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Challenges for tensile stresses

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Abstract. This essay describes some interesting phenomena related to flexible (hanging) structures in nature and engineering. Despite of the seeming simplicity, the behaviour of such structures may be complicated due to non-linear effects.

Key words: tension, hanging structures, spider's web.

1. INTRODUCTION

Engineering designs start from the wish to construct certain machines, buildings, structures, and other useful things. Given the basic idea, the implementing starts from the rather general assumptions. Two most important factors in designing the loaded structural elements in engineering are the material properties from one side and the loading conditions from the other.

Keeping this simple arguing, the materials are brittle or ductile. The loading conditions may vary considerably but we still witness several rather firmly kept engineering solutions. The elements that should resist mostly the compression are designed and built more solid. Structures made of concrete blocks are a good example. Indeed, concrete resists well compression while tensile stresses cause microcracks in concrete structures even at low loading rates. This is why steel reinforcement is used, because steel as a ductile material is well suited to resist tension. So the slender steel bars reinforce the tensile zones in concrete beams. But given a slender steel bar under compression along its axis, we immediately confront the stability problems. If, however, this bar is under tension only, its strength can be used up to the limit.

Such a rather naive presentation omits certainly many details, including complex stress states, complex materials, etc. Nevertheless, one is clear – there are certain structural elements well suited to work under tensile stresses. This essay considers some aspects of such elements.

The reason for discussing this fascinating problem is obvious – the leading papers in this issue are devoted to the analysis of hanging roofs whose main elements are cables under tension. The long-time studies of these structures have been carried out under the supervision of Valdek Kulbach [1]. The author of this essay has been his student and it will be a pleasure to pay a tribute to the teacher. The main idea to write these lines is to show that beside hanging roofs there are also other extremely interesting structures in engineering and nature that bear tensile stresses. However, not everything is simple ...

2. BASIC MECHANICS

A thin elastic filament is a three-dimensional body with cross-section dimensions much smaller than its length. Such a model is widely used in explaining many technological processes like the kinking of telephone cables or describing long molecular structures including proteins, DNA, and bacterial fibres [2]. Its dynamics is governed by the well-known Kirchhoff equations [2,3]. A filament can also resist bending stresses and thus the conservation of motion and angular momentum provide the governing equations – altogether six coupled non-linear partial differential equations of the second order in time and arc-length (measured along the filament). If, however, the rigidity of a filament is negligible, then the model of a flexible string can be used. Again, there is a wide area of usage of strings, starting from piano strings to textile yarn manufacturing processes and reinforcements of composite materials or to space structures (cf. ref. in [2,4]). A special case is a helical fibre or helix which also has many applications: helical cables and ropes, models of DNA molecules, audio tapes, etc. [4,5]. The analogy with vortex filaments in fluids is obvious [6]. So beside cables, used for hanging structures in civil engineering, similar elements are of interest in many fields of science and in many applications.

A flexible string of mass density ρ per unit length can be described by the equation of motion [7]

$$\rho \frac{\partial^2 \mathbf{R}(s,t)}{\partial t^2} = \frac{\partial}{\partial s}\left(\frac{T(s,t)}{A(s,t)}\frac{\partial \mathbf{R}(s,t)}{\partial s}\right), \tag{1}$$

where $\mathbf{R}(s,t)$ is the position vector and s is the coordinate along the string. The relation between the tension force T and the stretch of the fibre A should be given through a constitutive equation. This is one possible source of non-linearity while another source is geometry – the possible large deflections of the string (or helix). The latter is usually more important, influencing the behaviour of a string even at low and moderate loadings.

Here we give some examples of fascinating dynamical phenomena in strings. One is the solitonic behaviour of waves propagating in a spacial helix. The position vector $\mathbf{R}(s,t)$ in Eq. (1) can then be decomposed into two – the longitudinal component $R_x(s,t)$ and the second component $R_t(s,t)$, lying in the cross-section of the helix. Leaving aside mathematics (for details see [7]), the spatial shape of the deformed helix can be calculated. The stretch within the helix, projected onto the x-axis, is a soliton as shown in Fig. 1. The speed of the soliton depends here on the initial geometry.

In principle, the general rotation of the helix can also be easily introduced. An interesting problem is whether under certain conditions a chaotic solution exists or not. Indeed, the problem is non-linear, so the prerequisites for a chaotic regime are satisfied. It has been shown [8] that statically such a helix can indeed take chaotic or quasiperiodic forms. Such a deformed shape is shown in Fig. 2. An interesting application of helices is to use them in composite materials for reinforcement [5].

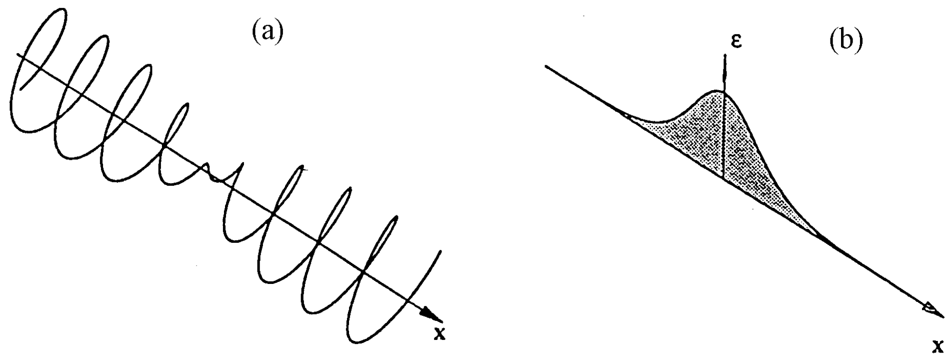

Fig. 1. Soliton in a helix: (a) deformed shape; (b) non-dimensional stretch ϵ along the x-axis (after Krylov and Rosenau [7]).

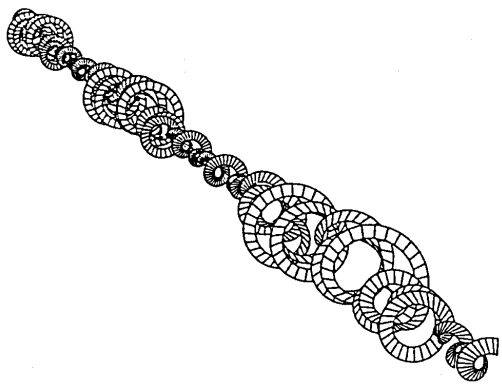

Fig. 2. Three-dimensional deformed shape of a helix (after Davies and Moon [8]).

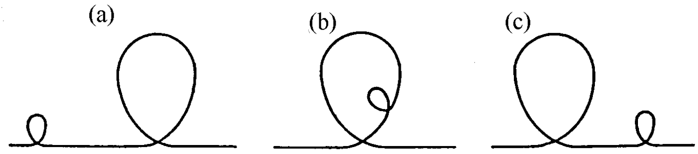

Fig. 3. Loop solitons for three successive time moments (from left to right).

The second example concerns the existence of the so-called loop solitons in loose strings. The governing equation is then the following [9]:

$$\frac{\partial^2 w}{\partial x \partial t} + \operatorname{sign}\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right) \frac{\partial^2}{\partial x^2} \quad \left[\frac{\partial^2 w/\partial x^2}{(1 + (\partial w/\partial x)^2)^{3/2}}\right] = 0, \tag{2}$$

where $w(x,t)$ is the transverse motion in a straight half-finite $(x > 0)$ string (elastica) subject to an excitation at the end $x = 0$, and s is the arc length along the solution curve. This equation has a solution in the form of a loop propagating along the string with an amplitude-dependent velocity (Fig. 3). These loops are of solitary character, i.e. they propagate without changing their shape. This is a remarkable property of waves in non-linear structures due to the balanced effects of non-linearity and dispersion. In some sense such waves are comparable with particles that justifies their common name – solitons (see the seminal paper [10]). The loop solitons have a remarkable property: the smaller the amplitude, the larger the velocity; i.e. the smaller loop overtakes the larger one (Fig. 3). On the contrary, the celebrated Korteweg-de Vries solitons [11] behave vice versa – the larger the amplitude, the larger the velocity; i.e. larger solitons overtake the smaller ones. Loop solitons are analysed in detail in [12].

We have agreed to forget about compression but what happens if beside tensile stresses also torques are applied to strings? The last example described briefly here is a case of a filament which is loaded by both tension and twist. The phenomenon that can then be observed is called *the twist to writhe conversion* [2] which is emphasized by generation of supercoiling (additional loops). Such an experiment can easily be carried out manually by twisting the end of a rubber tape or a telephone cord. It can then be clearly observed that such a loading indeed results in strange additional loops called snarls. This phenomenon has remarkable consequences not only for structural elements but also in biological context. As shown in [2], it concerns the behaviour of tendrils in climbing plants (growth and handedness of loops or snarls) and also the self-assembly of certain bacterial filaments.

3. STRINGS IN NATURE

Much in engineering is learned from natural objects. In our context of hanging structures, one should look how spiders spin their webs made of threads of biological origin. The webs are usually either vertical or horizontal, sheet-like and rather symmetric. Their shapes are really fantastically beautiful. The most "classical" shape is depicted in Fig. 4. In Fig. 5a a horizontal web is shown which really resembles a hanging roof. What is remarkable from the engineering viewpoint is the flatness of the structure. Another spatial "fishing-net" – type structure is shown in Fig. 5b. Actually, keeping in mind the famous hanging membranes of Frey Otto, the German engineer, nature surely overpasses the technological ideas. Last, a system of webs shown in Fig. 6, characterizes the "fantasy" of their creators. The material of webs – the spider silk – has remarkable properties [14]. One should mention first that it has a history of manufacturing about 400 million years [15]; so the web-engineering is really rather sophisticated. Spider dragline silks are exceptionally strong and extensible. The strength of the silk is about 1.1 GPa compared with 1.3 GPa of a typical steel but its relative density is only 1.39 g/cm^3 compared with 7.89 g/cm^3 of steel. It is of interest, however, to compare the properties of spider dragline silk with those of man-made fibres of Kevlar. According to [14], the comparative data of silk vs Kevlar yarn are the following: silk (*Nephila edulis*) – diameter $3.35 \pm 0.63\mu\mathrm{m}$, max strain 0.39 ± 0.08, max stress 1.15 ± 0.20 GPa, elasticity modulus 7.9 ± 1.8 GPa; Kevlar 81 high-tenacity yarn – diameter 12μm, max strain 0.05, max stress 3.6 GPa, elasticity modulus 90 GPa. The conclusion from [14] sounds: "Thus Kevlar is 3 times stronger but spider silk is 5 times tougher because it is 8 times more extendible".

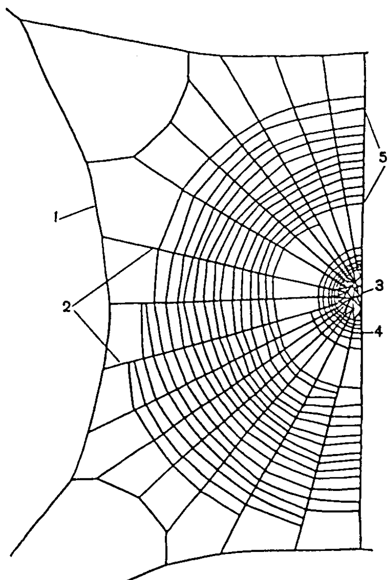

Fig. 4. The spiral web of *Araneus diadematus*: 1 – the frame; 2 – the rays; 3 – central part; 4, 5 – spiral draglines (after [13]).

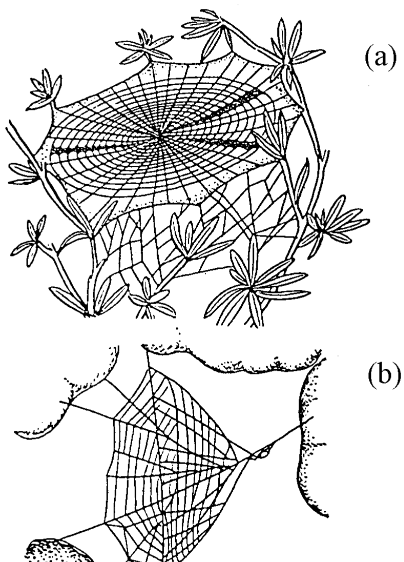

Fig. 5. (a) horizontal web of *Uloborus*; (b) three-dimensional spacial web of *Theridiosoma gemmosum* (after [13]).

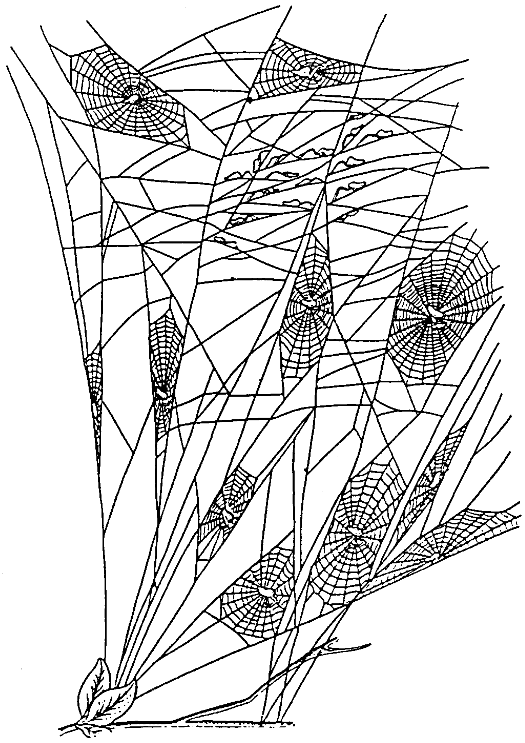

Fig. 6. A web of webs (after [13]).

Knowing the advanced spinning "technology" of spiders, the engineering technology can much learn how to produce fibres with similar properties. Remarkable in the natural "technology" are the low temperature, low energy cost and low spinning speeds. The webs are quite resistant even in our climate and "designed" in an optimal way, saying nothing about the beauty.

Certainly the spider's webs are not the only examples from nature. Contemporary science looks more and more to smaller scales, that is to molecular structures. Mentioned already in the previous section were the DNA chains like helices. The excitations along those helices are modelled using the soliton concept; for example, the so-called Davydov solitons are thought to be responsible for transferring energy along the bonded spines of a helix [16]. While this model is effectively one-dimensional and corresponds to an intuitive understanding of a string or a helix, then peptide chains in general terms are at least two-dimensional [17]. Actually the theory described in [17] is a remarkable example how ideas of continuum mechanics are able to describe complicated biological structures, here – the propagation of waves along chains of molecules.

4. DYNAMICAL HAZARDS

As noted above, the structural elements designed to resist tensile stresses are not jeopardized by the loss of stability characteristic to structures under compression. However, this does not mean that everything is much simpler. There is another physical phenomenon related to the high flexibility of such structures – during aeroelastic excitations (wind loading) flexible structures may maintain large-scale oscillations. Actually there are several mechanisms for such a behaviour like unimodal galloping, bimodal flutter, and vortex resonance [18]. It is argued that the famous failure of the Tacoma Narrows suspension bridge was caused namely by such effects because the designers had overlooked the existence of an unstable limit cycle during certain excitations. As an example of such a modelling, let us consider the behaviour of a non-linear aeroelastic oscillator [18]. A suitable model consists of a square prism fixed by a spring and a dashpot to a foundation so that the structure can move vertically (axis y is directed downwards). Let the wind blow with a velocity V past the prism parallel to axis x. The equation of motion is then [18];

$$m\ddot{y} + r\dot{y} + ky = \frac{1}{2}\rho V^2 a \ \left[A_1(\dot{y}/V) - A_3(\dot{y}/V)^3 + \ A_5(\dot{y}/V)^5 - A_7(\dot{y}/V)^7\right], \tag{3}$$

where the right-hand side represents the highly non-linear aerodynamic force. Further, m, r, k, ρ, a are the physical constants (a is the frontal area of the prism) and $A_1, ..., A_7$ are empirical constants. Such a simple model structure undergoes the Hopf bifurcation at a certain value of V and at higher values of wind velocities there are three limit cycles, the middle one of which is unstable. It is clear that if this structure is loaded within the domain of the attraction of the unstable limit cycle, then the large-amplitude oscillations will be generated even at moderate excitations.

By changing the cross-sections, the coefficients of the model (3) will be changed but the phenomenon is the same. This example could serve as a model for ice-coated power cables [19]. Modelling of the atmospheric icing is of interest in many areas including civil engineering and air industry. From the viewpoint of structural design it involves the determination of ice loads. For example, the rime on a 22 kV power cable could lead to the ice load as high as about 300 kg/m [19]. The increase of the cross-section with the growth of mass changes then dramatically the dynamical behaviour of such structures and may cause their failure with serious consequences.

5. FINAL REMARKS

Hanging structures are economical and beautiful. But one should know a lot about their behaviour because their seeming simplicity may be deceptive. Truly the arguing may start from understanding that it is not only the strength of materials and the cross-sections of structural elements which are decisive in design, the loss of stability rules sometimes the situation. Then clearly the elements that are meant to resist only tensile stresses could be used if possible in order to get rid of stability problems. One may be tempted to simplify the design rules but other hazards should not be forgotten. Due to high flexibility, flutter or galloping may occur at dynamical loadings. Looking for the reasons of such behaviour, we come to the concept of non-linearity. Non-linearity is a very important notion in contemporary science, meaning that the processes are not additive like they are within the framework of linear theories. Even more, non-linearity gives rise to many novel physical phenomena including chaos (cf., for example, a large number of references on that topic in [[11]]). This is also the case of hanging structures, possessing the richness of natural and technological world. It also means that despite of the seeming simplicity of hanging structures, their analysis should be carried out with a care not neglecting the non-linear effects (cf. non-linear equations (1)–(3) and [[1]]).

It addition, one should not forget that natural and technological problems are interwoven into a complicated system where the analogies and links between phenomena could really be stunning. It is not by chance that examples in this essay are mostly drawn from physics in order to demonstrate how closely engineering problems are related to other physical phenomena. It is not difficult, for example, to link the design problems of hanging roofs with spider's webs and the design of cables with modelling of icing and icicles.

REFERENCES

1. Kulbach, V. Investigation of prestressed cable structures at Tallinn Technical University. *Proc. Estonian Acad. Sci. Eng.*, 2002, **8**, 68–83.
2. Goriely, A. and Tabor, M. The nonlinear dynamics of filaments. *Nonlinear Dyn.*, 2000, **21**, 101–133.
3. Dill, E. H. Kirchhoff's theory of rods. *Arch. History Exact Sci.*, 1992, **44**, 2–23.
4. Krylov, V. and Slepyan, L. I. Binary wave in a helical fiber. *Phys. Rev. B*, 1997, **55**, 14067–14070.
5. Slepyan, L., Krylov, V., and Parnes, R. Solitary waves in a helix. *Proc. Estonian Acad. Sci. Phys. Math.*, 1995, **44**, 29–39.
6. Keener, J. P. Knotted vortex filament in an ideal fluid. *J. Fluid Mech.*, 1990, **211**, 629–651.
7. Krylov, V. and Rosenau, P. Solitary waves in an elastic string. *Phys. Lett. A*, 1996, **217**, 31–42.
8. Davies, M. A. and Moon, F. C. 3-D spacial chaos in the elastica and the spinning top: Kirchhoff analogy. *Chaos*, 1993, **3**, 93–99.
9. Wadati, M., Konno, K., and Ischikawa, Y. W. New integrable nonlinear evolution equations. *J. Phys. Soc. Japan*, 1979, **47**, 1689–1700.
10. Zabusky, N. J. and Kruskal, M. D. Interaction of "solitons" in a collisionless plasma and recurrence of initial states. *Phys. Rev. Lett.*, 1965, **15**, 240–243.
11. Engelbrecht, J. *Nonlinear Wave Dynamics: Complexity and Simplicity*. Kluwer, Dordrecht, 1997.
12. Konno, K. and Jeffrey, A. Some remarkable properties of two loop soliton solutions. *J. Phys. Soc. Japan*, 1983, **52**, 1–3.
13. *Loomade elu. Selgrootud III*. Valgus, Tallinn, 1984.

14. Vollrath, F. and Knight, D. P. Liquid crystalline spinning of spider silk. *Nature*, 2001, **410**, 541–548.
15. Shear, W. A., Palmer, J. M., Coddington, J. A., and Bonamo, P. M. A Devonian spinneret: early evidence of spider and silk use. *Science*, 1989, **246**, 479–481.
16. Davydov, A. S. *Solitons in Molecular Systems*. Reidel, Dordrecht, 1985.
17. Zorski, H. and Infeld, E. Continuum dynamics of a peptide chain. *Int. J. Non-Linear Mech.*, 1997, **32**, 769–801.
18. Thompson, J. M. T. and Stewart, H. B. *Nonlinear Dynamics and Chaos*. J. Wiley, Chichester, 1986.
19. Makkonen, L. Models for the qrowth of rime, glaze, icicles and wet snow on structures. *Philos. Trans. R. Soc. Lond. A*, 2000, **358**, 2913–2939.

Väljakutse tõmbepingetele

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Essees on kirjeldatud mitmeid füüsikalisi probleeme, mis on seotud ripp-konstruktsioonidega nii looduses kui ka inseneriasjanduses. Hoolimata näilisest lihtsusest võib selliste tõmbele töötavate konstruktsioonide käitumine olla keeruline, seda eeskätt mittelineaarsete efektide tõttu.