Proc. Estonian Acad. Sci. Eng., 1999, 5, 2, 112–121
https://doi.org/10.3176/eng.1999.2.02

MODELLING OF TWO-DIMENSIONAL ELASTIC WAVE PROPAGATION WITH CONTINUOUS CELLULAR AUTOMATA

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Received 1 October 1998

Abstract. A novel approach to construct an algorithm for simulation of wave propagation in elastic media is presented. The method is based on the idea that each element of a continuum can be considered as a cell, the state of which is determined as a thermodynamic state of the corresponding element of the medium. The concept of discrete thermodynamic systems is used for thermodynamic description of the non-equilibrium states of these elements. In the framework of this concept, the contact quantities are introduced, which determine the interaction of an element with its neighbourhood. The interaction between elements of the medium is described in the Gibbsian phase space instead of the space of physical variables. Specification of contact quantities depends on the considered process. As a result, relations are obtained which couple the non-equilibrium state of an element with states of its neighbours. Therefore, we can formulate the rules of evolution for the elements of a continuum to model the process by means of the cellular automata technique. The proposed method is a tool for direct simulation of a process rather than for solution of partial differential equations. Nevertheless, in simple cases it can be reduced to classical finite-difference schemes. Results of numerical experiments are presented.

Key words: cellular automata, elastic wave propagation, thermodynamics of discrete systems.

1. INTRODUCTION

The problem of elastic wave propagation is well studied both theoretically and experimentally [1–4]. It is a part of the theory of elasticity, which, in its turn, is a part of the general theory of continuum mechanics [5,6].

In the case of linear elasticity, stress and strain tensors are coupled by Hooke's law [6]

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij}, \tag{1}$$

where σ_{ij} are components of the Cauchy stress tensor, ε_{ij} are components of the strain tensor, λ and μ are the Lamé coefficients, δ_{ij} is the Kronecker delta, and $\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$ in view of the summation convention.

Deformation rate is determined only by linear terms [5]

$$\frac{\partial \varepsilon_{ij}}{\partial t} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2}$$

where t is time and u_i are components of the vector of the deformation velocity.

The Newton principle of linear momentum can be written in the form [5,6]

$$\rho_0 \frac{\partial u_i}{\partial t} = \frac{\partial \sigma_{ij}}{\partial x_j} \tag{3}$$

where ρ_0 is density of the medium.

Introducing the time derivatives of the stress tensor according to Hooke's law (1)

$$\frac{\partial \sigma_{ij}}{\partial t} = \lambda \delta_{ij} \frac{\partial \varepsilon_{kk}}{\partial t} + 2\mu \frac{\partial \varepsilon_{ij}}{\partial t}, \tag{4}$$

we can rewrite the full system of equations of linear elasticity in terms of stresses and velocities of deformation. In the two-dimensional case we have [7]

$$\frac{\partial \sigma_{11}}{\partial t} = (\lambda + 2\mu)\frac{\partial u}{\partial x} + \lambda \frac{\partial v}{\partial y}, \tag{5}$$

$$\frac{\partial \sigma_{22}}{\partial t} = \lambda \frac{\partial u}{\partial x} + (\lambda + 2\mu)\frac{\partial v}{\partial y}, \tag{6}$$

$$\frac{\partial \sigma_{12}}{\partial t} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \tag{7}$$

$$\rho_0 \frac{\partial u}{\partial t} = \frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y}, \tag{8}$$

$$\rho_0 \frac{\partial v}{\partial t} = \frac{\partial \sigma_{12}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y} \tag{9}$$

where u and v are the components of the deformation velocity.

If we could obtain a solution of the system of equations (5)–(9), which satisfies corresponding initial and boundary conditions, the problem of elastic wave propagation would be solved. Unfortunately, exact solution of these equations is known only in a few simple cases [1–4]. In practice, numerical methods are usually applied. Here we return to a discrete representation of the continuous medium, which is governed by a certain approximation of Eqs. (5)–(9). In this way, we can try to simulate the process directly instead of the improvement of the numerical approximations.

The main goal of this paper is to develop a method for the modelling of elastic wave propagation by means of cellular automata technique. It should be noted that classical cellular automata are discrete dynamic systems [8]. The discreteness means that space, time, and properties of the automaton can have only a finite, countable number of states. The main idea is not to describe a complex system with complex equations, but to let the complexity emerge by interaction of simple elements following simple rules [8].

Cellular automata are built up from identical cells which are arranged in a two-dimensional lattice. The future state of each cell depends only on the current state of the cell and on the states of the cells in the neighbourhood. The development of each cell is defined by rules which describe the interaction between cells.

We shall try to extend the main ideas of cellular automata for the case of the continuous space of state which is only suitable for the modelling of the elastic wave propagation in a continuous medium. To define the state space of cells representing elements of a continuum, we shall use the concept of discrete thermodynamic systems [9].

2. DISCRETE SYSTEMS

Discrete system is a generalization of the concept of a thermodynamic system which allows us to take into consideration non-equilibrium states of the system [9]. The interaction between neighbouring systems can be described by intensive non-equilibrium contact quantities, namely, the contact temperature Θ, the dynamic pressure p, and the dynamic chemical potentials $\boldsymbol{\mu}$ [10], if the interaction consists of heat exchange $\dot{Q}$, of work exchange $\dot{W}$, and of time rates of the mole numbers of different species $\dot{\boldsymbol{n}}^e$ due to external material exchange.

The extended state space of a discrete system in a stationary frame and in absence of external deformations can be chosen as follows [10]:

$$Z = \{V, \boldsymbol{n}, U, \Theta, p, \boldsymbol{\mu}; T^*, p^*, \boldsymbol{\mu}^*\}. \tag{10}$$

Here U is the internal energy of the system, V is volume, T^*, p^*, and $\boldsymbol{\mu}^*$ correspond to the equilibrium environment.

The contact quantities provide a complete thermodynamic description of non-equilibrium states of a separated discrete system [11].

3. ELASTIC MEDIUM

In the case of an isothermal elastic medium, we describe the external deformation of an element by linearized strain per unit mass $\rho_0^{-1}\varepsilon_{ij}$ [6]. The non-equilibrium element interacts with its surroundings during an irreversible process through a transfer of work and heat. In the isothermal case, this interaction is described by the Cauchy stresses. The work exchange is determined as follows [12]

$$\dot{W} = -\frac{1}{\rho_0}\sigma_{ij}\dot{\varepsilon}_{ij}. \tag{11}$$

To generalize the concept of discrete thermodynamic systems for the elastic media, we need to introduce dynamic stresses Σ_{ij} by the following inequalities

$$\dot{\varepsilon}_{ij}\left(\Sigma_{ij}-\sigma_{ij}^{*}\right)\geq 0 \quad (\dot{Q}=0;\ \dot{n}^{e}=0), \tag{12}$$

where σ_{ij}^{*} is the stress tensor in the equilibrium environment.

After introducing the deformation variables, we come to the extended state space of a discrete system representing an element of isothermal elastic medium without mass exchange

$$Z=\{\varepsilon_{ij},U,\Sigma_{ij};\sigma_{ij}^{*}\}. \tag{13}$$

The time rate of internal energy is determined by the work exchange

$$\dot{U}=\frac{\partial U}{\partial\varepsilon_{ij}}\dot{\varepsilon}_{ij}. \tag{14}$$

It can be shown that additional conditions should be fulfilled between neighbouring systems to provide the independence of the thermodynamic description from the chosen size of the cells:

$$\frac{\partial(U^{(n)}+E^{(n)})}{\partial\varepsilon_{ij}}=\frac{\partial(U^{(n+1)}+E^{(n+1)})}{\partial\varepsilon_{ij}}, \tag{15}$$

where $E^{(n)}$ is the interaction energy which corresponds to one of the neighbouring systems.

Consequently, the conditions of interaction (15) can be expressed for each pair of interacting cells in the form

$$\sigma_{ij}^{(n)}+\Sigma_{ij}^{(n)}=\sigma_{ij}^{(n+1)}+\Sigma_{ij}^{(n+1)}, \tag{16}$$

if we suppose that the interaction energy has elastic nature.

It should be noted that the dynamic stresses represent contact forces acting on the chosen element or cell. Their action results in the variation of deformation velocity according to the Newton principle of linear momentum. The latter can be expressed in the two-dimensional case as follows

$$\rho_0 u_{ij}^{k+1}=\rho_0 u_{ij}^{k}+a\left[(\Sigma_{11}^{+})_{ij}^{k}-(\Sigma_{11}^{-})_{ij}^{k}+(\Sigma_{12}^{+})_{ij}^{k}-(\Sigma_{12}^{-})_{ij}^{k}\right], \tag{17}$$

$$\rho_0 v_{ij}^{k+1}=\rho_0 v_{ij}^{k}+b\left[(\Sigma_{21}^{+})_{ij}^{k}-(\Sigma_{21}^{-})_{ij}^{k}+(\Sigma_{22}^{+})_{ij}^{k}-(\Sigma_{22}^{-})_{ij}^{k}\right], \tag{18}$$

where Σ_{mn}^{+} and Σ_{mn}^{-} correspond to the opposite sides of the cell, respectively, and a and b are constants. Here superscript k corresponds to the time step, and subscripts ij point the cell number.

At the same time, we should bear in mind that stresses and strains are coupled by Hooke's law (1). An approximation of this law we can represent in the form which is convenient for using in cellular automata

$$(\sigma_{11})_{ij}^{k+1} = (\sigma_{11})_{ij}^{k} + c(\lambda+2\mu)((U^{+})_{ij}^{k} - (U^{-})_{ij}^{k}) + c\lambda((V^{+})_{ij}^{k} - (V^{-})_{ij}^{k}), \quad (19)$$

$$(\sigma_{22})_{ij}^{k+1} = (\sigma_{22})_{ij}^{k} + c(\lambda+2\mu)((V^{+})_{ij}^{k} - (V^{-})_{ij}^{k}) + c\lambda((U^{+})_{ij}^{k} - (U^{-})_{ij}^{k}), \quad (20)$$

$$(\sigma_{12})_{ij}^{k+1} = (\sigma_{12})_{ij}^{k} + d\mu((V1^{+})_{ij}^{k} - (V1^{-})_{ij}^{k} + (U1^{+})_{ij}^{k} - (U1^{-})_{ij}^{k}), \quad (21)$$

where c and d are constants and $U, U1$, and $V, V1$ are certain contact deformation velocities.

4. BUILDING A CELLULAR AUTOMATA ALGORITHM FOR ELASTIC MEDIA

There is a lot of possibilities for choosing the contact quantities in the case of an elastic medium. At the first look, the natural way is to identify the dynamic stresses with values of stress components for the corresponding cells in the neighbourhood. However, such a choice leads to non-monotonic calculation results. The best choice, in our opinion, is to use the exact solutions of the dynamic one-dimensional problem for each side of the cell [13]. In this case, the contact quantities are determined as follows

$$(\Sigma_{11}^{+})_{ij}^{k} = \frac{(\sigma_{11})_{ij}^{k} + (\sigma_{11})_{i+1j}^{k}}{2} - \rho_0 c_p \frac{u_{i+1j}^{k} - u_{ij}^{k}}{2}, \quad (22)$$

$$(\Sigma_{11}^{-})_{ij}^{k} = \frac{(\sigma_{11})_{ij}^{k} + (\sigma_{11})_{i-1j}^{k}}{2} - \rho_0 c_p \frac{u_{ij}^{k} - u_{i-1j}^{k}}{2}, \quad (23)$$

$$(\Sigma_{12}^{+})_{ij}^{k} = \frac{(\sigma_{12})_{ij}^{k} + (\sigma_{12})_{ij+1}^{k}}{2} - \rho_0 c_s \frac{u_{ij+1}^{k} - u_{ij}^{k}}{2}, \quad (24)$$

$$(\Sigma_{12}^{-})_{ij}^{k} = \frac{(\sigma_{12})_{ij}^{k} + (\sigma_{12})_{ij-1}^{k}}{2} - \rho_0 c_s \frac{u_{ij}^{k} - u_{ij-1}^{k}}{2}, \quad (25)$$

$$(\Sigma_{22}^{+})_{ij}^{k} = \frac{(\sigma_{22})_{ij}^{k} + (\sigma_{22})_{ij+1}^{k}}{2} - \rho_0 c_p \frac{v_{ij+1}^{k} - v_{ij}^{k}}{2}, \quad (26)$$

$$(\Sigma_{22}^{-})_{ij}^{k} = \frac{(\sigma_{22})_{ij}^{k} + (\sigma_{22})_{ij-1}^{k}}{2} - \rho_0 c_p \frac{v_{ij}^{k} - v_{ij-1}^{k}}{2}, \quad (27)$$

$$(\Sigma_{21}^{+})_{ij}^{k} = \frac{(\sigma_{21})_{ij}^{k} + (\sigma_{21})_{i+1j}^{k}}{2} - \rho_0 c_s \frac{v_{i+1j}^{k} - v_{ij}^{k}}{2} \quad (28)$$

$$(\Sigma_{21}^-)_{ij}^k = \frac{(\sigma_{21})_{ij}^k + (\sigma_{21})_{i-1j}^k}{2} - \rho_0 c_s \frac{v_{ij}^k - v_{i-1j}^k}{2}, \tag{29}$$

$$(U^+)_{ij}^k = \frac{u_{i+1j}^k + u_{ij}^k}{2} - \frac{(\sigma_{11})_{i+1j}^k - (\sigma_{11})_{ij}^k}{2\rho_0 c_p}, \tag{30}$$

$$(U^-)_{ij}^k = \frac{u_{ij}^k + u_{i-1j}^k}{2} - \frac{(\sigma_{11})_{ij}^k - (\sigma_{11})_{i-1j}^k}{2\rho_0 c_p}, \tag{31}$$

$$(V^+)_{ij}^k = \frac{v_{ij+1}^k + v_{ij}^k}{2} - \frac{(\sigma_{22})_{ij+1}^k - (\sigma_{22})_{ij}^k}{2\rho_0 c_p}, \tag{32}$$

$$(V^-)_{ij}^k = \frac{v_{ij}^k + v_{ij-1}^k}{2} - \frac{(\sigma_{22})_{ij}^k - (\sigma_{22})_{ij-1}^k}{2\rho_0 c_p}, \tag{33}$$

$$(U1^+)_{ij}^k = \frac{u_{ij+1}^k + u_{ij}^k}{2} - \frac{(\sigma_{12})_{ij+1}^k - (\sigma_{12})_{ij}^k}{2\rho_0 c_s}, \tag{34}$$

$$(U1^-)_{ij}^k = \frac{u_{ij}^k + u_{ij-1}^k}{2} - \frac{(\sigma_{12})_{ij}^k - (\sigma_{12})_{ij-1}^k}{2\rho_0 c_s}, \tag{35}$$

$$(V1^+)_{ij}^k = \frac{v_{i+1j}^k + v_{ij}^k}{2} - \frac{(\sigma_{12})_{i+1j}^k - (\sigma_{12})_{ij}^k}{2\rho_0 c_s}, \tag{36}$$

$$(V1^-)_{ij}^k = \frac{v_{ij}^k + v_{i-1j}^k}{2} - \frac{(\sigma_{12})_{ij}^k - (\sigma_{12})_{i-1j}^k}{2\rho_0 c_s}, \tag{37}$$

where $c_p = \sqrt{\dfrac{\lambda + 2\mu}{\rho_0}}$ and $c_s = \sqrt{\dfrac{\mu}{\rho_0}}$.

Now we have all the equations which are needed for calculations. If we determine constants a, b, c, and d as equal to the ratio of the time step to the size of the cell, we come to the well-known explicit finite difference scheme [13].

5. BOUNDARY CONDITIONS

For the sake of simplicity, we consider a rectangular area limited by the straight lines $x = x_0, x = x_N, y = y_0, y = y_K$. To calculate all the desired quantities by means of the system of equations (17)–(21), we need to determine $(\Sigma_{11}^-)_{0j}^k$, $(\Sigma_{12}^-)_{0j}^k$, $(V1^-)_{0j}^k$, and $(U^-)_{0j}^k$ at the left boundary $x = x_0$, the values of $(\Sigma_{11}^+)_{Nj}^k$, $(\Sigma_{12}^+)_{Nj}^k$, $(V1^+)_{Nj}^k$, and $(U^+)_{Nj}^k$ at the right boundary $x = x_N$, the values of $(\Sigma_{22}^-)_{i0}^k$,

$(\Sigma_{21}^-)_{i0}^k$, $(V^-)_{i0}^k$, and $(U1^-)_{i0}^k$ at the lower boundary $y = y_0$, and the values of $(\Sigma_{22}^+)_{iK}^k$, $(\Sigma_{21}^+)_{iK}^k$, $(V^+)_{iK}^k$, and $(U1^+)_{iK}^k$ at the upper boundary $y = y_K$.

We start with the left boundary. Two first conditions for the contact quantities express the conservation of the Riemann invariants along corresponding characteristic lines [13]

$$(U^-)_{0j}^k - \frac{(\Sigma_{11}^-)_{0j}^k}{\rho_0 c_p} = u_{0j}^k - \frac{(\sigma_{11})_{0j}^k}{\rho_0 c_p}, \tag{38}$$

$$(V1^-)_{0j}^k - \frac{(\Sigma_{12}^-)_{0j}^k}{\rho_0 c_s} = u_{0j}^k - \frac{(\sigma_{11})_{0j}^k}{\rho_0 c_s}. \tag{39}$$

If boundary conditions

$$\alpha_1 u(x_0, y, t) + \beta_1 \sigma_{11}(x_0, y, t) = f_1(y, t), \tag{40}$$

$$\alpha_2 u(x_0, y, t) + \beta_2 \sigma_{12}(x_0, y, t) = f_2(y, t) \tag{41}$$

are prescribed at this boundary, we can approximate them by relations

$$\alpha_1 (U^-)_{0j}^k + \beta_1 (\Sigma_{11}^-)_{0j}^k = f_{1j}, \tag{42}$$

$$\alpha_2 (V1^-)_{0j}^k + \beta_2 (\Sigma_{12}^-)_{0j}^k = f_{2j}. \tag{43}$$

Here α_i and β_i are constants, and $f_i(y, t)$ are certain prescribed functions. Now we obtain the possibility to calculate all the needed quantities at this boundary.

At the right boundary, other invariants are

$$(U^+)_{Nj}^k + \frac{(\Sigma_{11}^+)_{Nj}^k}{\rho_0 c_p} = u_{Nj}^k + \frac{(\sigma_{11})_{Nj}^k}{\rho_0 c_p}, \tag{44}$$

$$(V1^+)_{Nj}^k + \frac{(\Sigma_{12}^+)_{Nj}^k}{\rho_0 c_s} = u_{Nj}^k + \frac{(\sigma_{11})_{Nj}^k}{\rho_0 c_s}. \tag{45}$$

Boundary conditions at this boundary are approximated in the same way as above.

Description of the boundary conditions for upper and lower boundaries is analogous.

6. NUMERICAL RESULTS

As an example, a finite aperture radiation into rectangular two-dimensional specimen with stress-free boundaries in various situations is presented. The excitation is made by prescription of the non-zero normal component of the stress tensor within one third of the length of a boundary only at the first time step. The physical properties of the medium are: $\lambda = 8, \mu = 3, \rho_0 = 1$.

The results of calculations in terms of surface plots for normal stress are given in Figs. 1–3. One can see dispersion, reflection, and interaction of elastic waves.

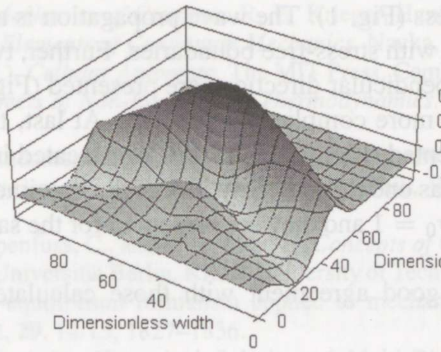

Fig. 1. Surface plot of the distribution of the normal stress in a rectangular specimen excited at the left boundary.

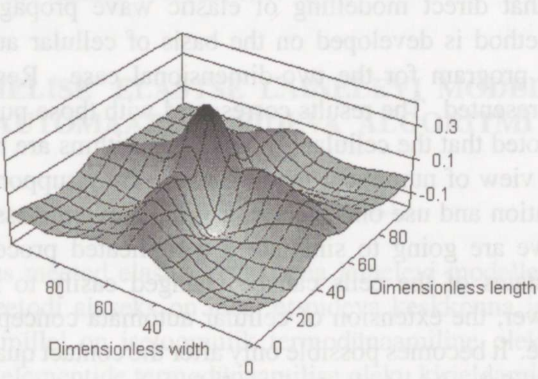

Fig. 2. Surface plot of the distribution of the normal stress in the case of perpendicularly propagating elastic waves.

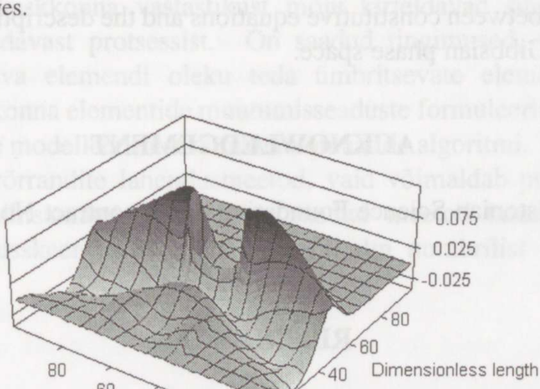

Fig. 3. Surface plot of the distribution of the normal stress in the case of interaction of the elastic wave with a crack in the middle of the specimen.

First, the propagation of one elastic wave is presented in terms of the corresponding normal stress (Fig. 1). The wave propagation is accompanied by its dispersion and interaction with stress-free boundaries. Further, two identical elastic waves propagating in perpendicular directions are presented (Fig. 2). Their mutual interaction forms slightly more complicated patterns. At last, the interaction of a wave with a crack is presented (Fig. 3). The crack was located in the middle of the specimen and its length was one third of the width of the specimen. All waves have the same initial intensity $\sigma_0 = 1$ and they are presented for the same time (after 150 time steps).

These results are in good agreement with those calculated by means of a different technique [14].

7. CONCLUSIONS

It is shown that direct modelling of elastic wave propagation in solids is possible. This method is developed on the basis of cellular automata technique and realized in a program for the two-dimensional case. Results of numerical experiments are presented. The results correspond with those published recently.

It should be noted that the cellular automata algorithms are not the best choice from the point of view of numerical mathematics. They support only the explicit method of calculation and use only a regular grid. Nevertheless, such algorithms are preferred if we are going to simulate a complicated process, because rules of updating the states of the cells can be changed easily to include necessary variations. However, the extension of cellular automata concept onto continuous media is not simple. It becomes possible only after the contact quantities are defined in the framework of the thermodynamics of discrete systems. The development of this theory is not finished yet and requires more efforts in understanding the correspondence between constitutive equations and the description of interaction of elements in the Gibbsian phase space.

ACKNOWLEDGEMENT

Support of Estonian Science Foundation under contract No. 3203 is gratefully acknowledged.

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KAHEMÕÕTMELISE ELASTSE LAINELEVI MODELLEERIMINE RAKUAUTOMAATIDE PIDEVA ALGORITMI ABIL

Arkadi BEREZOVSKI

On esitatud uus meetod elastse keskkonna lainelevi modelleerimise algoritmi koostamiseks. Meetodi aluseks on idee, et pideva keskkonna igat elementi saab vaadelda rakuna, millel on iseloomulik termodünaamiline olek. Selliste mittetasakaalus olevate elementide termodünaamilise oleku kirjeldamiseks on kasutatud diskreetsete termodünaamiliste süsteemide kontseptsiooni, mille raames viiakse sisse elemendi ja keskkonna vastastikust mõju kirjeldavad suurused. Nende valik sõltub vaadeldavast protsessist. On saadud tingimused, mis seovad iga mittetasakaalus oleva elemendi oleku teda ümbritsevate elementide olekuga. Selline pideva keskkonna elementide muutumisseaduste formuleerimine võimaldab kasutada protsesside modelleerimiseks rakuautomaatide algoritmi. Esitatud meetod ei ole diferentsiaalvõrrandite lahendusmeetod, vaid võimaldab protsesside otsest modelleerimist. Lihtsamatel juhtudel on saadud meetod taandatav üldtuntud numbrilistele arvutusskeemidele. On toodud kolm numbrilist näidet elastsete lainete levi kohta.