{x = \xi(u) \in c \mid \lambda(u)[\xi(u) - \lim_{u\to\infty} \xi(u)] \in c\}, \\ c_0^\lambda &= \{x = \xi(u) \in c \mid \lambda(u)[\xi(u) - \lim_{u\to\infty} \xi(u)] \in c_0\}. \end{aligned}$$

A function $x = \xi(u)$ is said to be summable by the method A with the speed λ (shortly A^λ-summable) if $Ax \in c^\lambda$. A function $x = \xi(u)$ is said to be A^λ-bounded if $Ax \in m^\lambda$.

2. RIESZ-TYPE FAMILIES OF SUMMABILITY METHODS

Let $\{A_\alpha\}$ be a family of summability methods A_α given by linear transformations of functions $x = \xi(u) \in \omega A_\alpha \subset X$ into functions $A_\alpha x = \eta_\alpha(u) \in X$, where α is a continuous parameter with values $\alpha > \alpha_0$ (α_0 is a fixed number). Suppose in the sequel that $\omega A_\alpha \subset \omega A_\beta$ for all $\beta > \alpha > \alpha_0$.

Definition 1. *A family $\{A_\alpha\}$ ($\alpha > \alpha_0$) is said to be a Riesz-type family if for every $x = \xi(u) \in \omega A_\alpha \subset X$ and $\beta > \alpha > \alpha_0$ the methods A_α and A_β are connected by the relation*

$$\eta_\beta(u) = \frac{M_{\alpha\beta}}{b_\beta(u)} \int_0^u (u-v)^{\beta-\alpha-1} b_\alpha(v)\eta_\alpha(v)\,dv \qquad (u > 0), \tag{2.1}$$

where $b_\alpha(u) \in X$, $b_\alpha(u) > 0$, and $M_{\alpha\beta}$ is a constant depending on α and β.

[2] The notations m, c, c_0, m^λ, c^λ, and c_0^λ are usually used for sets of sequences (see [10]), but by analogy we use them here for functions.

In other words, a Riesz-type family $\{A_\alpha\}$ is a family where every two methods A_α and A_β are connected by the relation

$$A_\beta = D_{\alpha\beta} \circ A_\alpha \qquad (\beta > \alpha > \alpha_0), \tag{2.2}$$

where $D_{\alpha\beta}$ is an integral method defined with the help of the functions

$$d_{\alpha\beta}(u,v) = \begin{cases} \dfrac{M_{\alpha\beta}}{b_\beta(u)}(u-v)^{\beta-\alpha-1} b_\alpha(v), & \text{if} \quad 0 \le v < u, \\ 0, & \text{if} \quad v \ge u. \end{cases} \tag{2.3}$$

Notice that the Riesz summability methods $A_\alpha = (R,\alpha)$ form a Riesz-type family with $b_\alpha(u) = u^\alpha$, $\alpha > 0$, and

$$M_{\alpha\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta-\alpha)}, \tag{2.4}$$

where $\Gamma(\cdot)$ is the gamma-function.

Indeed, it is known that the methods (R,α) and (R,β) are connected by the relation

$$\eta_\beta(u) = \frac{M_{\alpha\beta}}{u^\beta} \int_0^u (u-v)^{\beta-\alpha-1} v^\alpha \eta_\alpha(v)\, dv \qquad (\beta > \alpha > 0)$$

(see [2], Section 5.14).

Also, the family of integral Nörlund methods $A_\alpha = (N, p_\alpha(u), q(u))$ $(\alpha > 0)$, defined with the help of the positive functions $p(u) \in X$ and $q(u) \in X$ (see [11]), can be considered as a Riesz-type family.

A function $x = \xi(u) \in X$ is said to be summable by the integral Nörlund method $A_\alpha = (N, p_\alpha(u), q(u))$ if the limit

$$\lim_{u\to\infty} \eta_\alpha(u) = \eta$$

exists, where

$$\eta_\alpha(u) = \frac{1}{P_\alpha(u)} \int_0^u p_\alpha(u-v) q(v) \xi(v)\, dv \qquad (u > 0), \tag{2.5}$$

$\alpha > 0$, and

$$p_\alpha(u) = \int_0^u (u-v)^{\alpha-1} p(v)\, dv$$

and

$$P_\alpha(u) = \int_0^u p_\alpha(u-v) q(v)\, dv.$$

Notice that here $\omega A_\alpha = X$ for all $\alpha > 0$.

If $p(u) = q(u) \equiv 1$, then Nörlund methods $(N, p_\alpha(u), q(u))$ become Riesz methods $(R, \alpha + 1)$.

It is known (see [11]) that the Nörlund methods $A_\alpha = (N, p_\alpha(u), q(u))$ and $A_\beta = (N, p_\beta(u), q(u))$ are connected by the relation

$$\eta_\beta(u) = \frac{M_{\alpha-1,\beta-1}}{P_\beta(u)} \int_0^u (u-v)^{\beta-\alpha-1} P_\alpha(v)\eta_\alpha(v)\,dv \quad (\beta > \alpha > 0) \tag{2.6}$$

and the functions $P_\alpha(u)$ and $P_\beta(u)$ are connected by the relation

$$P_\beta(u) = M_{\alpha-1,\beta-1} \int_0^u (u-v)^{\beta-\alpha-1} P_\alpha(v)\,dv \quad (\beta > \alpha > 0),$$

where $M_{\alpha-1,\beta-1}$ is defined by the formula (2.4).

The relations (2.1) and (2.6) show that the family of Nörlund summability methods forms the Riesz-type family with $b_\alpha(u) = P_\alpha(u)$.

A survey of the research on integral Nörlund methods $(N, p(u), \mathbf{1})$ can be found in recent papers [12] and [13].

Remark 1. Note that further Riesz-type families can be constructed, for example, as follows:

(i) Let $\{A_\alpha\}$ be a Riesz-type family and A be a summability method such that $Ax \in \omega A_\alpha$ for all $x \in \omega A$ and $\alpha > \alpha_0$. Then also the family $\{B_\alpha\}$ with $B_\alpha = A_\alpha \circ A$ is a Riesz-type family. In particular, this way the normal Riesz methods $B_\alpha = (R, \rho, \alpha)$ $(\alpha > 0)$ for summing sequences $x = (\xi_n)$ can be defined (see [2], Section 4.16). Here $B_\alpha x = A_\alpha(Ax)$ with

$$Ax = \eta(u) = \begin{cases} \sum_{k=0}^n \xi_k, & \text{if} \quad \rho_n < u \leq \rho_{n+1}, \\ 0, & \text{if} \quad u \leq \rho_0, \end{cases}$$

where $0 \leq \rho_0 < \rho_1 < \rho_2 < \rho_3 \cdots < \rho_n \to \infty$ and $A_\alpha = (R, \alpha)$.

(ii) Let $D_{\alpha\beta}$ be the integral methods defined by (2.3), and $D_{\alpha\beta}x \in X$ $(x \in X)$ for all $\beta > \alpha > \alpha_0$, and A be a summability method. Fix α and define $A_\beta = D_{\alpha\beta} \circ A$. Then $\{A_\beta\}$ is a Riesz-type family with $\beta > \alpha$.

(iii) Let $\{A_\alpha\}$ be a Riesz-type family and $c_\alpha(u)$ be some functions with $c_\alpha(u) \in X$, $c_\alpha(u) > 0$ and $b_\alpha(u)/c_\alpha(u) \in X$. Then the family $\{B_\alpha\}$, where $B_\alpha x = c_\alpha(u)\eta_\alpha(u)$ and $\eta_\alpha(u) = A_\alpha x$, is a Riesz-type family too.

Proposition 1 gives us the conditions for regularity of methods $D_{\alpha\beta}$.

Proposition 1. *Let* $D_{\alpha\beta}$ $(\beta > \alpha > \alpha_0)$ *be the integral methods defined by* (2.3), *where* $b_\alpha(u) \in X$ *and* $b_\alpha(u) > 0$. *If for all* $\beta > \alpha > \alpha_0$ *the equality*

$$b_\beta(u) = M_{\alpha\beta} \int_0^u (u-v)^{\beta-\alpha-1} b_\alpha(v)\, dv \qquad (u > 0) \tag{2.7}$$

holds, then all methods $D_{\alpha\beta}$ *are regular.*

Proof. Due to Theorem 6 in [2], it is sufficient to verify that the conditions

$$\lim_{u\to\infty} \int_0^{v_0} d_{\alpha\beta}(u,v)\, dv = 0 \tag{2.8}$$

for every finite $v_0 > 0$,

$$\lim_{u\to\infty} \int_0^{u} d_{\alpha\beta}(u,v)\, dv = 1, \tag{2.9}$$

and

$$\int_0^{u} d_{\alpha\beta}(u,v)\, dv = O(1) \qquad (u > 0) \tag{2.10}$$

are fulfilled for the methods $D_{\alpha\beta}$.

Notice that the conditions (2.9) and (2.10) are satisfied due to (2.7).

So we need to verify only the condition (2.8). Fixing $\alpha' = (\alpha + \alpha_0)/2$ and supposing $v \le v_0 < u$, we have by the condition (2.7) that

$$\begin{aligned}
d_{\alpha\beta}(u,v) &= \frac{M_{\alpha\beta}(u-v)^{\beta-\alpha-1} b_\alpha(v)}{b_\beta(u)} = \frac{M_{\alpha\beta}(u-v)^{\beta-\alpha-1} b_\alpha(v)}{M_{\alpha'\beta}\int_0^u (u-v)^{\beta-\alpha'-1} b_{\alpha'}(v)\, dv} \\
&= O(1)\frac{(u-v)^{\beta-\alpha-1} b_\alpha(v)}{\int_0^{v_0} (u-v)^{\beta-\alpha'-1} b_{\alpha'}(v)\, dv + \int_{v_0}^u (u-v)^{\beta-\alpha'-1} b_{\alpha'}(v)\, dv} \\
&= O(1)\frac{(u-v)^{\beta-\alpha-1} b_\alpha(v)}{\int_0^{v_0} (u-v)^{\beta-\alpha'-1} b_{\alpha'}(v)\, dv} \\
&= O_{v_0}(1)\frac{(u-v)^{\beta-\alpha-1} b_\alpha(v)}{u^{\beta-\alpha'-1} b_{\alpha'+1}(v_0)} \\
&= O_{v_0}(1)\left(1 - \frac{v}{u}\right)^{\beta-\alpha-1} \frac{1}{u^{\alpha-\alpha'}} \to 0
\end{aligned}$$

(uniformly for $v \le v_0$ as $u \to \infty$). Hence, the condition (2.8) is also fulfilled. $\square$

The next result follows directly from Proposition 1 and the relation (2.2).

Proposition 2. *Let* $\{A_\alpha\}$ $(\alpha > \alpha_0)$ *be a Riesz-type family. If the condition* (2.7) *is fulfilled, then the inclusion*

$$cA_\alpha \subset cA_\beta \qquad (\beta > \alpha > \alpha_0) \tag{2.11}$$

is true, i.e. the implication

$$\lim_{u\to\infty} \eta_\alpha(u) = \eta \implies \lim_{u\to\infty} \eta_\beta(u) = \eta \qquad (\beta > \alpha > \alpha_0)$$

holds for every $x \in cA_\alpha$.

Notice that for the methods $A_\alpha = (R, \alpha)$ and $A_\alpha = (N, p_\alpha(u), q(u))$ $(\alpha > 0)$ the conditions of the last proposition are satisfied with $b_\alpha(u) = u^\alpha$ and $b_\alpha(u) = P_\alpha(u)$, respectively, and thus (2.11) holds.

3. CONVEXITY THEOREMS FOR A RIESZ-TYPE FAMILY

In this section we will extend Theorem A to a Riesz-type family and interpret it from the point of view of summability with speed.

Theorem 1. *Let* $\{A_\alpha\}$ $(\alpha > \alpha_0)$ *be a Riesz-type family and* $W_{\alpha\beta\gamma}(u)$ *be defined by*

$$W_{\alpha\beta\gamma}(u) = [U(u)]^{(\beta-\gamma)/(\beta-\alpha)}[V(u)]^{(\gamma-\alpha)/(\beta-\alpha)} \qquad (\alpha_0 < \alpha < \gamma < \beta). \tag{3.1}$$

Then the implications

(1) $b_\alpha(u)\eta_\alpha(u) = O(1)U(u)$ *and* $b_\beta(u)\eta_\beta(u) = O(1)V(u)$ *imply* $b_\gamma(u)\eta_\gamma(u) = O(1)W_{\alpha\beta\gamma}(u)$,

(2) $b_\alpha(u)\eta_\alpha(u) = O(1)U(u)$ *and* $b_\beta(u)\eta_\beta(u) = o(1)V(u)$ *imply* $b_\gamma(u)\eta_\gamma(u) = o(1)W_{\alpha\beta\gamma}(u)$,

and

(3) $b_\alpha(u)\eta_\alpha(u) = o(1)U(u)$ *and* $b_\beta(u)\eta_\beta(u) = O(1)V(u)$ *imply* $b_\gamma(u)\eta_\gamma(u) = o(1)W_{\alpha\beta\gamma}(u)$

are true for all $\beta > \gamma > \alpha > \alpha_0$ *and* $x \in \omega A_\alpha$ *as* $u \to \infty$.

Proof. Theorem 1 will be proved with the help of Theorem A. Denote first

$$\zeta(u) = b_\alpha(u)\eta_\alpha(u), \tag{3.2}$$

$\mu = \beta - \alpha$, $\varphi = \gamma - \alpha$ and afterwards (cf. (1.1))

$$\xi_\mu(u) = \int_0^u (u-v)^{\mu-1}\zeta(v)\,dv \tag{3.3}$$

and

$$\xi_\varphi(u) = \int_0^u (u-v)^{\varphi-1}\zeta(v)\,dv. \tag{3.4}$$

Now we can apply Theorem A to $\zeta(u)$ and $\mu > \varphi > 0$ (instead of $\xi(u)$ and $\beta > \alpha > 0$).

Let us prove our implication (1). Implication (1) of Theorem A gives us:

$$\zeta(u) = O(1)U(u),\ \xi_\mu(u) = O(1)V(u) \implies \xi_\varphi(u) = O(1)W_{\mu\varphi}(u), \tag{3.5}$$

where the function $W_{\mu\varphi}(u)$ has by (1.2) the form

$$\begin{aligned} W_{\mu\varphi}(u) &= [U(u)]^{1-\varphi/\mu}[V(u)]^{\varphi/\mu} = [U(u)]^{(\beta-\gamma)/(\beta-\alpha)}[V(u)]^{(\gamma-\alpha)/(\beta-\alpha)} \\ &= W_{\alpha\beta\gamma}(u) \quad (\alpha_0 < \alpha < \gamma < \beta). \end{aligned}$$

Now we shall show that the implication (3.5) can be rewritten as implication (1) of our theorem. Really, by (2.1), (3.2), and (3.3) we get that

$$\begin{aligned} b_\beta(u)\eta_\beta(u) &= M_{\alpha\beta}\int_0^u (u-v)^{\beta-\alpha-1}\zeta(v)\,dv \\ &= M_{\alpha,\alpha+\mu}\int_0^u (u-v)^{\mu-1}\zeta(v)\,dv = M_{\alpha,\alpha+\mu}\xi_\mu(u). \end{aligned}$$

Analogically, by (3.4) we get

$$b_\gamma(u)\eta_\gamma(u) = M_{\alpha,\alpha+\varphi}\xi_\varphi(u).$$

Thus, the implication (3.5) can be seen as our statement (1). The other two implications can be proved in the same way with the help of implications (2) and (3) of Theorem A. $\square$

Convexity Theorem 1 can be applied to the estimation of speeds of summability in a Riesz-type family.

Let us denote

$$\lambda_\alpha(u) = \frac{b_\alpha(u)}{U(u)}, \tag{3.6}$$

$$\mu_\beta(u) = \frac{b_\beta(u)}{V(u)}, \tag{3.7}$$

and

$$\Psi_{\alpha\beta\gamma}(u) = \frac{b_\gamma(u)}{W_{\alpha\beta\gamma}(u)}, \tag{3.8}$$

where $b_\delta(u) \in X$ are functions defined by the Riesz-type family $\{A_\alpha\}$ and $W_{\alpha\beta\gamma}(u)$ is defined by (3.1).

Theorem 2. *Let $\{A_\alpha\}$ $(\alpha > \alpha_0)$ be a Riesz-type family satisfying, for all $\beta > \alpha > \alpha_0$, the relation (2.7). Suppose the functions $\lambda_\alpha(u)$, $\mu_\beta(u)$, and $\Psi_{\alpha\beta\gamma}(u)$, defined by formulas* (3.6)–(3.8), *are monotonically increasing for some $\beta > \gamma > \alpha$. Then the implications*

(1) $A_\alpha x \in m^{\lambda_\alpha},\ A_\beta x \in m^{\mu_\beta} \implies A_\gamma x \in m^{\Psi_{\alpha\beta\gamma}}$,

(2) $A_\alpha x \in m^{\lambda_\alpha},\ A_\beta x \in c_0^{\mu_\beta} \implies A_\gamma x \in c_0^{\Psi_{\alpha\beta\gamma}}$,

and

(3) $A_\alpha x \in c_0^{\lambda_\alpha},\ A_\beta x \in m^{\mu_\beta} \implies A_\gamma x \in c_0^{\Psi_{\alpha\beta\gamma}}$

are true.

For proof of this theorem we need

Proposition 3. *Let $\{A_\alpha\}$ $(\alpha > \alpha_0)$ be a Riesz-type family satisfying, for all $\beta > \alpha > \alpha_0$, the relation (2.7) and the functions $\lambda_\alpha(u)$, $\mu_\beta(u)$, and $\Psi_{\alpha\beta\gamma}(u)$ be defined by* (3.6)–(3.8)*. Then the implications*

(1) $\lambda_\alpha(u)[\eta_\alpha(u) - \eta] = O(1)$ *and* $\mu_\beta(u)[\eta_\beta(u) - \eta] = O(1)$ *imply* $\Psi_{\alpha\beta\gamma}(u)[\eta_\gamma(u) - \eta] = O(1)$,

(2) $\lambda_\alpha(u)[\eta_\alpha(u) - \eta] = O(1)$ *and* $\mu_\beta(u)[\eta_\beta(u) - \eta] = o(1)$ *imply* $\Psi_{\alpha\beta\gamma}(u)[\eta_\gamma(u) - \eta] = o(1)$,

and

(3) $\lambda_\alpha(u)[\eta_\alpha(u) - \eta] = o(1)$ *and* $\mu_\beta(u)[\eta_\beta(u) - \eta] = O(1)$ *imply* $\Psi_{\alpha\beta\gamma}(u)[\eta_\gamma(u) - \eta] = o(1)$

are true for all $\beta > \gamma > \alpha > \alpha_0$, $\eta \in \boldsymbol{R}$ and $x \in \omega A_\alpha$ as $u \to \infty$.

Proof. Here we will use Theorem A again. Let us denote

$$\zeta(u) = b_\alpha(u)[\eta_\alpha(u) - \eta], \tag{3.9}$$

$\mu = \beta - \alpha$ and $\varphi = \gamma - \alpha$. Using the relations (2.7) and (3.9), we get

$$\begin{aligned}
b_\beta(u)[\eta_\beta(u) - \eta] &= b_\beta(u)\eta_\beta(u) - b_\beta(u)\eta \\
&= M_{\alpha\beta}\left(\int_0^u (u-v)^{\beta-\alpha-1} b_\alpha(v)\eta_\alpha(v)\,dv - \int_0^u (u-v)^{\beta-\alpha-1} b_\alpha(v)\eta\,dv\right) \\
&= M_{\alpha\beta}\int_0^u (u-v)^{\beta-\alpha-1}\zeta(v)\,dv = M_{\alpha,\alpha+\mu}\int_0^u (u-v)^{\mu-1}\zeta(v)\,dv \\
&= M_{\alpha,\alpha+\mu}\xi_\mu(u).
\end{aligned}$$

Now we can apply Theorem A to $\zeta(u)$ and $\mu > \varphi > 0$.

Implication (1) of Theorem A says that (3.5) holds, with $\xi_\mu(u)$ and $\xi_\varphi(u)$ defined by (3.3) and (3.4).

By (3.9) and (2.7) we get

$$b_\gamma(u)[\eta_\gamma(u) - \eta] = M_{\alpha,\alpha+\varphi}\xi_\varphi(u).$$

Thus, the implication (3.5) can be rewritten as implication (1) of our Proposition 3. Implications (2) and (3) can be proved in the same way by using implications (2) and (3) of Theorem A. □

Now we are able to prove Theorem 2.

Proof of Theorem 2. Notice first that implications (1)–(3) of Proposition 3 are true. If $A_\alpha x \in m^{\lambda_\alpha}$ or $A_\alpha x \in c_0^{\lambda_\alpha}$, then the limit $\lim_{u\to\infty}\eta_\alpha(u)$ exists, and

$$\lambda_\alpha(u)[\eta_\alpha(u) - \lim_{u\to\infty}\eta_\alpha(u)] = O(1)$$

or

$$\lambda_\alpha(u)[\eta_\alpha(u) - \lim_{u\to\infty}\eta_\alpha(u)] = o(1),$$

respectively. By Proposition 2 also $\lim_{u\to\infty}\eta_\beta(u)$ and $\lim_{u\to\infty}\eta_\gamma(u)$ exist, which are equal to $\lim_{u\to\infty}\eta_\alpha(u)$. To prove Theorem 2, it remains to apply implications (1)–(3) of Proposition 3 with

$$\eta = \lim_{u\to\infty}\eta_\alpha(u) = \lim_{u\to\infty}\eta_\beta(u) = \lim_{u\to\infty}\eta_\gamma(u).$$

Remark 2. As we have already told, the restrictions on the functions $U(u)$ and $V(u)$ in Theorem A have been modified by different authors. Although our paper is not focussed on improving conditions on the functions $U(u)$ and $V(u)$, our discussion above shows that Theorems 1 and 2 remain true whenever Theorem A remains true. In other words, our theorems remain true for all these conditions on the functions $U(u)$ and $V(u)$ which are available (for implications) in Theorem A.

4. THE COMPARATIVE INEQUALITIES FOR SPEEDS OF SUMMABILITY

Suppose $\{A_\alpha\}$ $(\alpha > \alpha_0)$ is a Riesz-type family and $\lambda_\alpha(u)$, $\mu_\beta(u)$, and $\Psi_{\alpha\beta\gamma}(u)$ are functions defined by (3.6)–(3.8).

In this section we shall prove some comparative estimations for these functions.

Proposition 4. *Let* $\{A_\alpha\}$ $(\alpha > \alpha_0)$ *be a Riesz-type family satisfying, for all* $\beta > \alpha > \alpha_0$, *the condition* (2.7). *Let the function* $b_\alpha(u)$ *be monotonically increasing for all* $\alpha > \alpha_0$ *and the functions* $\lambda_\alpha(u)$, $\mu_\beta(u)$, *and* $\Psi_{\alpha\beta\gamma}(u)$ $(u > 0)$ *be defined by* (3.6)–(3.8)*. Then the inequalities*

$$S_{\gamma\beta} u^{\gamma-\beta} \left[\frac{V(u)}{U(u)}\right]^{(\beta-\gamma)/(\beta-\alpha)} \mu_\beta(u) \le \Psi_{\alpha\beta\gamma}(u) \le T_{\alpha\gamma} u^{\gamma-\alpha} \left[\frac{U(u)}{V(u)}\right]^{(\gamma-\alpha)/(\beta-\alpha)} \lambda_\alpha(u) \tag{4.1}$$

hold for all $\beta > \gamma > \alpha > \alpha_0$ *and* $u > 0$ $(S_{\gamma\beta}$ *and* $T_{\alpha\gamma}$ *are suitable positive constants*).

Proof. Using the relation (2.7) and monotony of the function $b_\alpha(u)$, we get

$$b_\beta(u) \le \frac{M_{\alpha\beta}}{(\beta-\alpha)} b_\alpha(u) u^{\beta-\alpha}. \tag{4.2}$$

Let us prove the right-sided inequality in (4.1) with the help of inequality (4.2):

$$\Psi_{\alpha\beta\gamma}(u) = \frac{b_\gamma(u)}{W_{\alpha\beta\gamma}(u)} \le \frac{M_{\alpha\gamma} b_\alpha(u) u^{\gamma-\alpha}}{(\gamma-\alpha)[U(u)]^{(\beta-\gamma)/(\beta-\alpha)}[V(u)]^{(\gamma-\alpha)/(\beta-\alpha)}} \frac{U(u)}{U(u)}$$

$$= T_{\alpha\gamma} u^{\gamma-\alpha} \left[\frac{U(u)}{V(u)}\right]^{(\gamma-\alpha)/(\beta-\alpha)} \lambda_\alpha(u).$$

Using again the inequality (4.2), we can prove also the left-sided inequality in (4.1):

$$\Psi_{\alpha\beta\gamma}(u) = \frac{b_\gamma(u)}{W_{\alpha\beta\gamma}(u)} \ge \frac{b_\beta(u)(\beta-\gamma)}{M_{\gamma\beta} u^{\beta-\gamma}[U(u)]^{(\beta-\gamma)/(\beta-\alpha)}[V(u)]^{(\gamma-\alpha)/(\beta-\alpha)}} \frac{V(u)}{V(u)}$$

$$= S_{\gamma\beta} u^{\gamma-\beta} \left[\frac{V(u)}{U(u)}\right]^{(\beta-\gamma)/(\beta-\alpha)} \mu_\beta(u).$$

□

Proposition 5. *Let* $\{A_\alpha\}$ $(\alpha > \alpha_0)$ *be a Riesz-type family satisfying, for all* $\beta > \alpha > \alpha_0$ *and* $u > 0$, *the condition*

$$K_{\alpha\beta} u^{\beta-\alpha} \leq \frac{b_\beta(u)}{b_\alpha(u)} \leq N_{\alpha\beta} u^{\beta-\alpha}. \tag{4.3}$$

Then the functions $\lambda_\alpha(u)$, $\mu_\beta(u)$, *and* $\Psi_{\alpha\beta\gamma}(u)$, *defined by* (3.6)–(3.8), *satisfy the inequalities*

$$\begin{aligned} &K_{\alpha\beta\gamma} \, [\lambda_\alpha(u)]^{(\beta-\gamma)/(\beta-\alpha)} \, [\mu_\beta(u)]^{(\gamma-\alpha)/(\beta-\alpha)} \\ &\qquad \leq \Psi_{\alpha\beta\gamma}(u) \leq N_{\alpha\beta\gamma} \, [\lambda_\alpha(u)]^{(\beta-\gamma)/(\beta-\alpha)} \, [\mu_\beta(u)]^{(\gamma-\alpha)/(\beta-\alpha)} \end{aligned} \tag{4.4}$$

for all $\beta > \alpha > \alpha_0$ *and* $u > 0$.

Proof. It follows from the relations (3.6)–(3.8) that

$$\begin{aligned} &\Psi_{\alpha\beta\gamma}(u) \\ &= [\lambda_\alpha(u)]^{(\beta-\gamma)/(\beta-\alpha)} [\mu_\beta(u)]^{(\gamma-\alpha)/(\beta-\alpha)} \frac{b_\gamma(u)}{[b_\alpha(u)]^{(\beta-\gamma)/(\beta-\alpha)} [b_\beta(u)]^{(\gamma-\alpha)/(\beta-\alpha)}}. \end{aligned}$$

The condition (4.3) implies the inequalities

$$K_{\alpha\beta\gamma} \leq \frac{b_\gamma(u)}{[b_\alpha(u)]^{(\beta-\gamma)/(\beta-\alpha)} [b_\beta(u)]^{(\gamma-\alpha)/(\beta-\alpha)}} \leq N_{\alpha\beta\gamma}$$

and therefore the condition (4.4) is satisfied for all $\beta > \gamma > \alpha > \alpha_0$. $\square$

Proposition 6. *Let* $\{A_\alpha\}$ $(\alpha > \alpha_0)$ *be a Riesz-type family satisfying, for all* $\beta > \alpha > \alpha_0$, *the relation* (2.7). *Let the function* $b_\alpha(u)$ *be monotonically increasing for every* $\alpha > \alpha_0$ *and the functions* $U(u)$ *and* $V(u)$ *satisfy the condition*

$$V(u) \geq M U(u) u^\delta \qquad (u > 0) \tag{4.5}$$

with some $\delta > 0$. *Then the functions* $\lambda_\alpha(u)$, $\mu_\beta(u)$, *and* $\Psi_{\alpha\beta\gamma}(u)$, *defined by* (3.6)–(3.8), *satisfy the inequalities*

$$K_{\alpha\beta\gamma} \mu_\beta(u) \leq \Psi_{\alpha\beta\gamma}(u) \leq N_{\alpha\beta\gamma} \lambda_\alpha(u) \qquad (u > 0) \tag{4.6}$$

for all $\alpha + \delta = \beta > \gamma > \alpha > \alpha_0$.

Proof. Using (3.6)–(3.8), (2.7), and (4.5), we get

$$\begin{aligned}\Psi_{\alpha\beta\gamma}(u) &= \lambda_\alpha(u)\frac{b_\gamma(u)}{b_\alpha(u)}\left[\frac{U(u)}{V(u)}\right]^{(\gamma-\alpha)/(\beta-\alpha)} \\ &\le \lambda_\alpha(u)\frac{M_{\alpha\gamma}}{(\gamma-\alpha)}u^{\gamma-\alpha}\left[\frac{U(u)}{MU(u)u^{\beta-\alpha}}\right]^{(\gamma-\alpha)/(\beta-\alpha)} = N_{\alpha\beta\gamma}\lambda_\alpha(u).\end{aligned}$$

Thus, the right-sided inequality in (4.6) is proved. The left-sided inequality in (4.6) can be proved analogically:

$$\begin{aligned}\Psi_{\alpha\beta\gamma}(u) &= \mu_\beta(u)\frac{b_\gamma(u)}{b_\beta(u)}\left[\frac{V(u)}{U(u)}\right]^{(\beta-\gamma)/(\beta-\alpha)} \\ &\ge \mu_\beta(u)\frac{(\beta-\gamma)}{M_{\gamma\beta}u^{\beta-\gamma}}\left[\frac{MU(u)u^{\beta-\alpha}}{U(u)}\right]^{(\beta-\gamma)/(\beta-\alpha)} = K_{\alpha\beta\gamma}\mu_\beta(u).\end{aligned}$$

□

Remark 3. Propositions 3, 4, 5, 6, and Theorem 2 remain true if we replace the functions $\lambda_\alpha(u)$, $\mu_\beta(u)$, and $\Psi_{\alpha\beta\gamma}(u)$, defined by (3.6)–(3.8), by those which satisfy the inequalities

$$M_\alpha\frac{b_\alpha(u)}{U(u)} \le \lambda_\alpha(u) \le N_\alpha\frac{b_\alpha(u)}{U(u)},$$

$$R_\beta\frac{b_\beta(u)}{V(u)} \le \mu_\beta(u) \le Q_\beta\frac{b_\beta(u)}{V(u)},$$

and

$$S_{\alpha\beta\gamma}\frac{b_\gamma(u)}{W_{\alpha\beta\gamma}(u)} \le \Psi_{\alpha\beta\gamma}(u) \le T_{\alpha\beta\gamma}\frac{b_\gamma(u)}{W_{\alpha\beta\gamma}(u)},$$

where $\alpha_0 < \alpha < \gamma < \beta$ and $u > 0$.

5. SOME EXAMPLES ON ESTIMATION OF SPEEDS OF SUMMABILITY

To complete our paper, we give some numerical examples as illustrations to our results.

Let $\{A_\alpha\}$ be a Riesz-type family obeying (2.7) for all $\beta > \alpha > \alpha_0$, and $\lambda = \lambda(u)$ and $\mu = \mu(u)$ be any two speeds of summability. For example, statement (2) in Theorem 2 can be reformulated in the form of the following proposition if we take $\lambda = \lambda_\alpha(u)$ and $\mu = \mu_\beta(u)$.

Proposition 7. *Suppose that for some function* $x = \xi(u)$ *and numbers* α *and* β $(\beta > \alpha)$ *we have*

$$A_\alpha x \in m^\lambda$$

and

$$A_\beta x \in c_0^\mu,$$

where $\lambda = \lambda(u)$ *and* $\mu = \mu(u)$ *are such speeds that the functions*

$$U(u) = \frac{b_\alpha(u)}{\lambda(u)} \tag{5.1}$$

and

$$V(u) = \frac{b_\beta(u)}{\mu(u)} \tag{5.2}$$

are monotonically increasing for $u \in [0, \infty)$*. Then*

$$A_\gamma x \in c_0^{\Psi_{\alpha\beta\gamma}}$$

for $\gamma \in (\alpha, \beta)$ *provided that* $\Psi_{\alpha\beta\gamma}(u)$, *defined by* (3.8), *is monotonically increasing.*

Example 1.

(a) Let us consider two Riesz methods (R, α) and (R, β), where $0 < \alpha < \beta$ and two speeds of summability $\lambda = \lambda(u)$ and $\mu = \mu(u)$.

Suppose that

$$A_\alpha x \in m^\lambda, \; A_\beta x \in c_0^\mu$$

for some function $x = \xi(u)$.

We will estimate the speed of summing function x by the method (R, γ), where $\gamma \in (\alpha, \beta)$.

As $b_\alpha(u) = u^\alpha$ for Riesz methods, by (5.1) and (5.2) we get

$$U(u) = \frac{u^\alpha}{\lambda(u)}$$

and

$$V(u) = \frac{u^\beta}{\mu(u)}.$$

If the functions $U = U(u)$ and $V = V(u)$ are monotonically increasing, we have by Proposition 7 that

$$A_\gamma x \in c_0^{\Psi_{\alpha\beta\gamma}}$$

for $\gamma \in (\alpha, \beta)$, where

$$\Psi_{\alpha\beta\gamma}(u) = [\lambda(u)]^{(\beta-\gamma)/(\beta-\alpha)}[\mu(u)]^{(\gamma-\alpha)/(\beta-\alpha)} \tag{5.3}$$

by (3.8) (see also Proposition 5).

(b) For a numerical example we choose first the speeds $\lambda(u) = u^8$ and $\mu(u) = u^8 \log u$ and afterwards the numbers $\alpha = 8$, $\beta = 11$, and $\gamma = 10$. In this case we have the monotonically increasing functions

$$U(u) \equiv 1$$

and

$$V(u) = \frac{u^3}{\log u}.$$

So, we have

$$A_8 x \in m^\lambda, \ A_{11} x \in c_0^\mu \implies A_{10} x \in c_0^{\Psi_{8,11,10}},$$

where

$$\Psi_{8,11,10}(u) = u^8 \log^{2/3} u$$

by (5.3).

This example illustrates also Propositions 4 and 5.

Example 2. Here we consider the Nörlund methods $A_\alpha = (N, p_\alpha(u), q(u))$ (see (2.5)), where $p(u) = u$ and $q(u) = \sqrt{u}$.

(a) Let us have two methods $A_\alpha = (N, p_\alpha(u), q(u))$ and $A_\beta = (N, p_\beta(u), q(u))$ with $0 < \alpha < \beta$ and two speeds of summability $\lambda = \lambda(u)$ and $\mu = \mu(u)$.

Suppose that

$$A_\alpha x \in m^\lambda, \ A_\beta x \in c_0^\mu$$

for some function $x = \xi(u)$.

Estimate the speed of summability of function x by the method $A_\gamma = (N, p_\gamma(u), q(u))$, where $\gamma \in (\alpha, \beta)$. We will use Proposition 7 again.

First we find the functions $p_\alpha(u)$ and $P_\alpha(u)$ $(\alpha > 0)$. Integrating by parts we get

$$p_\alpha(u) = \int_0^u (u-v)^{\alpha-1} p(v)\, dv = \int_0^u (u-v)^{\alpha-1} v\, dv = \frac{u^{\alpha+1}}{\alpha(\alpha+1)}.$$

Now we can find the function $P_\alpha(u)$:

$$\begin{aligned} P_\alpha(u) &= \int_0^u p_\alpha(u-v) q(v)\, dv \\ &= \int_0^u \frac{(u-v)^{\alpha+1}}{\alpha(\alpha+1)} \sqrt{v}\, dv = \mathrm{B}\left(\frac{3}{2}, \alpha+2\right) \frac{u^{\alpha+5/2}}{\alpha(\alpha+1)}, \end{aligned}$$

where $\mathrm{B}(\cdot,\cdot)$ is the beta-function.

As $b_\alpha(u) = P_\alpha(u)$, by (5.1) and (5.2) we can take

$$U(u) = \frac{u^{\alpha+5/2}}{\lambda(u)}$$

and

$$V(u) = \frac{u^{\beta+5/2}}{\mu(u)}.$$

If the functions $U = U(u)$ and $V = V(u)$ are monotonically increasing, by Proposition 7 we have

$$A_\gamma x \in c_0^{\Psi_{\alpha\beta\gamma}}$$

for $\gamma \in (\alpha, \beta)$, where $\Psi_{\alpha\beta\gamma}(u)$ is defined by (5.3).

(b) In particular, choosing the speeds $\lambda(u) = u^4$ and $\mu(u) = u^5 \log u$ and afterwards the numbers $\alpha = 3$, $\beta = 6$, and $\gamma = 4$, we get the monotonically increasing functions

$$U(u) = \frac{u^{3+5/2}}{u^4} = u^{3/2}$$

and

$$V(u) = \frac{u^{6+5/2}}{u^5 \log u} = \frac{u^{7/2}}{\log u}.$$

Thus, we have

$$A_3 x \in m^\lambda,\ A_6 x \in c_0^\mu \implies A_4 x \in c_0^{\Psi_{3,6,4}},$$

where

$$\Psi_{3,6,4}(u) = u^{13/3} \log^{1/3} u,$$

by (5.3).

This example illustrates also Propositions 4 and 5.

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M. Rieszi kumerusteoreemist

Veera Pavlova ja Anne Tali

On vaadeldud tuntud kumerusteoreemi, mille tõestas M. Riesz oma töös [1]. See teoreem annab teatavad kumerustingimused Rieszi menetluste (R, α) jaoks. Hiljem on mitmed autorid Rieszi kumerusteoreemi üldistanud, modifitseerides kumerustingimusi (nt [4] ja [5]) ja ka menetluste (R, α) definitsiooni (nt [6] ja [7]).

Artiklis on üldistatud Rieszi kumerusteoreemi laiemale summeerimismenetluste klassile ja rakendatud seda summeerimiskiiruste hindamisel. Põhitulemusteks on teoreemid 1 ja 2 koos lausetega 4–6.