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ON THE ROBUST CONTROLLER DESIGN VIA SCHUR INVARIANT TRANSFORMS

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Abstract. By the use of a multiparametric Schur invariant transform on the coefficient space of closed-loop characteristic polynomials, a family of stable discrete-time systems is obtained, starting from a nominal stable system. Only linear transforms have been used to formulate some necessary stability conditions and to introduce a rough stability measure. A straightforward robust controller design procedure is proposed for interval plants.

Key words: discrete-time systems, robust control, stability, polynomials, transfer functions.

1. INTRODUCTION

During the last decade, the robust controller design for interval plants has been a problem of great interest. The most significant results in this field are Kharitonov's theorems [1], the edge theorem [2], and the box theorem [3]. These theorems formulate the sufficient stability conditions for interval systems. Unfortunately, Kharitonov's theorems do not hold for discrete-time systems [4,5], and that is why the discrete analog of the box theorem [6] is considerably weaker than the continuous one. The edge theorem holds for both continuous-time and discrete-time systems, but it suffers from a dimensionality curse [5]. However, the crucial fact is that the Kharitonov-like approach does not give any measure for stability robustness. Therefore these theorems suit very well for checking stability of interval systems but not for designing a robust controller, particularly in the discrete-time case.

On the other hand, only few attempts have been made to bring together the well-known robust control methods like H_∞-optimization or structured singular value approach and the Kharitonov-like tests [7].

Our aim is to find a simple tool for comparing robust controller candidates in the discrete-time case. The key idea is to use such a multilinear transform of the coefficients of a polynomial which does not alter the stability of a discrete-time system. By the use of this transform, we can generate a family of stable systems, starting from a nominal stable system. We shall use only linear transforms to obtain some simple necessary stability conditions and to introduce a rough stability measure.

The following problems will be considered. First, we introduce a Schur invariant multiparametric transform $S: R^{n+1} \times R^r \to R^{n+1}$, which maps a polynomial $f(z)$ into a family of polynomials $f(z,\xi) = S(\xi)f(z)$. Every member of the family $f(z,\xi)$ will be Schur stable if $f(z)$ is Schur stable and $\xi_i \in (-1,1), i = 1,...,r$.

Second, we use this transform to formulate some necessary stability conditions for polytopes $\mathcal{P}[f_j(\xi)]$ of the polynomials $f_j(z,\xi), j = 1,...,N$. The stability measure ρ is then introduced as a minimal distance between the nominal point $f(z)$ and the stable vertices of the polytope $\mathcal{P}[f_j(\xi)]$.

The third problem is concerned with controller design by the use of the measure ρ. Starting from an interval plant and a desired characteristic polynomial of a closed-loop system, a heuristic procedure is proposed to choose a candidate for a robust controller. Because the procedure is based on necessary stability conditions and deals only with vertex polynomials, the final check via some sufficient stability test is needed.

2. INVARIANT TRANSFORMS OF SCHUR POLYNOMIALS

A polynomial of the degree n

$$f(z) = f_n z^n + ... + f_1 z + f_0$$

is called Schur stable if all its roots lie in the unit circle, $|\lambda_i| < 1, i = 1,...,n$. Let us introduce polynomials $\bar{f}(z)$ and $h(z)$ of the degree n and $n-1$, respectively

$$\bar{f}(z) = z^n f(z^{-1}) = f_0 z^n + f_1 z^{n-1} + \cdots + f_{n-1} z + f_n,$$

$$h(z) = z^{-1}\left[f(z) - \frac{f_0}{f_n}\bar{f}(z)\right] \tag{1}$$

and let us recall the lemma which allows us to reduce the degree of a polynomial without losing stability.

Lemma 1 [8]. *If $f(z)$ satisfies $|f_n| > |f_0|$, then $h(z)$ will be Schur stable if and only if $f(z)$ is Schur stable.*

We can obtain a similar statement for increasing the degree of a polynomial without losing stability. Let us define a polynomial $g(z)$ of the degree $n+1$

$$g(z,\xi) = zf(z) + \xi\bar{f}(z), \qquad \xi \in R. \tag{2}$$

Lemma 2 [9]. *If $\xi \in (-1,1)$, then $g(z,\xi)$ will be Schur stable if and only if $f(z)$ is Schur stable.*

By repeated use of Lemmas 1 and 2, we can introduce a *Schur invariant transform* $S : R^{n+1} \times R^r \to R^{n+1}$ on the coefficients space $f \in R^{n+1}$ of polynomials $f(z)$ with r-vectors ξ and ν of independent parameters $\xi, \nu \in R^r$, $r \leq n$. Let

$$f(z,\xi,\nu) = \sum_{i=0}^{n} f_i(\xi,\nu) z^i, \qquad f_i(\xi,\nu) \in R$$

and

$$f(\xi,\nu) = S(\xi,\nu) f, \tag{3}$$

where

$$\begin{aligned} f &= [f_0, ..., f_n]^T, \\ f(\xi,\nu) &= [f_0(\xi,\nu), ..., f_n(\xi,\nu)]^T. \end{aligned}$$

Then according to (1) and (2), we can define a $(n+1) \times (n+1)$ matrix $S(\xi,\nu)$ of $2r$ variables $\xi_1, ..., \xi_r$ and $\nu_1, ..., \nu_r$

$$S(\xi,\nu) = R(\xi) P(\nu),$$

where $R(\xi)$ and $P(\nu)$ are $(n+1) \times (n-r+2)$ and $(n-r+2) \times (n+1)$ matrices, respectively

$$R(\xi) = R_{n+1}(\xi_1) \begin{bmatrix} 0^T \\ R_n(\xi_2) \end{bmatrix} \cdots \begin{bmatrix} 0^T \\ R_{n-r+2}(\xi_r) \end{bmatrix}, \tag{4}$$

$$P(\nu) = [0 \vdots P_{n-r+2}(\nu_1)] \ldots [0 \vdots P_n(\nu_{r-1})] P_{n+1}(\nu_r) \tag{5}$$

and

$$R_j(\xi.) = I_j + \xi.E_j,$$

$$P_j(\nu.) = I_j - \nu.E_j,$$

where I_j is a $j \times j$ unit matrix and $E_j = [e_j \vdots \ldots \vdots e_1]$, $\quad e_i = (\underbrace{0...0}_{i-1} 1\ 0...0)^T$.

By repeated use of Eq. (1), we obtain

$$h^{(j+1)}(z) = z^{-1} \left[h^{(j)}(z) - \frac{h_0^{(j)}}{h_{n-j}^{(j)}} z^{n-j} h^{(j)}(z^{-1}) \right],$$

where $h^{(0)}(z) = f(z)$. Obviously,

$$f(z,\nu) = \sum_{i=0}^{n} f_i(\nu) z^i = z^r h^{(r)}(z),$$

where $f(\nu) = P(\nu) f$, if

$$\nu_{r-j} = \frac{h_0^{(j)}}{h_{n-j}^{(j)}}, \qquad j = 0, ..., r-1. \tag{6}$$

If $r = n$ and $f(z)$ is monic, then $\nu_j, j = 1, ..., n$ are called *reflection coefficients* of the polynomial $f(z)$ [10]. Recall that for a Schur polynomial, all the reflection coefficients have $\nu_j \in (-1, 1)$.

Lemma 3 [9]. *The polynomial*

$$f(z, \xi, \nu) = \sum_{i=0}^{n} f_i(\xi, \nu) z^i, \qquad f(\xi, \nu) = S(\xi, \nu) f$$

with ν_j from (6) *will be Schur stable if and only if*

1) *polynomial $f(z)$ is Schur stable,*

2) $\xi_j \in (-1, 1), \quad j = 1, ..., r.$

Next, we shall use the transform $S(\xi, \nu)$ with fixed $\nu_j, j = 1, ..., r$ according to (6) and with $\xi_j \in (-1, 1), j = 1, ..., r$. Therefore, we call it a Schur invariant transform and denote by $S(\xi)$.

3. NECESSARY STABILITY CONDITIONS VIA LINEAR SCHUR INVARIANT TRANSFORM

Lemma 3 formulates the sufficient stability conditions for Schur polynomials, starting from a stable polynomial $f(z)$. By the use of the transform $S(\xi)$, we can give also some simple necessary stability conditions.

Let us consider a polytope of N polynomials $f_1(z), ..., f_N(z)$

$$\mathcal{P}(f_1, ..., f_N) = \gamma_1 f_1(z) + ... + \gamma_N f_N(z),$$

$$\sum_{k=1}^{N} \gamma_k = 1, \qquad |\gamma_k| \leq 1, \quad k = 1, ..., N.$$

For $f_k(z) = f(z, \xi_k, \nu)$, according to Lemma 3, we obtain

Corollary 1. *For the stability of a polytope $\mathcal{P}(f_1, ..., f_N)$ with*

$$f_k(z) = f(z, \xi_k, \nu), \qquad f_k = S(\xi_k, \nu) f, \qquad k = 1, ..., N; \qquad \xi_k, \nu \in \mathcal{R}^r$$

and ν from (6), *it is necessary that*

1) *polynomial $f(z)$ is Schur stable,*

2) $\xi_{kj} \in (-1, 1), \quad j = 1, ..., r.$

The transform $S(\xi)$ is nonlinear in respect of polynomial coefficients $f_0, ..., f_n$ and multilinear in respect of independent parameters $\xi_{k1}, ..., \xi_{kr}$. To obtain a simple necessary condition for the Schur stability of a polytope $\mathcal{P}(f(z, \xi_k, \nu), k = 1, ..., N)$, we have to fix a stable polynomial $f(z)$ and to choose the parameters $\xi_{kj}, j = 1, ..., r$, so that the transform $S(\xi)$ will reduce to a linear one. The simpliest way to accomplish the latter is to fix $r-1$ parameters of the r-vector ξ.

Let us fix the parameters vector ξ_k as follows:

$$\xi_{ki} = \begin{cases} 1 & \text{if } k = i, \\ -1 & \text{if } k = n+i \\ \nu_i & \text{if } k \neq i, k \neq n+i, \\ & i = 1, ..., n; k = 1, ..., 2n, \end{cases} \tag{7}$$

where ν_i is the reflection coefficients of the polynomial $f(z)$.

Now we can generate a family of n line segments $\mathcal{P}(f_i, f_{n+i})$ through the point $f(z)$. Along the line segment $\mathcal{P}(f_i, f_{n+i})$ between the points f_i and f_{n+1}, only the i-th reflection coefficient ν_i is varying $\nu_i \in (-1, 1)$. According to Corollary 1, the polytope $\mathcal{P}(f_k, k = 1, ..., 2n)$ with ξ_{ki} and $\nu_i, i = 1, ..., n$ from (6) and (7) is a candidate for a stable polyhedron around the point $f(z)$.

We need some measure to compare the candidates for stable polyhedrons. The most natural measure would be the minimal distance ρ^* from the point $f(z)$ to the stability boundary. As we know only some points of the stability boundary, we choose the measure as the minimal distance between the point f and the points $f(\xi_k, \nu) = S(\xi) f$

$$\rho = \min_k |f_k - f|. \tag{8}$$

With the linear invariant transform $S(\xi)$ and Lemma 3, we can formulate necessary stability conditions for several different polytopes, which include $f(z)$. For example, let us consider the polynomials $f_{k_1}(z) = f(z, \xi_k, \xi_{k_1}, \nu, \nu_k)$ with coefficient vectors

$$f_{k_1} = S(\xi_{k_1}, \nu_k) f_k = S(\xi_{k_1}, \nu_k) S(\xi_k, \nu) f, \tag{9}$$

where ν and ν_k are the reflection coefficients vectors of the polynomials $f(z)$ and $f_k(z)$, respectively, $\xi_k, \nu \in \mathcal{R}^r$; $\xi_{k_1}, \nu_k \in \mathcal{R}^{r_1}$; $r \leq n$, $r_1 \leq n$.

Corollary 2. *For the stability of a polytope* $\mathcal{P}(f, f_{k_1}, k_1 = 1, ..., 2n)$ *with* f_{k_1} *from* (9), *it is necessary that*

1) *polynomial f(z) is Schur stable,*

2) $\xi_{kj} \in (-1, 1)$, $j = 1, ..., r$; $r \leq n$; $k = 1, ..., 2n$,

3) $\xi_{k_1 j_1} \in (-1, 1)$, $j_1 = 1, ..., r_1$; $r_1 \leq n$.

Let ξ_{jk} be fixed according to (7) and $n-1$ parameters of the vector ξ_{k_1} satisfy $\xi_{k_1 i} = \nu_{ki}$, while $\xi_{k_1 j} \in (-1, 1)$, $j \neq i$; $i, j = 1, ..., n$. Then we produce n line segments through the point $f_k(z)$. Because one of the reflection coefficients ν_{ki} has

$|\nu_{ki}| = 1, n-1$ of these line segments lie on the stability boundary. The n-th line segment is stable and goes through the point $f(z)$.

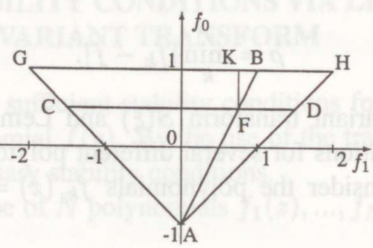
Now we can modify the stability measure ρ, taking into account the new information about the stability boundary. Instead of separate points $f_k(z)$, we have a couple of $M \leq 2n(n-1)$ line segments $f_{k_1} = S(\xi_{k_1}) f_k$, $\xi_{k_1} \in (-1, 1)$ on the stability boundary. Let us define the stability measure $\tilde{\rho}$ as the minimal distance between the point $f(z)$ and the line segments $f_{k_1}(\xi_k)$, $\xi_{kj} \in (-1, 1)$; $\xi_{k_1 i} = \nu_{ki}, i \neq j;\ \ i, j = 1, ..., n$:

$$\tilde{\rho} = \min_{\xi_{k_1}} |f_{k_1}(\xi_{k_1}) - f|. \tag{10}$$

Obviously, $\rho \geq \tilde{\rho} \geq \rho^*$.

In a similar way, we can produce a variety of stable line segments through the points $f_{k_1}(z)$ and modify the measure $\tilde{\rho}$.

Let us illustrate the above statements for $n = 2$ (monic polynomials) (Figure). According to (7), we generate the line segments AB $(-1 < \xi_1 < 1, \xi_2 = \nu_2)$ and CD $(\xi_1 = \nu_1, -1 < \xi_2 < 1)$ through the nominal point F. Conformably to Corollary 1, the quadrangle ADBC is a candidate for the stability region around the point F. By (8), we find $\rho = \overline{FB}$. On the second step, we will find the line segments AH, AG, and GH on the stability boundary through the points A, C, D, and B. According to Corollary 2, the triangle AGH will be a candidate for the stability region. Thus, $\tilde{\rho} = \overline{FK} = \rho^*$.



Stability boundaries and linear invariant transforms (n = 2).

4. ROBUST CONTROLLER DESIGN FOR INTERVAL PLANT

In the standard feedback system, suppose that the nominal single input single output plant

$$W_{p0}(z) = \frac{b^0(z)}{a^0(z)} = \frac{b^0_{m-1} z^{m-1} + \cdots + b^0_1 z + b^0_0}{a^0_m z^m + \cdots + a^0_1 z + a^0_0}$$

is stabilized by the controller

$$W_c(z) = \frac{q(z)}{r(z)}.$$

It means that the closed-loop system

$$W_s(z) = \frac{g(z)}{f(z)} = \frac{b^0(z)q(z)}{a^0(z)r(z) + b^0(z)q(z)}$$

has a Schur characteristic polynomial $f(z)$ of the degree $n \geq m$.

Our aim is to find a suitable controller for an interval plant $W_p(z)$ with $a_i^- \leq a_i \leq a_i^+$, $b_i^- \leq b_i \leq b_i^+$, $i = 1, ..., m$, and $b_m^- = b_m^+ = 0$.

The suitability of a controller $W_c(z) = q(z)/r(z)$ can be found by the use of the stability measure ρ (or $\tilde{\rho}$) for all of the corner polynomials $f^c(z)$ of a closed-loop system

$$f^c(z) = a^c(z)r(z) + b^c(z)q(z), \tag{11}$$

where $a_i^c \in \{a_i^-, a_i^+\}, b_i^c \in \{b_i^-, b_i^+\}, i = 1, ..., m$.

The problem is that some of characteristic polynomials $f(z)$ of the closed-loop system may be placed very close to the stability boundary. So we shall seek a controller which will produce equidistant corner polynomials $f^c(z)$ in the sense of the measure ρ.

Suppose we have calculated the distances ρ (or $\tilde{\rho}$) by (8) (or by (10)) for all of the corner polynomials $f^{c_j}(z), j = 1, ..., 2^{n+1}$. It is reasonable to modify the desired closed-loop characteristic polynomial $f^0(z)$ by weighting the corner polynomials $f^c(z)$ proportionally to the distances ρ_j

$$f^1(z) = \alpha f^0(z) + \alpha_{c_j} f^{c_j}(z), \tag{12}$$

where

$$\alpha_{c_j} = \beta_c \rho_j, \tag{13}$$

$$\alpha + \sum_j \alpha_{c_j} = 1 \tag{14}$$

and α is the fixed weight of the nominal plant $0 < \alpha < 1$, $\rho_j > 0$ if f^{c_j} is stable and $\rho_j < 0$ if f^{c_j} is unstable, β_c is a constant which satisfies (13) and (14).

The design procedure consists of the following basic steps :

1. Starting from a desired characteristic polynomial $f^l(z)$ of the closed-loop system and the nominal plant $W_{p0}(z)$, find the controller $W_c(z, l)$, $l = 0, 1, ...$.

2. Find the corner polynomials $f^{c_j}(z, l)$ of the closed-loop system with the controller $W_c(z, l)$ according to (11), $j = 1, ..., 2^{n+1}$.

3. By the use of the linear invariant transform $S(\xi_k)$ and necessary stability constraints (7), find the points $f_k^{c_j}(z, l)$, $k = 1, ..., 2n$, on the stability boundary for every corner polynomial $f^{c_j}(z, l)$.

4. Calculate by (8) the distances $\rho^{c_j}(l)$. Find the minimal and maximal distances $\rho_{\min}^{c_j}(l)$ and $\rho_{\max}^{c_j}(l)$.

5. Compare the distances $\rho^{c_j}_{\min}(l)$ and $\rho^{c_j}_{\min}(l-1)$.
If $\rho^{c_j}_{\min}(l) \le \rho^{c_j}_{\min}(l-1)$, then
a) make use of the modified distance $\tilde{\rho}$ or
b) increase the weighting constant α.
If $\rho^{c_j}_{\min}(l) > \rho^{c_j}_{\min}(l-1)$ and
a) if $\frac{\rho^{c_j}_{\max}(l)}{\rho^{c_j}_{\min}(l)} \ge \gamma, \gamma > 1$, go to step 6,
b) if $\frac{\rho^{c_j}_{\max}(l)}{\rho^{c_j}_{\min}(l)} < \gamma$, stop.

6. Check the stability of corner polynomials $f^{c_j}(z,l)$. If some of them are unstable, put $\rho_j = -\rho_j$.

7. Find the new desired closed-loop characteristic polynomial $f^{l+1}(z)$ by formulas (12)–(14) and return to step 1 with $l = l+1$.

It is worth pointing out that the proposed procedure gives only a candidate for the robust controller. It is necessary to check this candidate via some sufficient stability criterion (e.g., edge theorem).

Example. Let us have the interval plant

$$W_{p0}(z) = \frac{0.5}{z^2 + 0.5z + 0.5}$$

with $0.2 \ge b_0 \ge 0.8$, $0.3 \ge a_1 \ge 0.7$, $0.4 \ge a_0 \ge 0.6$, and we are seeking a suitable first-order controller

$$W_c(z) = \frac{q_1 z + q_0}{z + r_0}.$$

Let us choose the constants $\alpha = 0.2$, $\gamma = 1.5$ and the closed-loop characteristic polynomial ($n = 3$)

$$f^0 = z^3 + 0.1z^2 + 0.2z + 0.5.$$

Then

$$W_c(z,0) = \frac{0.2z + 1.0}{z}$$

and

$$\rho^c_{\min}(0) = 0.356 \qquad \rho^c_{\max}(0) = 0.917$$
$$\rho^c_{\min}(1) = 0.512 \qquad \rho^c_{\max}(1) = 0.933$$
$$\rho^c_{\min}(2) = 0.593 \qquad \rho^c_{\max}(2) = 0.936$$
$$\rho^c_{\min}(3) = 0.646 \qquad \rho^c_{\max}(3) = 0.938$$

For $l = 3$ we have

$$f^3(z) = z^3 + 0.267z^2 + 0.15z + 0.094$$

and $\frac{\rho^c_{\max}(3)}{\rho^c_{\min}(3)} < \gamma$. Thus, the candidate for the robust controller is

$$W_c(z,3) = \frac{0.137z + 0.508}{z + 0.026}.$$

5. CONCLUSIONS

By the use of a multiparametric transform on the coefficients space of the closed-loop characteristic polynomial $f(z)$, a family of stable discrete-time systems $f(z, \xi)$ is obtained, starting from a nominal stable system. This Schur invariant transform is multilinear in respect of independent parameters $\xi_1, ..., \xi_r$. Only linear Schur invariant transforms have been used to formulate some necessary stability conditions and to introduce a stability measure ρ.

The proposed robust controller design procedure is quite straightforward but needs a final stability check.

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ROBUSTSE REGULAATORI SÜNTEES SCHURI INVARIANTSE TEISENDUSE ABIL

Ülo NURGES

Kasutades mitme vaba parameetriga Schuri invariantset teisendust suletud süsteemi karakteristliku võrrandi kordajate ruumis ja lähtudes stabiilsest nominaalsüsteemist on saadud stabiilsete diskreetsete süsteemide hulk. Lineaarse teisenduse abil on leitud stabiilsuse tarvilikud tingimused polütoopide kujul. Nende polütoopide tippude kaudu on defineeritud stabiilsuse mõõt, mis võimaldab esitada lihtsa protseduuri robustse regulaatori võimaliku variandi valikuks.