

ON RADIAL DEFLECTIONS IN CYLINDRICAL SHELLS UNDER LOCAL LOADING

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Abstract. This paper discusses the results of computing of the displacements of cylindrical shells under local loading. The loading, assumed to be perpendicular to the shell axis, is distributed along the line of intersection with the elliptic cylinder. The method of series expansion of solution was applied. The results published earlier and those obtained using the finite element method are compared.

Key words: elasticity, cylindrical shells.

1. STATEMENT OF THE PROBLEM

On a simply supported closed cylindrical shell, as shown in Fig. 1, a line load is applied. The load is equally distributed along the cross-section of the elliptic cylinder, whose axis intersects that of the cylindrical shell investigated. Both axes are assumed to be perpendicular. The load is applied along the intersection line of the two cylinders. The objective is to determine the radial displacements of the shell.

2. METHOD OF SOLUTION

Next, we shall briefly describe the method presented in [1]. The equilibrium equation for the cylindrical shell can be expressed in terms of radial deflections

$$L(w) = \frac{r^3}{D} [\Delta^2(q) - L_1(X) - L_2(Y)], \quad (1)$$

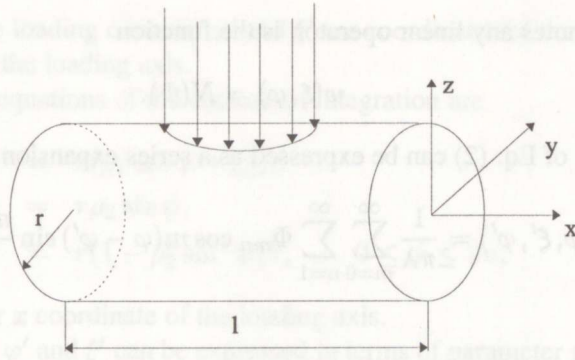


Fig. 1. Loading conditions.

where

$$\begin{aligned}
 L &\equiv \Delta^4 + 2\nu \frac{\partial^6}{\partial \xi^6} + 6 \frac{\partial^6}{\partial \xi^4 \partial \varphi^2} + 2(4 - \nu) \frac{\partial^6}{\partial \xi^2 \partial \varphi^4} + \\
 &+ 2 \frac{\partial^6}{\partial \varphi^6} + \frac{1 - \nu^2}{\epsilon} \frac{\partial^4}{\partial \xi^4} + 2(2 - \nu) \frac{\partial^4}{\partial \xi^2 \partial \varphi^2} + \frac{\partial^4}{\partial \varphi^4}; \\
 L_1 &\equiv 2\nu \frac{\partial^3}{\partial \xi^3} - \frac{\partial^3}{\partial \xi \partial \varphi^2} + \epsilon \left(\frac{\partial^5}{\partial \xi \partial \varphi^4} - \frac{\partial^5}{\partial \xi^5} \right); \\
 L_2 &\equiv (2 + \nu) \frac{\partial^3}{\partial \xi^2 \partial \varphi} + \frac{\partial^3}{\partial \varphi^3} - 2\epsilon \left(\frac{\partial^5}{\partial \xi^4 \partial \varphi} + \frac{\partial^5}{\partial \xi^2 \partial \varphi^3} \right).
 \end{aligned}$$

Let Φ be the 2π -periodical fundamental solution of the operator L , i.e. the solution of the operator equation

$$L(\Phi) = \delta_{2\pi}(\xi - \xi', \varphi - \varphi'), \quad (2)$$

where

$$\begin{aligned}
 \delta_{2\pi}(\xi - \xi', \varphi - \varphi') &= \sum_{m=-\infty}^{\infty} \delta(\xi - \xi', \varphi - \varphi' - 2\pi m) = \\
 &= \frac{1}{\pi \lambda} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (2 - \delta_{0m}) \cos m(\varphi - \varphi') \sin \frac{n\pi \xi'}{\lambda} \sin \frac{n\pi \xi}{\lambda}, \quad (3)
 \end{aligned}$$

$$\xi' \neq 0, \quad \xi' \neq \lambda.$$

Then the solution of the equation

$$L(w) = N(\delta_{2\pi}), \quad (4)$$

where N denotes any linear operator, is the function

$$w(\xi, \varphi) = N(\Phi). \quad (5)$$

The solution of Eq. (2) can be expressed as a series expansion

$$\Phi(\xi, \varphi, \xi', \varphi') = \frac{1}{\pi\lambda} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \Phi_{mn} \cos m(\varphi - \varphi') \sin \frac{n\pi\xi'}{\lambda} \sin \frac{n\pi\xi}{\lambda}, \quad (6)$$

where

$$\begin{aligned} \Phi_{mn} &= \frac{(2 - \delta_{0m})\lambda^8}{c_{mn}}, \\ c_{mn} &= a_{mn}^2(a_{mn} - \lambda^2)^2 + \frac{1 - \nu}{\epsilon}(\lambda\pi n)^2[(1 + \nu)(\lambda\pi n)^2 - 2\epsilon b_{mn}] - (\lambda\pi n)^4, \\ a_{mn} &= (\lambda m)^2 + (\pi n)^2, \quad b_{mn} = \lambda^4 m^2(m^2 - 1) - (\pi n)^4. \end{aligned}$$

To solve Eq. (1) for any particular case of loading, one can construct the linear operator N and then, according to Eq. (5), apply that operator to function Φ .

3. SOLUTION OF THE PROBLEM

According to the method described above, the first goal is to determine the linear operator N . Let p_0 be the intensity of the line-load. Thus, if the load is equally distributed along the ellipse, the canonical equation of which is $(x/a)^2 + (y/b)^2 = 1$, the total load P can be expressed as

$$P = p_0 r \ell_0, \quad (7)$$

where

$$\begin{aligned} \ell_0 &= \int_0^{2\pi} (\rho_1^2 \sin^2 t + \rho_2^2 \cos^2 t)^{\frac{1}{2}} dt, \\ \rho_1 &= a/r, \quad \rho_2 = b/r. \end{aligned}$$

The components of the line-load are

$$\begin{aligned} X &= 0, \\ Y(\xi, \varphi) &= \frac{p_0}{r^2} \int_L \cos^2(\varphi' - \varphi_0) \delta_{2\pi}(\xi - \xi', \varphi - \varphi') dl, \\ q(\xi, \varphi) &= \frac{p_0}{r^2} \int_L \sin(\varphi' - \varphi_0) \cos(\varphi' - \varphi_0) \delta_{2\pi}(\xi - \xi', \varphi - \varphi') dl, \end{aligned} \quad (8)$$

where L denotes the loading contour; φ' and ξ' are coordinates of the line element dl ; φ_0 is rotation of the loading axis.

The parametric equations of the contour of integration are

$$\begin{aligned} x &= r(\rho_1 \cos \psi + \xi_0), \\ y &= r\rho_2 \sin \psi, \\ z &= r(1 - \rho_2^2 \sin^2 \psi)^{\frac{1}{2}}, \quad 0 \leq \psi \leq 2\pi, \end{aligned} \quad (9)$$

where $r\xi_0$ stands for x coordinate of the loading axis.

The coordinates φ' and ξ' can be expressed in terms of parameter ψ as follows:

$$\begin{aligned} \varphi' &= \varphi_0 + \arcsin(\rho_2 \sin \psi), \\ \xi' &= \xi_0 + \rho_1 \cos \psi. \end{aligned} \quad (10)$$

Now it is possible to express the contour integrals (8) in terms of definite integrals

$$\begin{aligned} X &= 0, \\ Y(\xi, \varphi) &= \frac{p_0}{r^2} \int_0^{2\pi} f_1(\rho_2, \psi) f_2(\rho_1, \rho_2, \psi) \delta_{2\pi}(\xi - \xi', \varphi - \varphi') d\psi, \\ q(\xi, \varphi) &= \frac{p_0}{r^2} \int_0^{2\pi} f_3(\rho_2, \psi) f_2(\rho_1, \rho_2, \psi) \delta_{2\pi}(\xi - \xi', \varphi - \varphi') d\psi, \end{aligned} \quad (11)$$

where

$$\begin{aligned} f_1(\rho_2, \psi) &= \rho_2 \sin \psi, \\ f_2(\rho_1, \rho_2, \psi) &= (\rho_1^2 \sin^2 \psi + \rho_2^2 \cos^2 \psi - \rho_1^2 \rho_2^2 \sin^4 \psi)^{\frac{1}{2}}, \\ f_3(\rho_2, \psi) &= (1 - \rho_2^2 \sin^2 \psi)^{\frac{1}{2}}. \end{aligned} \quad (12)$$

The next step is to use the expressions for load components (11) in Eq. (1). The linear operator N for Eq. (4) can be written in the form

$$N \equiv \frac{p_0 r^2}{D} \int_0^{2\pi} [\Delta^2(\dots) f_3(\rho_2, \psi) f_2(\rho_1, \rho_2, \psi) - L_2(\dots) f_1(\rho_2, \psi) f_2(\rho_1, \rho_2, \psi)] d\psi. \quad (13)$$

To obtain residual deflections according to (5), one can apply that operator to function Φ . The result is

$$w(\xi, \varphi) = \frac{p_0 r^2 \lambda^3}{\pi D} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (w_{mn}^1 \cos m\varphi + w_{mn}^2 \sin m\varphi) \sin \frac{n\pi\xi}{\lambda}, \quad (14)$$

where

$$w_{mn}^1 = \frac{(2 - \delta_{0m})d_{mn}^1}{c_{mn}}, \quad w_{mn}^2 = \frac{(2 - \delta_{0m})d_{mn}^2}{c_{mn}},$$

$$d_{mn}^1 = \int_0^{2\pi} f_2(\rho_1, \rho_2, \psi) [f_3^{mn}(\rho_2, \psi) \cos m\varphi' + f_1^{mn}(\rho_2, \psi) \sin m\varphi'] \sin \frac{n\pi\xi'}{\lambda} d\psi,$$

$$d_{mn}^2 = \int_0^{2\pi} f_2(\rho_1, \rho_2, \psi) [-f_1^{mn}(\rho_2, \psi) \cos m\varphi' + f_3^{mn}(\rho_2, \psi) \sin m\varphi'] \sin \frac{n\pi\xi'}{\lambda} d\psi,$$

$$f_1^{mn}(\rho_2, \psi) = f_1(\rho_2, \psi) [2a_{mn}\epsilon\pi^2 m n^2 + \lambda^2 m (a_{mn} + (1 + \nu)\pi^2 n^2)],$$

$$f_3^{mn}(\rho_2, \psi) = f_3(\rho_2, \psi) a_{mn}^2.$$

4. NUMERICAL RESULTS

In the limit process $\rho_1 \rightarrow 0$ and $\rho_2 \rightarrow 0$, the loading described above tends to the concentrated load in the point (ξ_0, φ_0) . To compare the results with those in [1, 2], the calculations for $\varphi_0 = 0, \xi_0 = \lambda/2, \rho_1 = \rho_2 = 10^{-6}$ were carried out. Tables 1 and 2 present the quantities $w_0 = P^{-1} E r^2 w(\varphi_0, \xi_0)$ as well as the relative deviations $r = |w_1 - w_2|/w_1$ from the results in [1].

Another limit process occurs when only $\rho_1 \rightarrow 0$ may approximate the line load along the arc $\xi = \text{const}, \varphi_1 \leq \varphi \leq \varphi_2$. The results for $\rho_1 = 10^{-6}, \varphi = \varphi_0 = 0, \xi = \xi_0 = \lambda/2$ and the comparison with the results in [1] are shown in Tables 3–5.

In all calculations, the expansion series were calculated for the indices $m = 0, 1, \dots, 40$ and $n = 1, 2, \dots, 81$. The integrals were calculated using the Simpson-method with 41 nodal points.

Table 1
 $\gamma = 100$

λ	w_0	$r(\%)$
1	10897	2.49
3	19665	2.46
5	25952	2.44
7	31248	2.48
9	35806	2.46
11	40497	2.46
13	45457	2.39
15	50140	2.39
17	54059	2.45
19	57108	2.45

Table 2
 $\lambda = 5$

γ	w_0	$r(\%)$
20	747	2.48
25	1210	2.23
33	2221	4.63
50	5582	2.51
100	25952	2.44
200	120158	3.00

Table 3
 $\gamma = 100,$
 $\rho_2 = \sin(\pi/10)$

λ	w_0	$r(\%)$
1	3040	3.61
3	9466	3.10
5	15033	3.17
7	20000	3.22
9	24444	3.30
11	29105	3.36
13	34052	3.42
15	38745	3.47
17	42713	3.49
19	45857	3.52

Table 4

$\lambda = 5, \rho_2 = \sin(\pi/10)$

γ	w_0	$r(\%)$
20	594	4.96
25	931	3.42
33	1631	5.34
50	3793	3.31
100	15033	3.17
200	57309	3.04

Table 5

$\gamma = 100, \lambda = 5$

ρ_2	w_0	$r(\%)$
$\sin(\pi/4)$	4672	3.34
$\sin(\pi/6)$	8837	3.25
$\sin(\pi/8)$	12367	3.42
$\sin(\pi/10)$	15033	3.17
$\sin(\pi/12)$	17041	2.97
$\sin(\pi/14)$	18563	2.80
$\sin(\pi/16)$	19732	2.57

5. COMPARATIVE ANALYSIS

In addition, we used the FEM-system ANSYS 5.0A in our calculations. The results obtained as well as those presented in [1] and displayed in Tables 1–3, and 5 are shown on the graphs in Figs. 2–5. On these bases, we can conclude that the radial displacements in all three cases practically coincide.

Thus, the method of the series expansion is fully applicable if no FEM-programs are available. On the other hand, to develop more complex methods is not reasonable if a FEM-program can be used because it guarantees satisfactory accuracy of the solution.

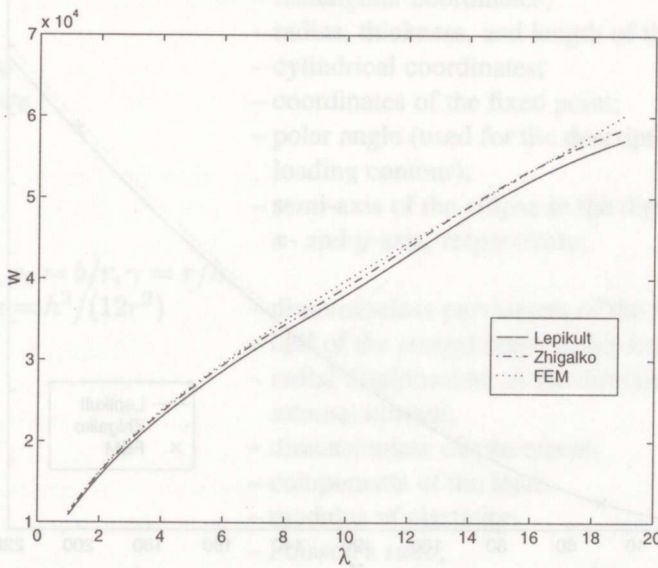


Fig. 2. Concentrated load, $\gamma = 100$.

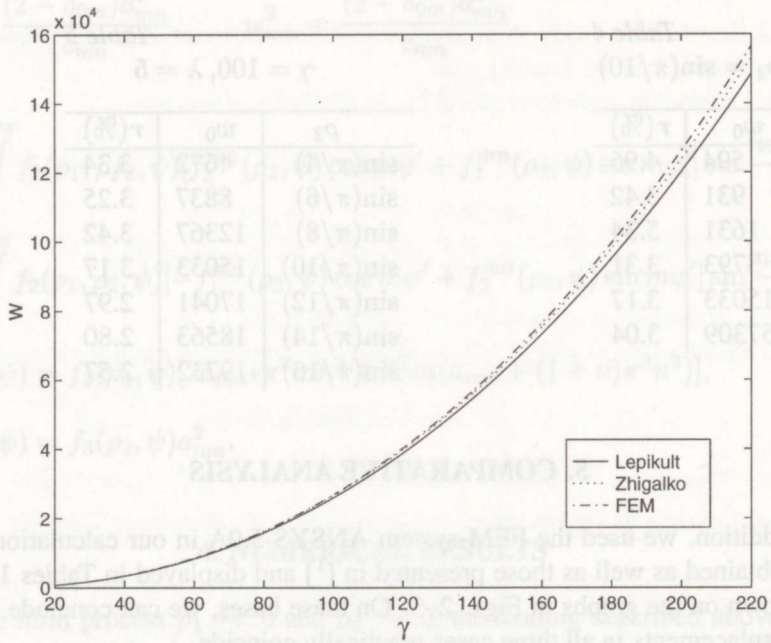


Fig. 3. Concentrated load, $\lambda = 5$.

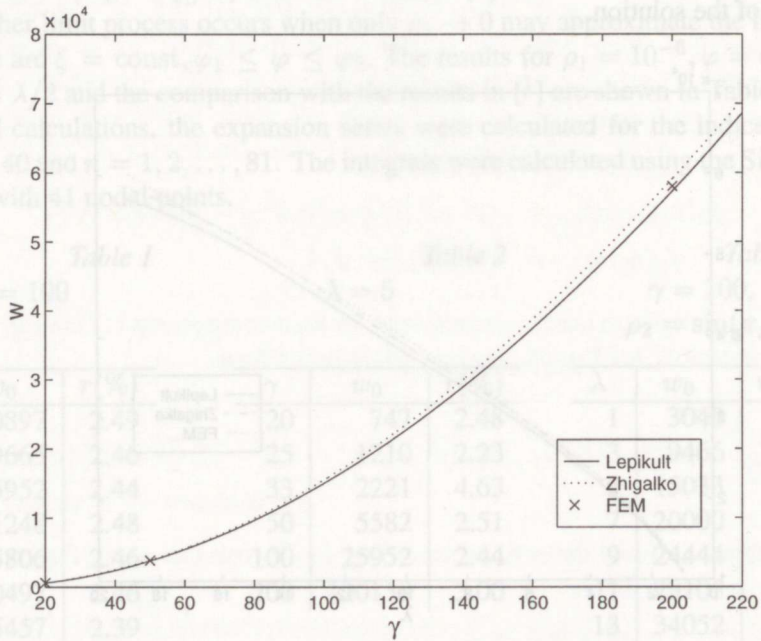


Fig. 4. Pressure along circular arc, $\lambda = 5$, $\beta = \pi/10$.

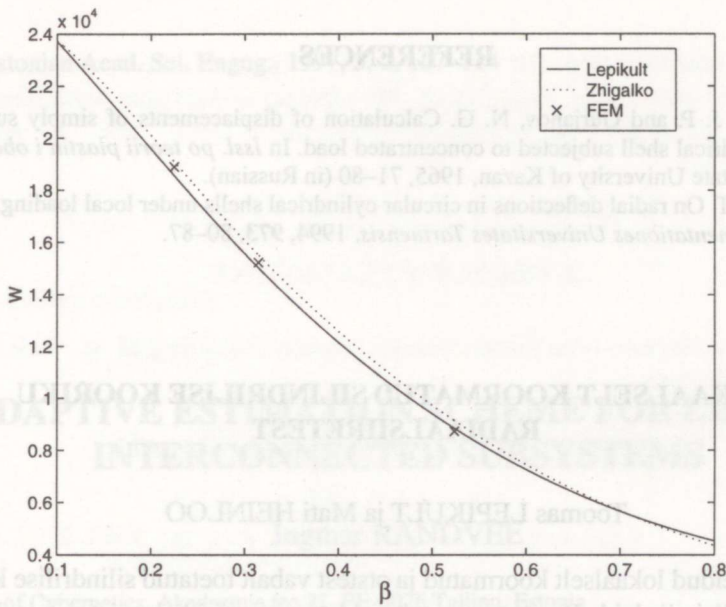


Fig. 5. Pressure along circular arc, $\lambda = 5$, $\gamma = 0.01$.

NOTATIONS

- | | |
|--|---|
| x, y, z | – rectangular coordinates; |
| r, h, l | – radius, thickness, and length of the shell; |
| $\xi = x/l, \varphi$ | – cylindrical coordinates; |
| $\xi', \varphi', \xi_0, \varphi_0$ | – coordinates of the fixed point; |
| ψ | – polar angle (used for the description of loading contour); |
| a, b | – semi-axis of the ellipse in the direction of the x - and y -axis, respectively; |
| $\rho_1 = a/r, \rho_2 = b/r, \gamma = r/h,$
$\lambda = l/r, \epsilon = h^2/(12r^2)$ | – dimensionless parameters of the shell; |
| β | – half of the central angle of the loaded arc; |
| w' | – radial displacement in the direction of internal normal; |
| $w = w'/r$ | – dimensionless displacement; |
| X, Y, q | – components of the load; |
| E | – modulus of elasticity; |
| ν | – Poisson's ratio; |
| $D = Eh^3/(12(1 - \nu^2))$ | – cylindrical stiffness of the shell; |
| δ_{0m} | – Kronecker's symbol; |
| $\Delta = \partial^2/\partial\xi^2 + \partial^2/\partial\varphi^2$ | – Laplace's operator. |

