Proc. Estonian Acad. Sci. Engng., 1997, **3**, 3, 171–184
https://doi.org/10.3176/eng.1997.3.03

TIME-DEPENDENT PROPERTIES OF A VISCOUS VORTEX RING

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Received 21 February 1997, revised 3 April 1997 and 11 April 1997, accepted 15 May 1997

Abstract. This research involves the unsteady process of vortex ring development in an unbounded viscous fluid. Earlier studies resulted in the linear solution of the Navier–Stokes equations valid for a low Reynolds number in the form of vorticity distribution. In this paper, a new integral expression for streamfunction is derived, and the time-dependent properties of a viscous vortex ring are obtained. A new expression is found for the translation velocity of the ring, which agrees both with the long-time asymptotic drift velocity and the Saffman result for rings with small cross-sections. The result is compared with the experimental data and with the obtained velocity of the trajectory of maximum vorticity.

Key words: vorticity flows, vortex ring, Navier–Stokes equations.

1. INTRODUCTION

Vortex rings are implemented in underwater drilling and modelling the downburst. In addition, they are of great practical interest to aeronautical engineers. Their properties have been studied for over a century. In [1] Shariff and Leonard sum up numerous theoretical and experimental results.

In particular, it can be noted that vortex rings pass through the formation and the post-formation stages during their development. In the post-formation stage, the impact of initial conditions vanishes, and the vortex ring is translated due to its own induced velocity. To achieve a comprehensive description of the post-formation stage, it is necessary to find the time-dependent vorticity distribution and the corresponding streamfunction, which are solutions of the Navier–Stokes equations, and to define the time-dependent properties of this kind of flow motion. However, the most important theoretical results obtained have described only the initial and final phases of this stage. To describe the initial motion of a vortex ring in the post-formation stage, a circular line vortex or the Gaussian distribution of vorticity is commonly used [2]. Saffman [3] described an extension of the expression for the velocity of the vortex ring in an ideal fluid with an arbitrary distribution of vorticity to the viscous case. On these bases, using the Gaussian distribution of vorticity, he obtained the velocity for a vortex ring with a small cross-section, which can be considered the velocity of the ring for an early stage of its evolution. In [4,5] the problem of the velocity definition of the ring was investigated, considering the dynamics in the potential flow region that surrounds the vortical domain. The expression for the ring's translation velocity derived by such a method was close to the result [3]. By using this expression and the vorticity distribution for the final self-similar phase of the ring's evolution [6], the asymptotic drift velocity of the vortex ring was found. Kambe and Oshima [7] made an attempt to generalize the self-similar vorticity distribution to a wider range of time variation. They took into account the second-order effects of the non-linear convective terms of the vorticity equation and obtained new unsteady vorticity distributions. But these distributions have no uniform validity. Hence, the Phillips' vorticity distributions and the streamfunction [6] have remained the sole example of the solution of the Navier–Stokes equations with the conservation of the total impulse for the problem of a viscous vortex ring [8]. The linear solution for this problem, which conserves the total impulse, was proposed in the form of vorticity distribution in [9,10]. Later, this solution was treated as the first order term of expansion in the powers of the Reynolds number [11]. However, the complete description of the evolution of the vortex ring by the use of this solution was not achieved because the corresponding streamfunction was not derived and the important properties, like the ring's translation velocity, were not found. The main goal of this study is to obtain these quantities.

The next section describes the problem and its linear solution in the form of vorticity distribution. In the third and fourth sections, new expressions for the corresponding streamfunction and for the ring's time-dependent properties are derived. Results are discussed in the fifth section.

2. LINEAR SOLUTION OF THE NAVIER–STOKES EQUATIONS

The flow is assumed to be axisymmetric with the constant density ρ and viscosity ν. We use ς to denote the vorticity. The behaviour of a vortex ring in an incompressible fluid is described by the evolution equation [12]

$$\frac{\partial \varsigma}{\partial t}+\frac{\partial}{\partial r}(v\varsigma)+\frac{\partial}{\partial x}(u\varsigma)=v\left[\frac{\partial^2 \varsigma}{\partial x^2}+\frac{\partial^2 \varsigma}{\partial r^2}+\frac{1}{r}\frac{\partial \varsigma}{\partial r}-\frac{\varsigma}{r^2}\right] \tag{1}$$

which corresponds to the axisymmetric problem; x, r are the axes of a cylindrical coordinate system and t is time.

The Stokes streamfunction Ψ can be introduced as follows:

$$u=\frac{1}{r}\frac{\partial \Psi}{\partial r}+U(t),\ v=-\frac{1}{r}\frac{\partial \Psi}{\partial x}\ . \tag{2}$$

Here $U(t)$ is the velocity of the frame moving with the vortex ring given by $U(t)=\frac{\mathrm{d}x_0(t)}{\mathrm{d}t}$, where $x_0(t)$ is the distance, which the vortex ring passes from the initial moment t_0. To employ equations (2), we assume that the vortex ring moves as a whole and does not shed the impulse into its wake.

The vorticity is related to the streamfunction Ψ by the equation

$$\frac{\partial^2 \Psi}{\partial r^2}+\frac{\partial^2 \Psi}{\partial x^2}-\frac{1}{r}\frac{\partial \Psi}{\partial r}=-r\varsigma. \tag{3}$$

Boundary conditions follow from the symmetry about the x axis and the decay of Ψ, ς at the infinity

$$\text{(i)}\quad \Psi(0,x)=\varsigma(0,x)=0\ , \tag{4}$$

$$\text{(ii)}\ \Psi,\varsigma\to 0,\ (x^2+r^2)^{1/2}\to\infty\ . \tag{5}$$

Integration of (1) under the boundary condition provides the conservation for the impulse:

$$I=\pi\rho\int_0^\infty\int_{-\infty}^\infty r^2\varsigma\,\mathrm{d}x\mathrm{d}r\ . \tag{6}$$

By introducing the dimensionless variables

$$\sigma=\frac{r}{L},\quad \eta=\frac{x-x_0(t)}{L},\quad \tau=\frac{R_0}{L},\quad \Phi=\frac{\Psi}{\varsigma_0 L^2},\quad \omega=\frac{\varsigma}{\varsigma_0},\quad L=(2\nu t)^{1/2},$$

$$\varsigma_0=A(M,\nu,R_0)t^{-\alpha},\ M=\frac{I}{\rho}\ , \tag{7}$$

where R_0 is initial radius of the vortex ring, we can obtain the corresponding equation of vorticity in the form

$$-2\alpha\omega-\sigma\frac{\partial\omega}{\partial\sigma}-\eta\frac{\partial\omega}{\partial\eta}-\tau\frac{\partial\omega}{\partial\tau}+\mathrm{Re}\left[\frac{\partial}{\partial\sigma}\left(-\frac{1}{\sigma}\frac{\partial\Phi}{\partial\eta}\omega\right)+\frac{\partial}{\partial\eta}\left(\frac{1}{\sigma}\frac{\partial\Phi}{\partial\sigma}\omega\right)\right]$$

$$=\frac{\partial^2\omega}{\partial\sigma^2}+\frac{\partial^2\omega}{\partial\eta^2}+\frac{1}{\sigma}\frac{\partial\omega}{\partial\sigma}-\frac{\omega}{\sigma^2}, \tag{8}$$

where $\mathrm{Re}=\varsigma_0 L^2/\nu$. In [11] solution of (8) was found for small Reynolds numbers in the form of asymptotic expansions

$$\omega(\sigma,\eta,\tau;\mathrm{Re})=\omega_1(\sigma,\eta,\tau)+\mathrm{Re}\,\omega_2(\sigma,\eta,\tau)+\ldots,$$

$$\Phi(\sigma,\eta,\tau;\mathrm{Re})=\Phi_1(\sigma,\eta,\tau)+\mathrm{Re}\,\Phi_2(\sigma,\eta,\tau)+\ldots, \tag{9}$$

which are valid as $\mathrm{Re}\to 0$ for fixed σ,η,τ. The first order term was obtained as follows:

$$\omega_1=\exp\left(-\tfrac{1}{2}(\sigma^2+\eta^2+\tau^2)\right)I_1(\sigma\tau), \tag{10}$$

where the values of ς_0 and Re were given by the following expressions

$$\varsigma_0=\frac{2M}{(4\pi\nu t)^{3/2}R_0},\quad \mathrm{Re}=\frac{M}{2(\pi\nu)^{3/2}(t)^{1/2}R_0}=\frac{M\tau}{2^{1/2}(\pi)^{3/2}\nu R_0^2}, \tag{11}$$

I_1 denotes the first-order modified Bessel function of the first kind. The direct substitution of (10) in (6) shows that the obtained solution fulfils the conditions of the total impulse conservation. All the results obtained below have the same range of validity as the solution (10) obtained earlier. Furthermore, this allows for formal description of the motion of the starting ring when the initial Reynolds number is very small (initial Reynolds number can be introduced as $\mathrm{Re}_0=\dfrac{M}{\pi R_0^2\nu}=\dfrac{\Gamma_0}{\nu}$, see below expression (18)). Subscript 1 is omitted below.

3. FIELD OF THE STREAMFUNCTION

According to the vorticity distribution (10), the field of the streamfunction Ψ from the Poisson integral solution is given by Batchelor [12] as

$$\Psi = \frac{\sigma\varsigma_0 L^3}{4\pi}\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi}\frac{\omega(\tau,\sigma^1,\eta^1)\sigma^1\cos\theta d\theta d\sigma^1 d\eta^1}{p^2}, \tag{12}$$

where $p^2 = (\eta-\eta^1)^2+(\sigma^1)^2+\sigma^2-2\sigma\sigma^1\cos\theta$.

However, it is possible to simplify the calculation of Ψ in our case by applying the method of the integral transforms. The usefulness of this method for the problem considered was shown in [9]. The Fourier–Hankel integral transforms of the vorticity ω from (10) and of the function $f = \frac{\Phi}{\sigma}$ from the equation (3) are

$$\bar{\omega} = \exp\left(-\frac{\mu^2+\alpha^2}{2}\right)J_1(\tau\mu), \tag{13}$$

$$\bar{f} = \frac{\exp\left(-\frac{\mu^2+\alpha^2}{2}\right)}{\mu^2+\alpha^2}J_1(\tau\mu), \tag{14}$$

where J_1 denotes the first-order Bessel function.

Using the inverse Fourier–Hankel integral transform for $\bar{f}$, we obtain Ψ as follows:

$$\Psi = \frac{2\varsigma_0 L^3\sigma}{(2\pi)^{1/2}}\int_0^{\infty}\int_0^{\infty}\frac{\mu\exp\left(\frac{-\mu^2-\alpha^2}{2}\right)}{\mu^2+\alpha^2}J_1(\tau\mu)J_1(\sigma\mu)\cos(\alpha\eta)d\mu d\alpha .$$

The integration with respect to α presented in [13] gives

$$\int_0^{\infty}\frac{\exp\left(\frac{-\alpha^2}{2}\right)}{\mu^2+\alpha^2}\cos(\alpha\eta)d\alpha = \left(\frac{\pi\exp\left(\frac{\mu^2}{2}\right)}{4\mu}\right)F(\mu,\eta), \tag{15}$$

where $F(\mu,\eta) = \exp(-\eta\mu)\left(1-\operatorname{erf}\left(\frac{\mu-\eta}{\sqrt{2}}\right)\right)+\exp(\eta\mu)\left(1-\operatorname{erf}\left(\frac{\mu+\eta}{\sqrt{2}}\right)\right)$

and erf (z) is the error function.

Thus we have the expression for Ψ in the form of a single integral instead of the triple integral (12) as

$$\Psi = \frac{M\sigma}{4\pi R_0}\int_0^{\infty} F(\mu,\eta) J_1(\tau\mu) J_1(\sigma\mu)\,d\mu \tag{16}$$

and the two velocity components in the dimensionless form inside the moving vortex ring will become

$$v_t = -\frac{1}{\sigma}\frac{\partial\Phi}{\partial\eta} = -\frac{\sqrt{\pi}}{2\sqrt{2}}\int_0^{\infty}\mu\left\{\exp(-\eta\mu)\left(\operatorname{erf}\left(\frac{\mu-\eta}{\sqrt{2}}\right)-1\right) + \exp(\eta\mu)\left(1-\operatorname{erf}\left(\frac{\mu+\eta}{\sqrt{2}}\right)\right)\right\} J_1(\tau\mu) J_1(\sigma\mu)\,d\mu \quad ,$$

$$u_t = \frac{1}{\sigma}\frac{\partial\Phi}{\partial\sigma} = \frac{\sqrt{\pi}}{2\sqrt{2}}\int_0^{\infty}\mu^2 F(\mu,\eta) J_1(\tau\mu) J_0(\sigma\mu)\,d\mu \quad . \tag{17}$$

4. TIME-DEPENDENT PROPERTIES OF A VISCOUS VORTEX RING

Using (10), we can obtain time-dependent properties of a viscous vortex ring. In particular, the circulation and enstrophy are

$$\Gamma = \int_{-\infty}^{\infty}\int_0^{\infty} \varsigma\, dr\, dx = \frac{M}{\pi R_0^2}(1-\exp(-\tau^2)) = \Gamma_0(1-\exp(-\tau^2)) \quad , \tag{18}$$

$$S = \frac{1}{2}\int_0^{\infty}\int_{-\infty}^{\infty}\varsigma^2 r\, dx\, dr = \frac{M^2}{8(R_0(\pi)^{1/2})^5}\tau^3\exp\left(-\frac{\tau^2}{2}\right) I_1\left(\frac{\tau^2}{2}\right) \quad . \tag{19}$$

The velocity of an unsteady ring is defined as [3]

$$U = \frac{\int_0^{\infty}\int_{-\infty}^{\infty}(\Psi + 6xrv)\varsigma\, dx\, dr}{2\int_0^{\infty}\int_{-\infty}^{\infty} r^2\varsigma\, dx\, dr} \quad . \tag{20}$$

Saffman derived this formula using the Lamb's transformation [14] for the velocity of a ring in an ideal fluid and confirmed its validity for the viscous flow. Based on (20), by using (7), the velocity can be defined as

$$U = \frac{1}{2}\frac{\varsigma_0^2 L^5 \pi}{M}\left[E_1 - 6\int_{-\infty}^{\infty}\int_{0}^{\infty}\sigma\eta\frac{\partial f}{\partial\eta}\omega d\eta d\sigma\right],\quad f = \Phi/\sigma\,, \tag{21}$$

where $E = \pi\rho\int_0^{\infty}\int_{-\infty}^{\infty}\varsigma\Psi dx dr = \varsigma_0^2 L^5 \pi\rho\int_{-\infty}^{\infty}\int_0^{\infty}\sigma f\omega d\sigma d\eta = \varsigma_0^2 L^5 \pi\rho E_1$ is the kinetic energy.

In [4] the asymptotic drift velocity U has been found, which is valid for the self-similar regime. We will seek the result, which will give the time dependence of U not only for the self-similar regime, but in the range of validity of the solution (10). The second integral (21) can be rewritten using the integration by parts

$$-6\int_0^{\infty}\sigma\left\{f\omega\eta\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f\frac{\partial}{\partial\eta}(\eta\omega)d\eta\right\}d\sigma = -6\int_0^{\infty}\sigma\left\{-\int_{-\infty}^{\infty} f\omega d\eta + \int_{-\infty}^{\infty}\eta^2 f\omega d\eta\right\}d\sigma$$

$$= 6E_1 - 6\int_0^{\infty}\int_{-\infty}^{\infty}\sigma\eta^2 f\omega d\eta d\sigma\,. \tag{22}$$

The result is

$$U = \frac{\pi}{M}\frac{\varsigma_0^2 L^5}{2}\left[7E_1 - 6\int_0^{\infty}\int_{-\infty}^{\infty}\sigma\eta^2 f\omega d\eta d\sigma\right]. \tag{23}$$

The Fourier integral transform for $\eta^2\omega$ is

$$I_1(\sigma\tau)\exp\left(-\frac{\tau^2+\sigma^2}{2}\right)\int_{-\infty}^{\infty}\eta^2\exp\left(-\frac{\eta^2}{2}\right)\exp(i\eta\alpha)d\eta$$

$$= I_1(\sigma\tau)\exp\left(-\frac{\tau^2+\sigma^2+\alpha^2}{2}\right)(1-\alpha^2)\,. \tag{24}$$

Using the Hankel's integral transforms (13), (14) and the Parseval's theorem, we obtain

$$U = \frac{M\tau}{4\pi^2 R_0^3} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\mu \exp(-\mu^2 - \alpha^2) J_1^2(\tau\mu)}{\mu^2 + \alpha^2} (1 + 6\alpha^2) \mathrm{d}\mu \mathrm{d}\alpha \ . \tag{25}$$

Integration with respect to α gives the final result for U

$$U = \frac{M\tau}{4\pi^2 R_0^3} \int_0^{\infty} \{\pi(1 - \mathrm{erf}(\mu))(1 - 6\mu^2) \ + \ 6(\pi)^{1/2} \mu \exp(-\mu^2)\} J_1^2(\tau\mu) \mathrm{d}\mu. \tag{26}$$

The integral (26) can be expressed in the form

$$U = \frac{M\tau}{4\pi^2 R_0^3} \Big\{ 3(\pi)^{1/2} \exp\left(-\frac{\tau^2}{2}\right) I_1\left(\frac{\tau^2}{2}\right) + \frac{1}{12}(\pi)^{1/2} \tau^2 {}_2F_2\left[\left\{\frac{3}{2}, \frac{3}{2}\right\}, \left\{\frac{5}{2}, 3\right\}, -\tau^2\right] - \frac{3(\pi)^{1/2}}{5} \tau^2 {}_2F_2\left[\left\{\frac{3}{2}, \frac{5}{2}\right\}, \left\{2, \frac{7}{2}\right\}, -\tau^2\right] \Big\} ,$$

where ${}_2F_2$ is generalized hypergeometric function [13].

By substituting the expansion $J_1(\tau\mu)$ in series for low values of τ into the place of $J_1^2(\tau\mu)$ from [15], we can obtain the result for the asymptotic drift velocity of the vortex ring as follows:

$$U_1 = \frac{M\tau^3}{32\pi^2 R_0^3} \int_{-\infty}^{\infty} [\mathrm{Ei}(-\alpha^2)(\alpha^2 + 6\alpha^4) + (1 + 6\alpha^2)] \mathrm{d}\alpha$$

$$= \frac{M\tau^3}{32\pi^2 R_0^3} \frac{28(\pi)^{1/2}}{15} = \frac{7}{15} \frac{M}{(8\pi\nu t)^{3/2}} , \tag{27}$$

where Ei is an integral exponential function.

The formula (27) and the result in [4] are identical and valid for the final period of the vortex ring decay.

5. RESULTS AND DISCUSSION

Calculations were made for the stream function and the ring's translation velocity in the range $0 < \tau \le 10$. A typical plot of the stream function Ψ for $\tau = 5$ is shown in Fig. 1. The expression for a circular line vortex [12] in our designations is

$$\Psi_* = \frac{M\sqrt{\sigma}}{(2\pi^2\sqrt{\tau})R_0^2}\left\{\left(\frac{2}{k} - k\right)K(k) - \frac{2}{k}E(k)\right\}, \quad k^2 = \frac{4\tau\sigma}{\eta^2 + (\tau+\sigma)^2}, \tag{28}$$

where K and E are elliptical integrals of the first and second kind, respectively.

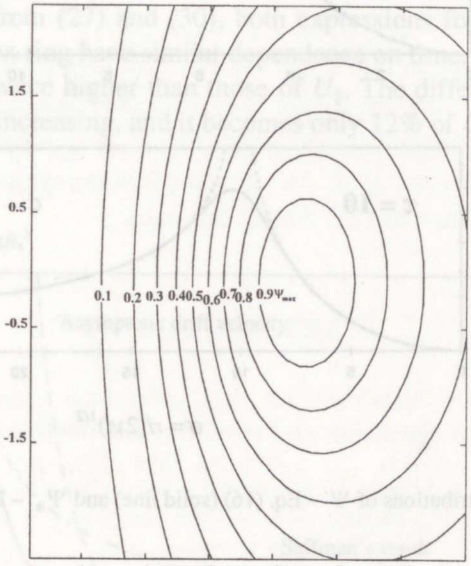

Fig. 1. Contour plots of streamfunction Ψ for $\tau = 5$.

The comparison of Ψ with Ψ_* for different values of $\tau = 2, 5, 10$ along σ-axis is shown in Fig. 2. The results are normalized by $M/(2\pi R_0)$. As shown in this figure, $\Psi \to \Psi_*$ as τ increases. The comparison of Ψ with Ψ_* along η-axis leads to the same results. The calculations by (16) for the low values of τ lead to the values identical with Phillips' self-similar solution. The tendency of the evaluated translation velocity as $\tau \to 0$ to the asymptotic drift velocity is consistent with this statement. Thus, the expression (16) describes the field of stream function, which is initially an approximation of a circular vortex line and finally transformed into the self-similar distribution in [6].

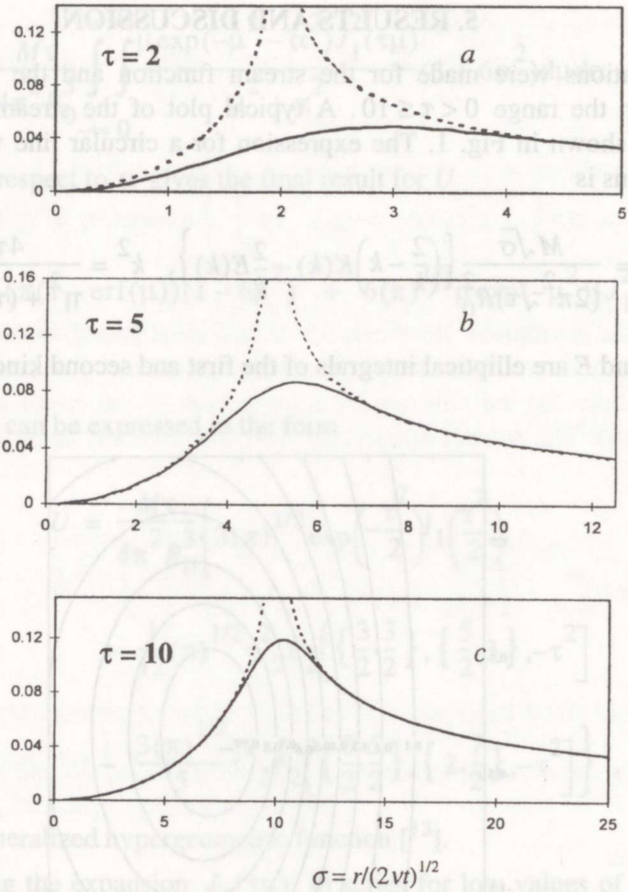

Fig. 2. Comparison of distributions of Ψ – Eq. (16) (solid line) and Ψ_* – Eq. (28) (dashed line) for different values of τ.

The translation velocity of the vortex ring, which is determined by (26) up to $\tau = 10$, is shown in Fig. 3. It is obtained by using the polynomial fitting of $J_1(\tau\mu)$ from [15]. In such a long time limit as $t^* = \dfrac{2t\nu}{R_0^2} = \dfrac{1}{\tau^2} \to \infty$, it coincides with a curve, which corresponds to the calculation of the asymptotic drift velocity (27), and for the early times ($\tau \to 10$) it tends to the values, which are predicted by the Saffman's formula for a vortex ring with a small cross-section. The difference in their values for $\tau = 10$ (the highest value of τ in our calculations) is less than 2%. All the results in Fig. 3 are normalized by $\Gamma_0 / R_0 = M / \pi R_0^3$, where Γ_0 is the initial circulation from (18). Having obtained the ring's translation velocity, the next step is to find the velocity of the trajectory of max ς.

By substituting $\sigma = \sigma_m$ into (17), we obtain the velocity in the location of the vorticity maximum

$$U_m = \frac{M\tau^2}{2\pi R_0^3}\int_0^\infty \left(1 - \mathrm{erf}\left(\frac{\mu}{(2)^{1/2}}\right)\right)\mu J_1(\tau\mu) J_0(\sigma_m \mu) d\mu. \quad (29)$$

Accordingly, the new expression for the asymptotic drift velocity of the vortex ring is

$$U_{m1} = \gamma \frac{M}{(8\pi\nu t)^{3/2}}, \gamma = 1.01098\,. \quad (30)$$

As it can be seen from (27) and (30), both expressions for the asymptotic drift velocity of the vortex ring have similar dependence on time, but the values of U_{m1} are approximately twice higher than those of U_1. The difference between U and U_m reduces with τ increasing, and it becomes only 12% of U for $\tau = 10$.

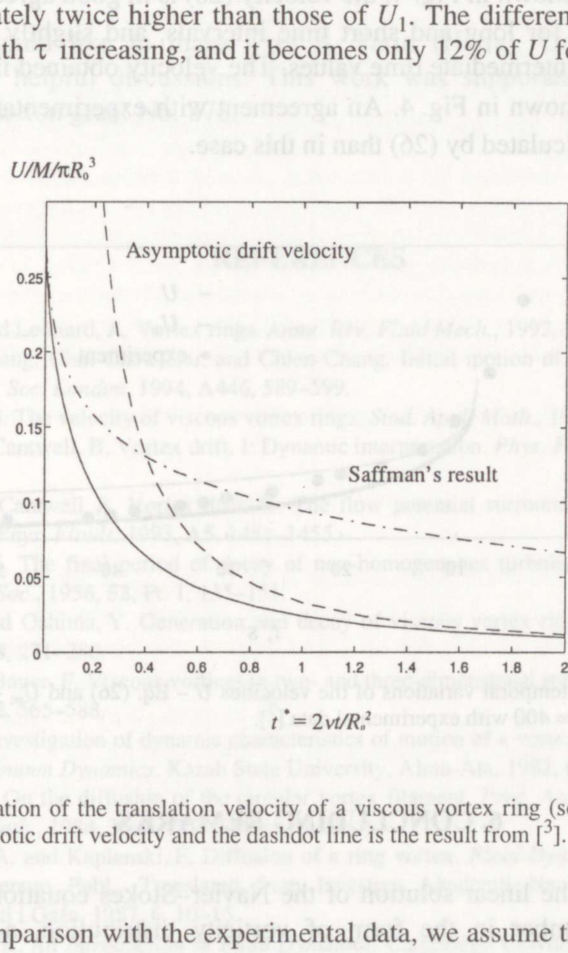

Fig. 3. Time variation of the translation velocity of a viscous vortex ring (solid line). Dashed line shows the asymptotic drift velocity and the dashdot line is the result from [3].

For the comparison with the experimental data, we assume that a virtual origin in time t_0 can be formally determined by the equality

$$\mathrm{Re} = \frac{M\tau}{(2)^{1/2}(\pi)^{3/2}\nu R_0^2} = \mathrm{Re}_0 = \frac{\Gamma_0}{\nu} = \frac{M}{\pi R_0^2 \nu}, \tag{31}$$

which leads to determining the value of t_0 as follows:

$$\tau_0 = (2\pi)^{1/2}, t_0 = \frac{R_0^2}{4\pi\nu}. \tag{32}$$

Using (32), theoretical and experimental results for different initial Reynolds numbers can be compared. Figure 4 shows the profiles of the translation velocity of the experimentally produced in water rings for $\mathrm{Re}_0 = 400$ [7], and the presented theoretical results (26) for $\Gamma_0 = 4$ c^2/s ($M = 4\pi$ c^4/s, $R_0 = 1$ c) and $\nu = 0.01$ c^2/s. As shown in Fig. 4, the velocity (26) is in good agreement with the experimental data for long and short time intervals, and slightly deviates from these data for the intermediate time values. The velocity obtained in the positions of $\max\varsigma$ is also shown in Fig. 4. An agreement with experimental data is better for the velocity calculated by (26) than in this case.

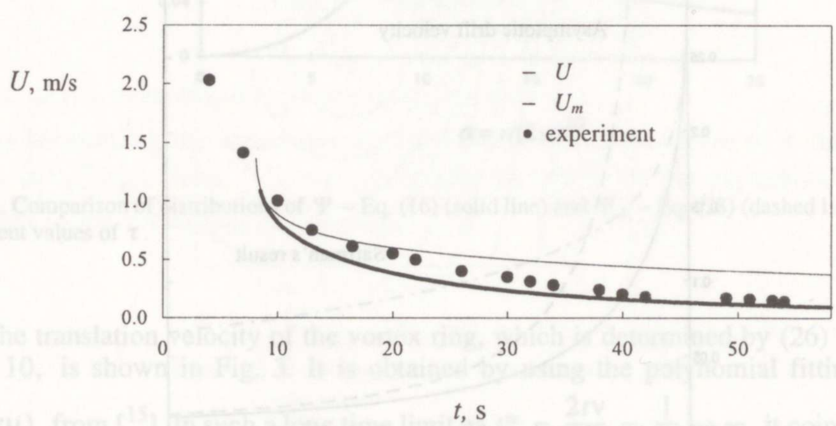

Fig. 4. Comparison of temporal variations of the velocities U – Eq. (26) and U_m – Eq. (29) for the Reynolds number $\mathrm{Re}_0 = 400$ with experimental data [7].

6. CONCLUDING REMARKS

In addition to the linear solution of the Navier–Stokes equations, valid for a low Reynolds number in the form of vorticity distribution, a new integral expression for the corresponding stream function was derived by applying the method of integral transforms. It was shown that the obtained vorticity and stream function transform into Phillips' results in a long time limit. This allowed us to consider them as an extension of the Phillips' solution to a wider range of time variation. Since the new evaluated expression of the stream function is free from singularities, it can be suggested as a useful approximation to the well-known expression of a circular line vortex frequently used in the vortex dynamics. Using the method of integral transforms and the Parseval's theorem, time-dependent translation velocity of the ring was obtained. The presented translation velocity showed better agreement with experimental data than the power-like functions used for this purpose earlier. Furthermore, this result allowed for the estimation of the distance $x_0(t)$, which the ring passes during its evolution.

ACKNOWLEDGEMENTS

Authors would like to thank Prof. A. Yarin (Haifa), Dr. Kh. Kestenboim (Moscow) for helpful discussions. This work was supported by the Estonian Science Foundation grant No. 878.

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VISKOOSSE KEERISRÕNGA LIIKUMISPARAMEETRID

Felix KAPLANSKI ja Ülo RUDI

Tuginedes Navier'–Stokesi võrrandite varem leitud lineaarsele lahendile keeriselisuse ja voolufunktsiooni jaotuste kujul, on esitatud mudel, mis iseloomustab keerisrõnga evolutsiooni viskoosses vedelikus. Selline lahend on rahuldavas kooskõlas senituntud lahenditega keerisrõnga evolutsiooni alg- ja lõppstaadiumis. On määratud keerisrõnga translatsioonikiirus, mis läheneb Saffmani valemiga määratud kiiruse väärtusele keerisrõnga evolutsiooni algstaadiumis ning langeb kokku keerisrõnga assümptootilise kiirusega keerisrõnga evolutsiooni lõppstaadiumis.