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ISOSPECTRAL VIBRATION OF QUASI-UNIFORM RODS AND STRINGS

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Abstract. The isospectral problem of quasi-uniform rods is solved by the perturbation method. For the shape function, a series representation by the influence functions, which are the eigenfunctions of certain Sturm–Liouville problem, is given. Similar results for non-homogeneous strings and for the Sturm–Liouville problem are described.

Key words: isospectral problem, perturbation method, vibration of rods and strings.

1. INTRODUCTION

The longitudinal vibrations of a straight elastic rod with the variable cross-section $A(x)$ are governed by the equation

$$(Au')' + \lambda A u = 0 \tag{1.1}$$

with $\lambda = \rho\omega^2 / E$, where E, ρ and ω are the Young's modulus, density and frequency.

With appropriate redefinitions of A, u and λ, this is also an equation governing vibrations of a thin rod in torsion, or of an acoustic horn.

Equation (1.1) can be easily reduced to the Sturm–Liouville equation

$$y'' + (\lambda - q(x))y = 0. \tag{1.2}$$

As the cross-section function $A(x)$ is positive, we can write

$$A = a^2, \qquad y = au. \tag{1.3}$$

Then

$$(Au')' = (a^2u')' = 2aa'u' + a^2u'' = ay'' - a''y. \tag{1.4}$$

Now, if we take

$$q(x) = \frac{a''}{a} = \frac{1}{2}\frac{A''}{A} - \frac{1}{4}\frac{A'^2}{A^2}, \tag{1.5}$$

from equation (1.1), we obtain the Sturm–Liouville Eq. (1.2).

Two systems given by differential operators are said to be isospectral with respect to the given regularity conditions and boundary conditions if their eigenvalue spectra are identical.

From the inverse Sturm–Liouville eigenvalue problem of determination of the coefficient functions in the Sturm–Liouville equation, we know that, in general, two spectra corresponding to two different boundary conditions are required to determine these functions [1–4]. Thus, a single spectrum for a given boundary conditions does not uniquely determine the coefficient function, and the same eigenvalue spectrum is obtained from operators with different coefficient functions.

We obtain simple isospectral sets for uniform rods from the transformation (1.3) and (1.5). For a given $A(x)$ there is a unique $q(x)$, but for a given $q(x)$ there are many $A(x)$ [5, 6]. If A_0 is one $A(x)$ corresponding to a given $q(x)$, then the general solution is

$$A = A_0\left(1 + b\int_0^x \frac{d\xi}{A_0(\xi)}\right)^2, \quad A(0) = 1, \quad b = \text{const}, \tag{1.6}$$
$$y = A_0^{1/2}u_0 = A^{1/2}u.$$

A complete characteristic of the isospectral potential $q(x)$ for the Sturm–Liouville problem (1.2) with various sets of boundary conditions is discussed in [7–10]. This analysis is extended to Eq. (1.1) in [11, 12]. In [13] some families of rods which have a common spectrum are described. In [14] it is shown that for some non-uniform rods and beams, the equation of motion can be transformed into the equation of motion for a uniform rod or beam with the same eigenvalues.

In this paper, the isospectral problem of quasi-uniform rods is solved with the perturbation method [15]. The series representation of the shape function by influence functions is given. For the influence functions, the Sturm–Liouville problem is constructed. Similar results are also presented for the non-uniform string and for the Sturm–Liouville problem.

2. THE EIGENVALUE PROBLEMS FOR INFLUENCE FUNCTIONS

Let us consider the eigenvalue problem for a vibrating rod

$$\begin{aligned}&(Au')' + \lambda Au = 0 \qquad x \in (0,\pi),\\ &u(0) = 0, \quad u(\pi) = 0\end{aligned} \tag{2.1}$$

in the case of quasi-uniform cross-section

$$A(x) = 1 + \varepsilon A_1(x) + \varepsilon^2 A_2(x) + \ldots, \tag{2.2}$$

where ε is a small parameter. We will try to find a function $A_1(x)$ for which the eigenspectrum of problem (2.1) is the same as for the uniform cross-section $A(x)=1$.

As the single spectrum is not sufficient to determine the function $A(x)$ uniquely, simultaneously with the problem (2.1), we consider the same problem on the vibrating rod with different boundary conditions

$$(Av')'+\mu Av=0 \qquad x\in(0,\pi), \\ v(0)=0, \quad v'(\pi)=0. \tag{2.3}$$

Now suppose that the solution to Eqs. (2.1) and (2.2) – $u(x,\varepsilon)$ and $\lambda(\varepsilon)$ can be expanded in a power series in ε

$$u(x,\varepsilon)=u_0(x)+\varepsilon u_1(x)+\varepsilon^2 u_2(x)+\ldots, \\ \lambda(\varepsilon)=\lambda_0+\varepsilon\lambda_1+\varepsilon^2\lambda_2+\ldots. \tag{2.4}$$

We look for the solution which gives the same spectrum as a uniform cross-section. Therefore we take here $\lambda(\varepsilon)=\lambda_0$ and $\lambda_1=0,\ \lambda_2=0,\ldots$

Substituting (2.4) into (2.1) and comparing the terms of equal powers in ε, we obtain

$$u_0''+\lambda_0 u_0=0, \tag{2.5}$$

$$u_0=0, \quad u_0(\pi)=0; \tag{2.6}$$

$$u_1''+\lambda_0 u_1=-(A_1u_0')'-\lambda_0 A_1 u_0, \tag{2.7}$$

$$u_1(0)=0, \quad u_1(\pi)=0. \tag{2.8}$$

If we use Eq. (2.5), then Eq. (2.7) takes the form

$$u_1''+\lambda_0 u_1=-A_1'u_0'. \tag{2.9}$$

Equation (2.9) has a solution if the term on the right is orthogonal to u_0. Consequently,

$$\int_0^\pi A_1'u_0'u_0\,dx=0. \tag{2.10}$$

For the influence function $u_0'u_0$, let us introduce here the notation

$$w=u_0'u_0. \tag{2.11}$$

Then

$$w'=u_0'^2+u_0'' \tag{2.12}$$

or using Eq. (2.5)

$$w'=u_0'^2-\lambda_0 u_0^2. \tag{2.13}$$

Similarly,

$$w' = 2u_0''u_0' - 2\lambda_0 u_0' u_0 \tag{2.14}$$

or

$$w'' = -4\lambda_0 u_0' u_0. \tag{2.15}$$

From Eqs. (2.11) and (2.15) we can see that the influence function $w(x)$ satisfies the differential equation

$$w'' + 4\lambda_0 w = 0. \tag{2.16}$$

From Eqs. (2.6) and (2.11) it follows that the boundary conditions for Eq. (2.16) have the form

$$w(0) = 0, \quad w(\pi) = 0. \tag{2.17}$$

In the same way, for the eigenvalue problem (2.3), we assume that

$$\begin{aligned} v(x, \varepsilon) &= v_0(x) + \varepsilon v_1(x) + \varepsilon^2 v_2(x) + \ldots, \\ \mu(\varepsilon) &= \mu_0 + \varepsilon \mu_1 + \varepsilon^2 \mu_2 + \ldots \end{aligned} \tag{2.18}$$

and obtain the equations and boundary conditions

$$v_0'' + \mu_0 v_0 = 0, \tag{2.19}$$

$$v_0(0) = 0, \quad v_0'(\pi) = 0; \tag{2.20}$$

$$v_1'' + \mu_0 v_1 = -A_1' v_0' - \mu_1 v_0, \tag{2.21}$$

$$v_1(0) = 0, \quad v_1'(\pi) = 0. \tag{2.22}$$

If we take

$$\int_0^\pi v_0^2 dx = 1, \tag{2.23}$$

from (2.21), it follows that

$$\mu_1 = -\int_0^\pi A_1' v_0' v_0 dx. \tag{2.24}$$

In exactly the same way as in the case of Eq. (2.10), we can introduce the influence function

$$w = v_0' v_0 \tag{2.25}$$

and obtain for $w(x)$ a differential equation

$$w'' + 4\mu_0 w = 0 \tag{2.26}$$

and boundary conditions (2.17).

Now we modify our original definitions (2.11) and (2.25) and define function $w(x)$ as a solution of the eigenvalue problem

$$w'' + \nu w = 0, \tag{2.27}$$

$$w(0)=0,\ w(\pi)=0. \tag{2.28}$$

From our construction, it follows that $\{u_{0n}'u_{0n}, 4\lambda_{0n}\}_0^\infty$ and $\{v_{0n}'v_{0n}, 4\mu_{0n}\}_0^\infty$ are eigensolutions of this problem. Let us now show that the eigenvalue problem (2.27) and (2.28) admit no other eigensolution, i.e.,

$$\{w_n, \nu_n\}_0^\infty = \{u_{0n}'u_{0n}, 4\lambda_{0n}\}_0^\infty \cup \{v_{0n}'v_{0n}, 4\mu_{0n}\}_0^\infty. \tag{2.29}$$

Let us first note that if $w(x)$ is the solution of the eigenvalue problem (2.27) and (2.28), then the integral $\int_0^x w dx$ saves the sign. Therefore we can regard this integral as nonnegative and write

$$\int_0^x w dx = \varphi^2. \tag{2.30}$$

Substituting (2.30) in Eq. (2.27) and boundary conditions (2.28), we obtain

$$6\varphi'\left(\varphi''+\frac{1}{4}\nu\varphi\right)+2\varphi\left(\varphi'''+\frac{1}{4}\nu\varphi'\right)=0, \tag{2.31}$$

$$\varphi(0)\varphi'(0)=0,\quad \varphi(\pi)\varphi'(\pi)=0. \tag{2.32}$$

From (2.30) it follows that $\varphi(0)=0$, and from (2.32) that either $\varphi(\pi)=0$ or $\varphi'(\pi)=0$. Therefore by (2.31) we conclude either $\varphi = u_{0n}$ and $\nu/4=\lambda_{0n}$ or $\varphi = v_{0n}$ and $\nu/4=\mu_{0n}$. Hence, the influence functions $\{u_{0n}'u_{0n}\}_0^\infty$ together with the influence functions $\{v_{0n}'v_{0n}\}_0^\infty$ form a complete system. As a result, we can write

$$\begin{aligned} w_{2k} &= u_{0k}'u_{0k}, & \nu_{2k} &= 4\lambda_{0k}, \\ w_{2k-1} &= v_{0k}'v_{0k}, & \nu_{2k-1} &= 4\mu_{0k}. \end{aligned} \tag{2.33}$$

Now Eqs. (2.10) and (2.24) can be written in the form

$$\int_0^\pi A_1' w_{2k} dx = 0, \tag{2.34}$$

$$\int_0^\pi A_1' w_{2k-1} dx = -\mu_{1k} \quad (k=1,2,\ldots). \tag{2.35}$$

3. ISOSPECTRAL QUASI-UNIFORM RODS

Let us consider the following isospectral problem: from the identical eigenvalue spectra of the problem (2.1) deduce the perturbation of the shape function $A_1(x)$ for quasi-uniform rods (2.2). In view of the results of the previous section, we must find such a function $A_1(x)$ that the conditions (2.34) hold.

As the set of influence functions $\{w_n\}_0^\infty$ is complete, an expansion of function $A_1(x)$ in terms of influence functions w_n is feasible. Thus

$$A_1{}'(x) = \sum_{n=1}^{\infty} c_n w_n(x), \tag{3.1}$$

where

$$c_n = \frac{\int_0^\pi A_1{}' w_n dx}{\int_0^\pi w_n^2 dx}. \tag{3.2}$$

From Eqs. (2.34) and (2.35), it follows that

$$c_{2k} = 0, \qquad c_{2k-1} = -\frac{\mu_{1k}}{\int_0^\pi w_{2k-1}^2 dx}. \tag{3.3}$$

The eigenvalue problems (2.5) and (2.6) for u_0 and (2.19) and (2.20) for v_0 have solutions

$$u_{0k} = \sqrt{\frac{2}{\pi}} \sin kx, \qquad v_{0k} = \sqrt{\frac{2}{\pi}} \sin\left(\frac{2k-1}{2} x\right). \tag{3.4}$$

From (2.33) we now get that the influence function takes the form

$$w_n = \frac{n}{2\pi} \sin nx. \tag{3.5}$$

Therefore

$$A_1{}'(x) = -\sum_{k=1}^{\infty} \frac{4\mu_{1k}}{2k-1} \sin(2k-1)x \tag{3.6}$$

and

$$A_1(x) = \sum_{k=1}^{\infty} \frac{4\mu_{1k}}{(2k-1)^2} \cos(2k-1)x + A_0. \tag{3.7}$$

Here we can choose arbitrarily the constants μ_{1k} for the isospectral problem (2.1).

If it is assumed that the shape function $A(x)$ must be symmetric around the midpoint of the rod, i.e. $A(\pi - x) = A(x)$, a single spectrum $\{\lambda_{1n}\}_0^\infty$ determines the function $A_1(x)$ uniquely. Then from (3.7) we can see that

$$A_1(x) = A_0. \tag{3.8}$$

It is obvious that if we look for the solution of the isospectral problem (2.3) with different boundary conditions and take the shape function $A(x)$ as (2.2), in the same way we get

$$A_1(x) = -\sum_{k=1}^{\infty} \frac{4\lambda_{1k}}{(2k)^2} \cos 2kx + A_0. \tag{3.9}$$

4. ISOSPECTRAL QUASI-HOMOGENEOUS STRINGS

Let us now consider the isospectral problem for strings. A vibration of a string of density $\rho(\xi)$ taut by a unit tension is governed by the eigenvalue problem

$$\eta'' + \kappa\rho\eta = 0 \qquad \xi \in (0, \pi), \tag{4.1}$$

$$\eta(0) = 0, \qquad \eta(\pi) = 0. \tag{4.2}$$

Equation (4.1) can be transformed by the transformation

$$x(\xi) = \frac{1}{K}\int_0^{\xi} \sqrt{\rho(z)}dz, \qquad \eta(\xi(x)) = u(x),$$
$$K = \frac{1}{\pi}\int_0^{\pi} \sqrt{\rho(z)}dz, \tag{4.3}$$

where

$$A(x) = \sqrt{\rho(\xi(x))}, \qquad \lambda = K^2\kappa \tag{4.4}$$

into Eq. (1.1).

Suppose that the density of the string is given in the form

$$\rho(\xi) = 1 + \varepsilon\rho_1(\xi) + \varepsilon^2\rho_2(\xi) + \dots . \tag{4.5}$$

We will try to find a function $\rho_1(\xi)$ in (4.5) for which the eigenspectrum of the problem (4.1) and (4.2) is the same as for uniform density $\rho(\xi) = 1$. As in Section 2, we expand the eigenfunctions $\eta(\xi, \varepsilon)$ and eigenvalues $\kappa(\varepsilon)$ in a power series in ε as in (2.4) and take $\kappa_1 = 0$, $\kappa_2 = 0, \dots$ Now exactly in the same way as in Section 2 we introduce the influence function $w(\xi)$ and get for $\rho_1(\xi)$ expansion in series by this influence functions. But the result is achieved more easily using the results obtained for rods and the transformation formulas (4.3) and (4.4).

Substituting Eqs. (4.5) and (2.2) into (4.3) and (4.4), we obtain

$$x = \xi, \quad K = 1, \quad A_1(x) = \frac{1}{2}\rho_1(x), \quad \lambda = \kappa. \tag{4.6}$$

Therefore

$$\rho_1(\xi) = -\sum_{k=1}^{\infty} \frac{8\mu_{1k}}{(2k-1)^2} \cos(2k-1)\xi. \tag{4.7}$$

For the isospectral problem of a string with boundary conditions

$$\eta(0) = 0, \quad \eta'(\pi) = 0, \tag{4.8}$$

we get from (4.6) and (3.9)

$$\rho_1(\xi) = -\sum_{k=1}^{\infty} \frac{8\lambda_{1k}}{(2k)^2} \cos 2k\xi. \tag{4.9}$$

Finally, we note that from solutions of the isospectral problem for rods and a string, we can also get the solution of the isospectral problem Sturm–Liouville equation in this case of iso-uniform potential function. If we take

$$q(x) = \varepsilon\, q_1(x) + \varepsilon^2\, q_2(x) + \ldots, \tag{4.10}$$

then from (1.5) it follows that

$$q_1(x) = \frac{1}{2} A_1''. \tag{4.11}$$

Therefore by (3.7), we have for boundary conditions

$$y(0) = 0, \quad y(\pi) = 0 \tag{4.12}$$

that

$$q_1(x) = 2\sum_{k=1}^{\infty} \mu_{1k} \cos(2k-1)x \tag{4.13}$$

and for boundary conditions

$$y(0) = 0, \quad y'(\pi) = 0 \tag{4.14}$$

that

$$q_1(x) = 2\sum_{k=1}^{\infty} \lambda_{1k} \cos 2kx. \tag{4.15}$$

Here, μ_{1k} and λ_{1k} are arbitrary constants.

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KVAASIÜHTLASTE VARRASTE JA KEELTE ISOSPEKTRAALNE VÕNKUMINE

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Häirituste meetodi abil on tuletatud kvaasiühtlaste varraste võnkumiste isospektraalprobleemile lahend. Varda kujufunktsioon on esitatud mõjufunktsioonide reana, kusjuures mõjufunktsioonid on saadud kui teatava Sturmi–Liouville'i probleemi omafunktsioonid. Analoogilised tulemused on toodud ka keele võnkumise isospektraalprobleemi kvaasihomogeense tiheduse jaoks ja Sturmi–Liouville'i isospektraalprobleemi kvaasikonstantse koefitsientfunktsiooni jaoks.