

IMPACT GRINDING AND DISINTEGRATORS

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Abstract. Comminution resulting from the collision of material particles against some moving objects is called grinding by collision. Its mechanics as well as its modes and instrumentation differ substantially from other refining methods, such as pushing and shifting. This type of grinding, implemented in twin-rotored machines, is called disintegrating, and the corresponding grinders are disintegrators. This paper discusses some essential features of disintegrating, its mechanics and modes, as well as some design problems and applications.

Key words: disintegrator, grinding technology, direct, selective, separative grinding, dynamic balancing.

1. INTRODUCTION

One of the predominant technologies in mining, in the production of minerals, and in materials treatment is grinding. Disintegrator, a grinding machine for soft materials, has been known for over a century. In Estonia, this development emerged in the 1960s, when Johannes Hint [1] used a disintegrator for producing silicate bricks from sand and lime. Later, a technology was created for a new building material – silicalcite. Silicalcite was produced from quartz and quicklime by the disintegrator technology. This technology was concrete-free.

A systematic research of grinding by collision was conducted by the Hans Rumpf's school [2]. The process of breaking single particles to pieces by collision was described by Priemer [3], who also proved Rumpf's model of breaking. Reiners [4] investigated grinding mixtures of different materials in a wide range of collision velocities (up to 950 m/s) and showed high selectivity of grinding at certain velocities. Lenkewitz [5] invented a mill with one-collision grinding action. The collision velocity was increased up to 400 m/s. Drögemeier and Leschonski [6] treated

ultrafine grinding with separation of highly pure limestone in a two-stage collision mill.

The investigations in [2-6] were mainly experimental, whereas in [1] a quasistatic case was derived on the basis of the Hertz static model. Ibidem, the corresponding formulae for calculating collision forces and contact time for a quartz sand particle colliding with a plane grinding surface were presented.

Besides the traditional areas, today grinding is applied to a variety of new tasks. Recycling has become a vital component of the life-cycle of any industrial product, and grinding has appeared to be one of the most important methods in any destructive activity, in particular, for selective approaches.

Grinding machines affect their destructive function usually by pushing, shifting or colliding the particles of the material. In addition, refining has to meet other requirements like homogeneity of the output or an exact distribution of the particle size in the final production (granulometry). This requires the comminution machine to operate also as a mixer.

Among the comminution machines are the traditional grinding devices, where a particle remains between the two grinding bodies and is broken by pushing (Fig. 1a) or shifting (Fig. 1b). The maximum generated stresses σ that occur in the particle are locally equal to σ_1 or more than the strength of the material $[\sigma]$. Ball- and hand-mills, querns, vibro mills, jaw crushers, and mortars are example of these grinders.

$$\sigma_1 \geq [\sigma]$$

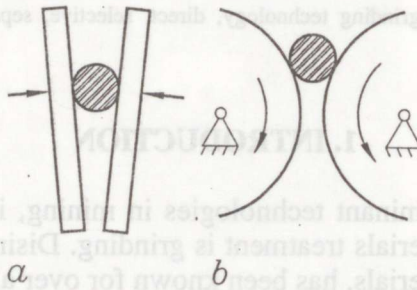


Fig. 1. Pushing (a) and shifting (b).

2. DISINTEGRATING

In addition, in some mills the particles are broken by colliding against the grinding elements of the rotating rotor. This principle is called disintegrating. Although known for a long time, it has not been adequately elaborated. At Tallinn Technical University, different theoretical aspects of collision milling have been studied, and the corresponding mills have been designed and developed. They are called DS disintegrators, and are operated not only for grinding minerals and metals but also for a wide variety of other materials, like metal chips, diamonds, gold content sand.

Disintegrators have been produced also by some other Estonian companies.

A disintegrator is a grinding mill made up from two rotors rotating in the opposite directions (Fig. 2). These rotors are equipped with one or more concentric rings, each locating a row of grinding bodies which are effective as targets for the colliding material and as accelerators for the next collision. In a disintegrator, the grinding is carried out by collision, i.e. by an unrestricted impact at a certain velocity. The values of velocities used in the disintegrator range from 30 to 200 mps. The impact of the particle against the grinding body causes an intensive stress wave of compression σ_1 to spread from the collision area (Fig. 3a). Since all materials work well on pressure, the particle remains intact during the propagation of the compression wave until it reaches the opposite side of the particle. After reflecting from the free surface of the particle, this wave propagates to the other direction as a tensile wave of the same intensity, and behind this wave with some delay the particle falls into pieces (Fig. 3b). The value of the stresses exceeds the strength of the material $[\sigma]$ about ten times.

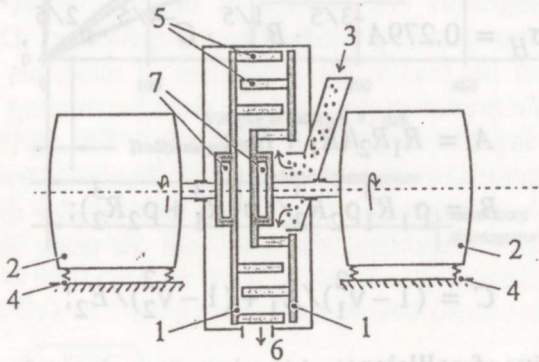


Fig. 2. Disintegrator. 1 – rotors, 2 – electric drives, 3 – material supply, 4 – elastic beds, 5 – grinding elements, 6 – output, and 7 – adaptive controller.

Figure 4 shows the change of the stress in quartz sand particles during collision. The velocity of collision is $v = 1$ mm/sec. The particles are assumed to be spherical and the stress is calculated by the formula $\sigma = \rho v^2$, where ρ is the density of the material. The stress is shown in the figure as a function of the distance from the collision point. The stress is zero at the collision point and increases to a maximum value of σ_1 at the opposite side of the particle. The stress then decreases to zero at the free surface of the particle. The stress is shown as a function of the distance from the collision point. The stress is zero at the collision point and increases to a maximum value of σ_1 at the opposite side of the particle. The stress then decreases to zero at the free surface of the particle.

Then the stress wave reflects from the free surface of the particle and propagates to the other direction as a tensile wave of the same intensity, $\sigma_2 \sim -\sigma_1$. Behind this wave with some delay the particle falls into pieces. The value of the stresses exceeds the strength of the material $[\sigma]$ about ten times.

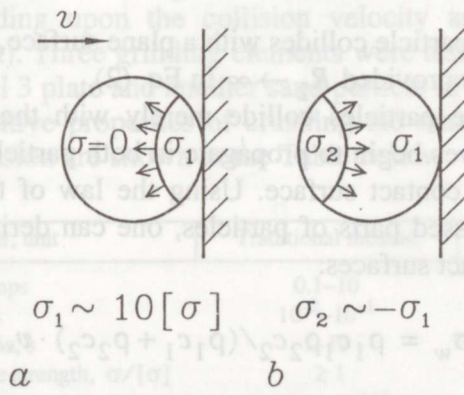


Fig. 3. Collision. Compression wave (a), tensile wave (b).

After particle break-up, in each piece, the stress waves undergo multiple reflection and refraction before they transform into plastic deformation and heat. This intensive transformation influences the crystal grids of materials, and their defects start to concentrate on the surfaces of the new small particles. As a result, the surface activity as well as the specific surface area increases and activates the material both mechanically and chemically. It has also been observed that the described destruction process produces the distribution of particles with a triple modal granulometry.

In the case of a clean rigid target, the contact time, i.e. the time the particle is in contact with the target, is very short, about 10^{-6} s, and with the grinding body covered by some treated material it is 10^{-4} s [7].

The dependence of stresses on the collision velocity can be calculated by two extreme models: either according to the quasistatic Hertz model applied to a spheric particle or according to the wave model meant for the particle that has a plane side and is colliding with this plane side.

Using the quasistatic Hertz model [1] one can derive formulae for calculating stresses σ_H of collision for the general case where both colliding particles are spheres:

$$\sigma_H = 0.279A^{-3/5} \cdot B^{1/5} \cdot C^{-4/5} \cdot v^{2/5}, \quad (1)$$

where

$$A = R_1 R_2 / (R_1 + R_2);$$

$$B = \rho_1 R_1^3 \rho_2 R_2^3 / (\rho_1 R_1^3 + \rho_2 R_2^3);$$

$$C = (1 - \nu_1^2) / E_1 + (1 - \nu_2^2) / E_2; \quad (2)$$

v – velocity of collision;

R_1, R_2 – radiuses of the colliding bodies;

ρ_1, ρ_2 – densities of the colliding body materials;

E_1, E_2 – Young modules;

ν_1, ν_2 – Poisson coefficients.

In case the spheric particle collides with a plane surface, the stresses could be calculated by (1) provided $R_2 \rightarrow \infty$ in Eq. (2).

Suppose that the particles collide merely with their plane surfaces. Then the stress waves begin to propagate in both particles to the opposite direction from the contact surface. Using the law of the momentum of motion for the stressed parts of particles, one can derive the formula of stresses in the contact surfaces:

$$\sigma_w = \rho_1 c_1 \rho_2 c_2 / (\rho_1 c_1 + \rho_2 c_2) \cdot v, \quad (3)$$

where c_1 and c_2 are velocities of the elastic waves,

$$c_i = (E_i/\rho_i)^{1/2}, i = 1, 2. \quad (4)$$

These stresses σ_w are called wave model stresses.

Since real particles differ from ideal spheres, and they do not collide with precisely plane surfaces, the Hertz model (1) and wave model (3) should be treated as boundary cases, and the actual real stresses are between these limit values (Fig. 4).

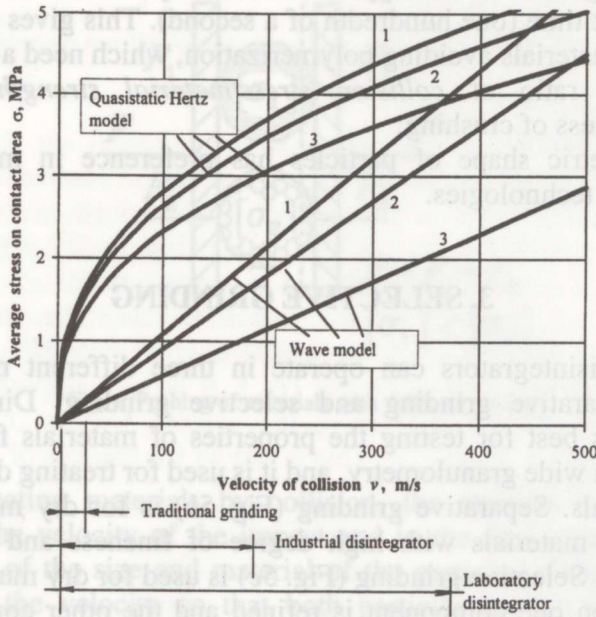


Fig. 4. Dependence of stress in quartz sand particle size of $d = 1$ mm on impact velocity. Target material: 1 - hard metal WC6 plate, 2 - steel 3 plate, 3 - quartz sand $d = 1$ mm particle.

Figure 4 shows the change of the stress in a quartz sand particle $d = 1$ mm depending upon the collision velocity and upon the target (grinding element). Three grinding elements were tested: a metal-ceramic plate WC6, a steel 3 plate and another sand particle of the same size [8].

Some comparative properties of crushing the materials by traditional methods and collision are shown in the Table below.

Parameter, unit	Traditional method	Collision
Velocity of loading, mps	0.1-10	30-200
Duration of loading, s	10^{-2} - 10^{-1}	10^{-6} - 10^{-5}
Duration in active zone, s	1-10	10^{-2}
Ratio of stresses to the strength, $\sigma/[\sigma]$	≥ 1	> 10
Character of stresses	compr. + shift	tension + shift
Shape of the particle	elongated or plane	isometric

According to this table and the formulae, one can draw the following conclusions:

- the average collision force of a particle with the mass of about 10–12 g and the velocity of loading 150 mps, reaches 400–600 kN (40–60 tons). The corresponding stresses caused by this force are at least one order of magnitude higher than required for the destruction of the particle. Part of this energy is accumulated in the material and activates it considerably;
- in disintegration technology, a material stays in the active zone for a very short time (one hundredth of a second). This gives preference to treating materials avoiding polymerization, which need a few seconds;
- the high ratio of *collision stress/material strength* shows the effectiveness of crushing;
- the isometric shape of particles has preference in many material treatment technologies.

3. SELECTIVE GRINDING

The DS disintegrators can operate in three different modes: direct grinding, separative grinding and selective grinding. Direct grinding (Fig. 5a) suits best for testing the properties of materials for producing materials with wide granulometry, and it is used for treating dry, damp and liquid materials. Separative grinding (Fig. 5b) is for dry materials only, and it yields materials with high degree of fineness and with narrow granulometry. Selective grinding (Fig. 5c) is used for dry multicomponent materials when one component is refined and the other components are left as intact as possible.

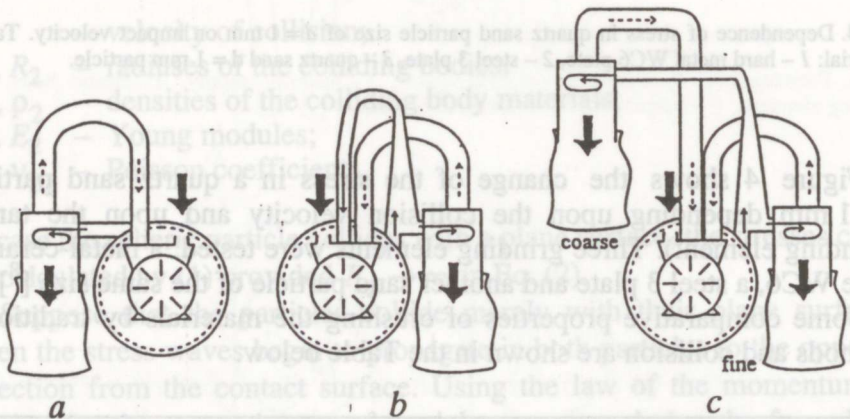


Fig. 5. Different modes of grinding. a – direct grinding, b – separative grinding, and c – selective grinding.

Selectivity is an essential characteristic of the traditional grinding methods, but it is particularly applicable to material treatment by collision.

Suppose a frail particle $[\sigma_1]$ and a hard particle $[\sigma_2]$ be simultaneously in an active grinding zone (Fig. 6). After the grinding bodies close in both particles, they apply some pressing force F and the stresses in the particles start growing until they exceed the strength of both materials. Despite $[\sigma_1] < [\sigma_2]$, both particles are broken, but the force of grinding these two pieces is distributed between them unevenly, for instance, 99.9% for the hard and 0.01% for the frail one.

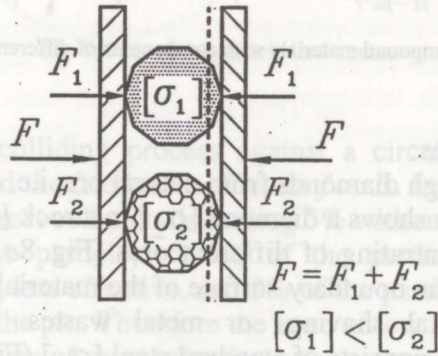


Fig. 6. Pushing of materials with different strength.

When treating materials by collision, the stresses in the particles depend on the velocity of the impact and in the first approximation are independent of the size and material of the particles. Consequently, one can control the velocity so that both particles remain unbroken (low velocity), or the hard particle remains intact and the frail particle breaks up (medium velocity), or both particles are broken (high velocity) (Fig. 7).

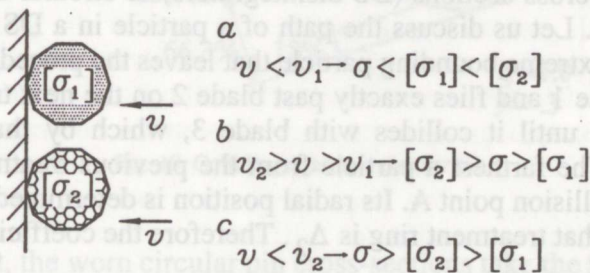


Fig. 7. Stresses induced by collision. *a* – both remain intact, *b* – the first breaks, the second remains intact, *c* – both break.

This phenomenon can be effective in many applications, e.g. when treating a material where a compound particle consists of different components, materials with different strength. In these cases, it is usually important to crush the compound particle so that the harder component remains intact whereas the more fragile parts are broken.

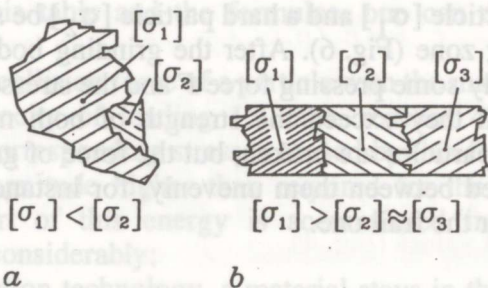


Fig. 8. Compound materials with components of different strength.

Some applications:

- selection of rough diamonds from natural ore, i.e. concentration of the yield. Figure 8a shows a diamond $[\sigma_2]$ in a rock $[\sigma_1]$;
- selective concentrating of different ores (Fig. 8a, b); while breaking appears along the boundary surface of the materials;
- cutting of metal shavings or metal wastes. The metal cutting instrumentation consists of standard steel $[\sigma_3]$ (Fig. 8b), alloyed steel $[\sigma_2]$ and some precious metals $[\sigma_1]$. It is very important to select these materials while treating the wastes without changing the properties of each material. This is one of the sources of raw material for powder metallurgy.

4. IMPACT GRINDING MECHANICS

The grinding elements of the double-rotored impact mills have usually parallelogram cross-sections (DS disintegrators) or circular cross-sections (Alpine-Mills). Let us discuss the path of a particle in a DS disintegrator (Fig. 9). The extreme bounding particle that leaves the preceding treatment ring from blade 1 and flies exactly past blade 2 on the next treatment ring and continues until it collides with blade 3, which by that time takes position 3'. The farthest a particle from the previous treatment ring can reach is the collision point A. Its radial position is determined by Δ_p . The radial size of that treatment ring is Δ_2 . Therefore the coefficient

$$\xi = \Delta_p / \Delta_2$$

shows which part of the grinding element's working surface is actually used for colliding in this particular mode. For example, if $\xi = .9$, then 90% of the blade surface is working and 10% serves as wear reserve without hazard to change the properties of the product. In case $\xi = 1.1$, 10% of the material bypasses this treatment ring without colliding. In some conditions where materials with extra wide granulometry are necessary, this mode is required [9].

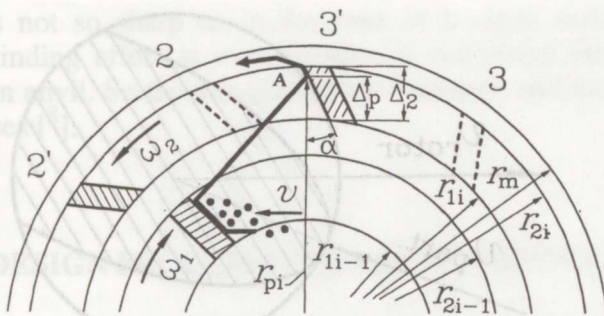


Fig. 9. Path of the particle in the rotor.

In Fig. 10, the colliding process against a circular pin is exposed, provided $\xi = 1$ and the friction coefficient $f = .3$. In this case, 28.7% of particles perform direct collision (including 5% without turning and 23.7% with turning of the particle). 66.3% of the particles perform slided collision, thus being less effective, and 3% of the particles, not exposed in Fig. 10, only touch the pin. Therefore the zone of low wear is 28.7% of the radial size, and the size of most intensive wear zone is 18%.

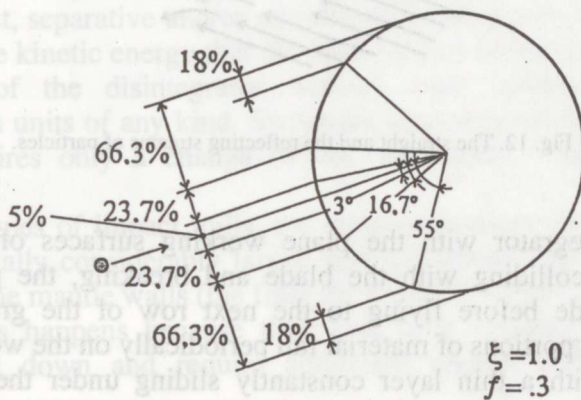


Fig. 10. Colliding against a circular pin.

As a result, the worn circular pin cross-sections take the form shown in Fig. 11 [1]. The mills with circular pins work mainly with slided collisions, i.e. with lower percentage of frontal collision than mills with plane grinding bodies. The stream of the material falls against the pin under a small angle α , and the reflecting stream disturbs the colliding effectiveness of the next stream (Fig. 12). So the particles are partially broken and partially worn, and therefore after such a treatment the product contains fine dusty particles and some amount of almost rough material.

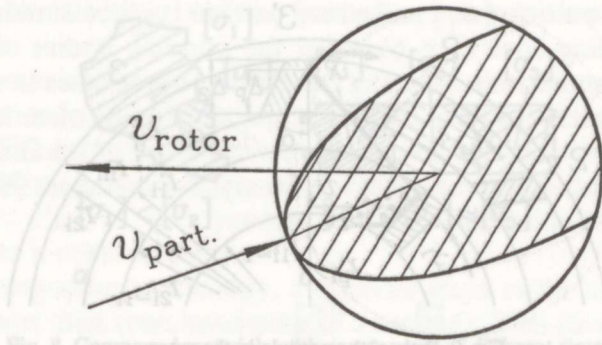


Fig. 11. Worn circular pin.

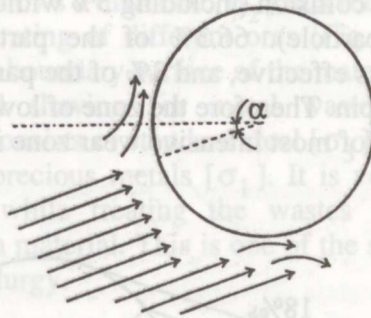


Fig. 12. The straight and the reflecting streams of particles.

In a disintegrator with the plane working surfaces of the grinding bodies, after colliding with the blade and breaking, the particle slides along the blade before flying to the next row of the grinding bodies (Fig. 9). Thus, portions of material fall periodically on the working surface covering it with a thin layer constantly sliding under the influence of centrifugal forces. If the time interval between the consecutive colliding portions is longer (e.g. $5 \cdot 10^{-3}$ s) than the time the portion takes to slide off the blade (e.g. $8 \cdot 10^{-4}$ s), then each portion will be treated individually. That means a portion falls onto the clean surface of the blade with the maximum impact. That is the case where the maximum activation and grinding effect is reached, but the wear of the surface reaches also the highest value. If, however, the next portion comes before the previous one has not completely left the working field, it partly covers the tail of it and starts to form a nonstationary moving layer. In practice, a stable layer appears after 12–16 portions, or in $8 \cdot 10^{-2}$ s that, in fact, is the stabilizing time of a disintegrator operation.

The presence of the described layer influences the treatment of the material considerably. Now the new particles impact against the layer of the ground material, they crush themselves as well as particles of the layer.

The impact is not so sharp as in the case of a clean surface, but the cumulating grinding effect is even greater. It resembles the effect of a hammer and an anvil. So the grinding effect increases and the wear of the blades decreases [8].

5. SOME DESIGN PECULIARITIES OF A DISINTEGRATOR

The ground material ejected from the rotor holds a significant kinetic energy [7, 10] that can be used for further material transportation. This is deliberately taken into account in disintegrator design: the distance between the rotor and the mantle is small, as compared to other types of impact grinders. It depends on the material and on the function of the particular disintegrator. Figure 9 shows the small difference between r_m and r_{2i} . So the ejected material falls on the inner side of the mantle at a small angle forming a moving layer. The new particles squeezing into the layer accelerate and press it forward. So the material moves with a certain velocity and kinetic energy together with some amount of air supplied by the rotor as a fan (Fig. 5). In DS disintegrators, this kinetic energy is used to transport material into the bunker for direct grinding (Fig. 5a) or into the classifier for separative or selective grinding (Fig. 5b, 5c). So one can compile direct, separative and/or selective-separative grinding modes as a reward for the kinetic energy that the material and air mixture flow has on the output of the disintegrator without using additional fans and transportation units of any kind. Switching from one mode of grinding to another requires only a change of the classifier. This is a simple operation.

In other types of impact mills, the distance between the rotor and the mantle is usually considerably larger, and that avoids the sticking of the particles on the mantle walls (the finer the particles the more adhesive they are). But this happens because they lose their velocity over the long distance, fall down and require additional energy to be transported onwards.

As already mentioned, the grinding process is characterized by considerable impact forces. In the case of rigid supports of the machine, these forces would have been transmitted to the bearings and supports directly inducing complicated problems with the life cycles of the bearings. That is why the disintegrator is supported by an elastic bed (Fig. 2), which reduces the dynamic forces in bearings by 4–5 orders of magnitude. This special design enables one to use standard electric drives with ordinary bearings, and one can fix the rotor directly on the motor's shaft. This simplifies the construction, decreases the dimensions, weight, and increases the performance stability of the mill.

Instead of a special foundation, a DS disintegrator can be designed, based on one elastic support point on the ground (Fig. 13). That forces the machine turn round a vertical axis and allows it to turn into positions, optimal for operating, serving and repairing.



Fig. 13. Disintegrator DSL-115 in the separative grinding system, which has one support point on the ground.

While rotating, the dynamic balance of the rotors is permanently disturbed by the ground material pieces sticking at random on the surface of the rotors as well as because of wear. These imbalancing forces are random variables in time and space, and their compensation is indispensable. An almost inertialess adaptive controller has been designed for dynamic balancing of the rotors operating with overcritical velocity (Fig. 2) [11]. The controller has a sensitivity threshold (lower balancing boundary) of imbalance being sufficient for normal operation of the mill and an upper balancing boundary that can be changed by means of a robust balancing unit [12].

6. APPLICATIONS

Along with the application of different disintegrator technologies and grinding different materials, a wide-scale experience has been gained. Each particular case is complicated and has its own peculiarities. For this reason, some examples are only very briefly described.

Technical diamonds are evaluated according to the number of crashes – the lower the number, the higher is the value of each particle. This number of defects can be reduced by selective disintegration whereby the nondefective diamonds do not break, but the others fall into smaller pieces without defects. For example, in a Russian depository, the fraction with particles $-4 > x > +2$ mm contained originally 59% diamonds with considerable defects, but after treatment in a disintegrator this percentage decreased to 3.8, obviously increasing the quality of output, although a finer fraction yielded.

In a goldfield, sand was processed by the wet flotation method and it was soon discovered that a significant number of gold particles, less than 400 microns, fell into the gangue: they could not be extracted. Another set of particles, sized $150-200 \times 10$ microns stuck to sand particles and did not precipitate. After appropriate disintegration, the gold slices, detached from the sand particles which broke into pieces, and the slices, being crushed by the strokes of the grinding elements, sank. This saved more than 90% of small particles formerly lost in the gangue and increased the production markedly.

In composite metal and alloy treatment, one of the goals is the selective and separative grinding, without disturbing the metallurgical properties. That means detachment of the components without heating them up, which can best be accomplished by disintegrating (the Table (very short time in the active zone), Fig. 5 (selection and separation of different fractions of particles), and Fig. 7 (distinguished destruction according to material strength)). This process is used to recycle metal wastes for powder metallurgy.

The grinding of the mineral material in a disintegrator activates the material. As an example, the clinker ground in a disintegrator gives the concrete about 15–18% stronger as compared to the clinker ground in a ball- or another mill.

In addition, our experiences cover selective crushing of slags and slimes together with the subsequent recovery of valuable components; grinding of leather wastes in the manufacturing of new artificial leather from natural fibre; disinfective destruction of parasite eggs in the fecal water active mud after biological treatment.

7. CONCLUSIONS

- As compared to other grinders, in the disintegrator, a material is left in the active zone for a very short time. This avoids heating up the

particles, causing, for instance, change of material properties and polymerization.

- Disintegration provides direct, separative and selective grinding, rendering wide predetermined granulometry of the output material. Separative disintegration gives high fineness of the output; and selective treatment is effective for multicomponent materials.
- The overall effectiveness of the process depends on the design of the grinding elements, the rotors, the mantle and the classifiers as well as on the special supporting elastic bed, reducing the dynamic forces significantly.
- The disintegrators described above are dynamically balanced by inertialess adaptive controllers, stabilizing vibrations of any kind.
- The remarkable phenomenon of activating the ground material has been utilized in several applications but not yet explained physically explicitly. This needs further fundamental research.

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LÖÖKJAHVATUSE PROBLEEMID JA DESINTEGRAATORID

Boris TAMM, Aleksei TŪMANOK

Materjali purustamist tema osakeste pörkumisel liikuvate objektide vastu nimetatakse löökjahvatuseks. See meetod erineb oluliselt nii oma mehaanika kui ka seadmete poolest teistest tuntud peenendusmeetoditest, mis põhinevad näiteks nihke- või tõukejõudude mõjul. Löökjahvatuse rakendamine kaksikrootorigega masinates on desintegreerimine ning vastavad jahvatid on desintegraatorid. Artiklis on käsitletud desintegreerimise põhinähtusi, selle mehaanikat ja erinevaid jahvatusviise, samuti mõningaid desintegraatori konstruktsiooni küsimusi. Lisatud on näited rakedustest.

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Abstract. The isospectral problem of quasi-uniform rods is solved by the perturbation method. For the shape function, a series representation by the Legendre functions, which are the eigenfunctions of certain Sturm-Liouville problem, is given. Similar results for non-homogeneous strings and for the Sturm-Liouville problem are described.

$$y'' + \lambda y = 0$$

Key words: isospectral problem, perturbation method, vibration of rods and strings. A complete characterization of the isospectral problem for the Sturm-Liouville problem (1.2) with various boundary conditions is discussed in [1]. In [2, 10] some families of rods which have a common spectrum are described. The longitudinal vibrations of a straight elastic rod with the variable cross-section $A(x)$ are governed by the equation of motion of a uniform rod or beam with the same eigenvalues.

In this paper, the isospectral problem for a non-uniform rod is solved with the perturbation method. The shape function is represented by the Legendre functions, which are the eigenfunctions of certain Sturm-Liouville problem. With appropriate redefinitions of $A(x)$ and λ , this is also an equation governing vibrations of a thin rod in torsion, or of an acoustic horn. Equation (1.1) can be easily reduced to the Sturm-Liouville equation

$$(2.1) \quad y'' + \lambda y = 0$$

2. THE EIGENVALUE PROBLEMS FOR INFLUENCE

As the cross-section function $A(x)$ is positive, we can write

(2.1) Let us consider the eigenvalue problem for a vibrating rod. Then

$$(2.2) \quad (Ay')' + \lambda y = 0, \quad y(0) = y(l) = 0$$

(2.3) Now, if we take

$$(2.4) \quad p(x) = \frac{1}{A(x)}, \quad q(x) = \frac{1}{A(x)}$$

$$(2.5) \quad A(x) = 1 + \lambda_1 x + \lambda_2 x^2 + \dots$$

from equation (1.1), we obtain the Sturm-Liouville Eq. (1.2).