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APPLICATION OF DIRECT INTEGRATION IN THE CASE OF EXTERNAL AND INTERNAL DAMPING

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Abstract. To simplify the dynamic analysis of structures supporting a moving load, the mass of the moving load or that of the supporting structures is neglected. It is well-known that dynamic stress values are influenced by external and internal damping. Their combined effects are only of importance when the dynamic system has a constant mass matrix. This paper presents an algorithm for the analysis of additional dynamic displacements of structures, whereby both the effects of the moving mass and those of internal friction must be considered. The algorithm and the numerical method were tested on examples. The factors mentioned showed important effects which justify their consideration in the analysis of real structures.

Key words: moving mass, internal friction, direct integration.

1. INTRODUCTION

In the dynamic analysis of structures, the determination of stresses in a structure due to a moving load is an important problem. It is well-known that the dynamic stress values are influenced both by external and internal damping. In [1] a suggestion is made to consider their combined effect, but only in the case of free vibration and in excitation by the harmonic forces.

An adequate numerical method for the analysis of structures with several degrees of freedom, permanent mass matrix under external damping is described in [2]. The effects of the moving mass are analysed in [3].

This paper presents a method of analysis for structures with several degrees of freedom exposed to external and internal damping. The developed algorithm and the numerical method were tested on examples. The above-mentioned factors showed important effects, which justify their consideration in the analysis of real structures.

2. APPLICATION OF DIRECT INTEGRATION

2.1. Consideration of a moving force under external damping

The second-order linear differential equation $\mathbf{M\ddot{u}} + \mathbf{C\dot{u}} + \mathbf{Ku} = \mathbf{r}$, describing the displacement of structures, expresses the dynamic equilibrium at any time in the considered time range.

Forces of inertia are expressed by $\mathbf{M\ddot{u}} = \mathbf{f}_I(t)$, damping forces by $\mathbf{C\dot{u}} = \mathbf{f}_D(t)$, stiffness forces by $\mathbf{Ku} = \mathbf{f}_E(t)$, while $\mathbf{r}(t)$ is the vector of external forces. (Matrices will be of order n.) The dynamic analysis is intended to solve the matrix differential equation under initial conditions $\dot{\mathbf{u}}_0, \mathbf{u}_0$ and $\ddot{\mathbf{u}}_0$ at a moment t_0, and once the displacements are determined, to find the dynamic stresses.

It is advisable to solve the initial value problem by the Wilson θ-method. Wilson assumes a linearly varying acceleration between the moments t and $t + \theta\,\Delta t$. (For $\theta = 1.4$, the procedure is definitely convergent.)

In this case,

$$\ddot{\mathbf{u}}_{t+\tau} = \ddot{\mathbf{u}}_t + \frac{\tau}{\theta\Delta t}\left(\ddot{\mathbf{u}}_{t+\theta\Delta t} - \ddot{\mathbf{u}}_t\right), \tag{1}$$

$$\dot{\mathbf{u}}_{t+\tau} = \dot{\mathbf{u}}_t + \ddot{\mathbf{u}}_t\tau + \frac{1}{2}\frac{\tau^2}{\theta\Delta t}\left(\ddot{\mathbf{u}}_{t+\theta\Delta t} - \ddot{\mathbf{u}}_t\right), \tag{2}$$

$$\mathbf{u}_{t+\tau} = \mathbf{u}_t + \dot{\mathbf{u}}_t\tau + \frac{1}{2}\ddot{\mathbf{u}}_t\tau^2 + \frac{1}{6}\frac{\tau^3}{\theta\Delta t}\left(\ddot{\mathbf{u}}_{t+\theta\Delta t} - \ddot{\mathbf{u}}_t\right). \tag{3}$$

Hence,

$$\ddot{\mathbf{u}}_{t+\theta\Delta t} = \frac{6}{(\theta\Delta t)^2}\left(\mathbf{u}_{t+\theta\Delta t} - \mathbf{u}_t\right) - \frac{6}{\theta\Delta t}\dot{\mathbf{u}}_t - 2\ddot{\mathbf{u}}_t, \tag{4}$$

$$\dot{\mathbf{u}}_{t+\theta\Delta t} = \frac{3}{\theta\Delta t}\left(\mathbf{u}_{t+\theta\Delta t} - \mathbf{u}_t\right) - 2\dot{\mathbf{u}}_t - \frac{\theta\Delta t}{2}\ddot{\mathbf{u}}_t. \tag{5}$$

Assuming $\mathbf{r}(t)$ to vary linearly during this period,

$$\mathbf{r}_{t+\theta\Delta t} = \mathbf{r}_t + \theta\left(\mathbf{r}_{t+\Delta t} - \mathbf{r}_t\right).$$

Displacements at time $t + \theta\,\Delta t$ result from

$$\left(\mathbf{K}+\frac{6}{(\theta\Delta t)^2}\mathbf{M}+\frac{3}{\theta\Delta t}\mathbf{C}\right)\mathbf{u}_{t+\theta\Delta t}=\mathbf{r}_{t+\theta\Delta t}+$$
$$+\mathbf{M}\left(\frac{6}{(\theta\Delta t)^2}\mathbf{u}_t+\frac{6}{\theta\Delta t}\dot{\mathbf{u}}_t+2\ddot{\mathbf{u}}_t\right)+$$
$$+\mathbf{C}\left(\frac{3}{\theta\Delta t}\mathbf{u}_t+2\dot{\mathbf{u}}_t+\frac{\theta\Delta t}{2}\ddot{\mathbf{u}}_t\right). \tag{6}$$

2.2. Consideration of a proportional internal damping

In the relationships above, the content of the matrix $\mathbf{C}$ has not been discussed. For an external damping, the matrix can be assembled if the individual damping elements related to the structure are known. For a damping due to frequency-independent internal friction, the matrix of an equivalent external damping – for different damping parameters of individual structural units – may be assumed if the complex stiffness matrix $\mathbf{K}_u + i\mathbf{K}_v$ is known in the form

$$\mathbf{C}=\mathbf{M}\mathbf{V}\langle\frac{1}{\omega_{ru}}\rangle\mathbf{V}^*\mathbf{K}_v\,, \tag{7}$$

using eigenvectors normalized to $\mathbf{M}$ of the eigenvalue problem

$$\mathbf{K}_u\mathbf{v}_r=\omega_{ru}^2\mathbf{M}\mathbf{v}_r\,.$$

Now, in the matrix differential equation of vibration, $\mathbf{K}$ will be replaced by $\mathbf{K}_u$ [1].

For the structural units with the same damping parameters (proportional damping), the equivalent damping matrix is

$$\mathbf{C}=\nu\mathbf{M}\mathbf{V}\langle\frac{1}{\omega_{ru}}\rangle\mathbf{V}^*\mathbf{K} \quad \text{and} \quad \mathbf{K}_u=u\mathbf{K}\,, \tag{8}$$

where

$$\nu=\frac{4\gamma}{4+\gamma^2}\,;\quad u=\frac{4-\gamma^2}{4+\gamma^2}\,;\quad \omega_{ru}=\frac{\omega_r}{\sqrt{1+\frac{\gamma^2}{4}}}\,;\quad \gamma=\frac{\vartheta}{\pi}\,.$$

Here, ϑ is the logarithmic decrement of damping, ω_r may be obtained from the r-th eigenvalue of the eigenvalue problem $\mathbf{K}\mathbf{v}=\omega^2\mathbf{M}\mathbf{v}$ for the undamped case, while $\mathbf{V}$ is a matrix containing eigenvectors normalized for $\mathbf{M}$. Obviously, in the case of internal damping, the direct integration problem has to be preceded by the solution of an eigenvalue problem. All these arguments must be considered in selecting the dynamic problem solution method and using the modal analysis.

3. APPLICATION OF MODAL ANALYSIS

3.1. Consideration without damping

The differential equation $\mathbf{M}\ddot{\mathbf{u}}+\mathbf{K}\mathbf{u}=\mathbf{r}$ must be solved. We are looking for the solution in the form $\mathbf{u}=\mathbf{V}\mathbf{x}$, where the eigenvalues and eigenvectors normalized for $\mathbf{M}$ ($\mathbf{V}^*\mathbf{M}\mathbf{V}=\mathbf{E}$) of the eigenvalue problem $\mathbf{K}\mathbf{v}=\omega^2\mathbf{M}\mathbf{v}$ are known. (In fact, initial conditions for $\mathbf{x}$ are $\mathbf{x}_0=\mathbf{V}^*\mathbf{M}\mathbf{u}_0$, $\dot{\mathbf{x}}_0=\mathbf{V}^*\mathbf{M}\dot{\mathbf{u}}_0$.)

After substitution and multiplying from the left by the transposed matrix $\mathbf{V}^*$

$$\mathbf{V}^*\mathbf{M}\mathbf{V}\ddot{\mathbf{x}}+\mathbf{V}^*\mathbf{K}\mathbf{V}\mathbf{x}=\mathbf{q}\ , \tag{9}$$

where

$$\mathbf{q}=\mathbf{V}^*\mathbf{r}=\mathbf{f}. \tag{10}$$

Due to orthogonality, theoretically, n single-unknown equations may be considered. It is known that in solving real technical problems, in a solution based on the eigenvectors, it is sufficient to involve a certain number ($m<n$) of eigenvectors computable by convenient procedures (e.g. subspace iteration) even for extended systems.

Equation r

$$\ddot{x}_r+\omega_r^2 x_r=q_r\,. \tag{11}$$

Accordingly,

$$\left(\omega_r^2+\frac{6}{(\theta\Delta t)^2}\right)x_{r_{t+\theta\Delta t}}=q_{r_{t+\theta\Delta t}}+$$

$$+\frac{6}{(\theta\Delta t)^2}x_{r_t}+\frac{6}{\theta\Delta t}\dot{x}_{r_t}+2\ddot{x}_{r_t}\ , \tag{12}$$

where

$$q_{r_{t+\theta\Delta t}}=f_{r_{t+\theta\Delta t}}\ ,$$

$$f_{r_{t+\theta\Delta t}}=f_{r_t}+\theta\left(f_{r_{t+\Delta t}}-f_{r_t}\right). \tag{13}$$

3.2. The case of proportional internal damping

For proportional internal damping, differential equation of motion is

$$\mathbf{M}\ddot{\mathbf{u}} + \left(\nu \mathbf{M}\mathbf{V} \left\langle \frac{1}{\omega_{ru}} \right\rangle \mathbf{V}^* \mathbf{K} \right) \dot{\mathbf{u}} + u\mathbf{K}\mathbf{u} = \mathbf{r}. \tag{14}$$

Using of eigenvalues and eigenvectors normalized for $\mathbf{M}$ of the eigenvalue problem $\mathbf{Kv} = \omega^2 \mathbf{Mv}$, solution may be sought for in the form $\mathbf{u} = \mathbf{Vx}$. After substitution and multiplying by transposed matrix $\mathbf{V}^*$ from the left

$$\mathbf{V}^*\mathbf{M}\mathbf{V}\ddot{\mathbf{x}} + \nu \mathbf{V}^*\mathbf{M}\mathbf{V} \left\langle \frac{1}{\omega_{ru}} \right\rangle \mathbf{V}^*\mathbf{K}\mathbf{V}\dot{\mathbf{x}} + u\mathbf{V}^*\mathbf{K}\mathbf{V}\mathbf{x} = \mathbf{q}, \tag{15}$$

where

$$\mathbf{q} = \mathbf{V}^*\mathbf{r} = \mathbf{f}. \tag{16}$$

Because of orthogonality, n single-unknown equations may be considered.

Equation r

$$\ddot{x}_r + \gamma\omega_{ru}\dot{x}_r + \omega_{ru}^2 x_r = q_r. \tag{17}$$

Accordingly,

$$\left(\omega_{ru}^2 + \frac{6}{(\theta\Delta t)^2} + \frac{3}{\theta\Delta t}\gamma\omega_{ru} \right) x_{r_{t+\theta\Delta t}} = q_{r_{t+\theta\Delta t}} + \frac{6}{(\theta\Delta t)^2} x_{r_t} + \frac{6}{\theta\Delta t}\dot{x}_{r_t} + 2\ddot{x}_{r_t} + \gamma\omega_{ru}\left(\frac{3}{\theta\Delta t} x_{r_t} + 2\dot{x}_{r_t} + \frac{\theta\Delta t}{2}\ddot{x}_{r_t} \right). \tag{18}$$

3.3. Computation for other than proportional internal damping or for composite internal and external damping

In this general case, the damping matrix cannot be diagonalized by means of eigenvectors for the undamped solution. So, inasmuch as diagonalization is to be made in the left-hand side of the matrix equation, the damping forces $\mathbf{C}\dot{\mathbf{u}} = \mathbf{f}_D(t)$ obtained by means of the damping matrix, involving the effects of external damping, have to appear in the right-hand side of the matrix equation. The dynamic equation may be written in the form

$$\mathbf{V}^*\mathbf{M}\mathbf{V}\ddot{\mathbf{x}}+\mathbf{V}^*\mathbf{K}\mathbf{V}\mathbf{x}=\mathbf{q},$$

$$\mathbf{q}=\mathbf{V}^*\mathbf{r}-\mathbf{V}^*\mathbf{C}\mathbf{V}\dot{\mathbf{x}}=\mathbf{f}-\mathbf{H}\dot{\mathbf{x}}. \qquad (19)$$

Solution may be obtained from Eq. (12) with

$$q_{r_{t+\theta\Delta t}}=f_{r_{t+\theta\Delta t}}-\mathbf{h}_r{}^*\dot{\mathbf{x}}_{t+\theta\Delta t}. \qquad (20)$$

Vectors $\dot{\mathbf{x}}_{t+\theta\Delta t}$ depend on vector $\mathbf{x}_{t+\theta\Delta t}$ of elements $x_{r_{t+\theta\Delta t}}$ in Eq. (20), requiring an iteration procedure.

The problem may also be solved without iteration. The unknowns belonging to the subspace may be obtained from an equation system of order m:

$$\left[\breve{\mathbf{D}}+\frac{3}{\theta\Delta t}\mathbf{H}\right]\mathbf{x}_{t+\theta\Delta t}=\mathbf{f}_{t+\theta\Delta t}+\mathbf{E}\left(\frac{6}{(\theta\Delta t)^2}\mathbf{x}_t+\frac{6}{\theta\Delta t}\dot{\mathbf{x}}_t+2\ddot{\mathbf{x}}_t\right)+$$

$$+(\mathbf{G}+\mathbf{H})\left(\frac{3}{\theta\Delta t}\mathbf{x}_t+2\dot{\mathbf{x}}_t+\frac{\theta\Delta t}{2}\ddot{\mathbf{x}}_t\right),$$

where $\breve{\mathbf{D}}$ and $\mathbf{G}$ are diagonal matrices, elements r of them $\omega_r^2+\frac{6}{(\theta\Delta t)^2}+\frac{3}{\theta\Delta t}\gamma\omega_{ru}$ and $\gamma\omega_{ru}$.

4. NUMERICAL RESULTS

4.1. The examined structure

In the numerical results, computations refer to a realistic structure. For a 30 m bridge spanning, cross-section type **a** simulates a bridge with reinforced concrete, and type **b** a bridge with steel structure, respectively (Fig. 1 and Table 1).

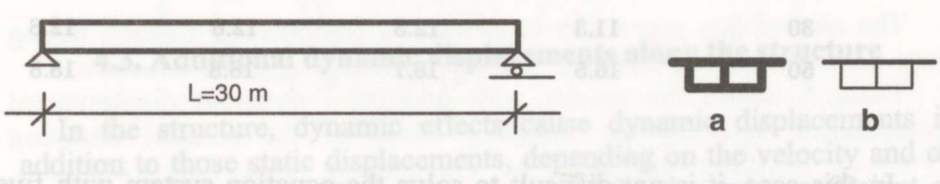

Fig.1. Arrangement of the examined structure.

Table 1

Mechanical characteristics of material and geometrical characteristics of structure

Characteristic	Type **a**	Type **b**
Cross-section area	3.12 m^2	0.4 m^2
Moment of inertia of cross-section	2.13 m^4	0.35 m^4
Elastic constant of the material	$2 \cdot 10^7$ kN/m^2	$2 \cdot 10^8$ kN/m^2
Poisson's ratio	0.166	0.3
Weight per unit volume	25 kN/m^3	150 kN/m^3*

* including the accessory weight of the bridge deck pavement.

The load moving on the bridge equals 800 kN. In different cases, the moving load velocities were assumed in the range of 0 to 50 m/s. Internal damping had a factor γ of 0.1.

4.2. The applied numerical method

The dynamic problem was solved by modal analysis. A numerical experiment was made to determine the number of eigenvectors needed to achieve the required accuracy. Displacement of the structure mid-point was tested by taking an ever increasing number of eigenvectors into consideration. Table 2 shows the percentages of additional displacements due to dynamic effect in the structure of type **a** for different velocities. Apparently, considering five eigenvectors yields additional dynamic displacements with an adequate accuracy.

Table 2

Additional displacements due to dynamic effect

Velocity, m/s	Number of eigenvectors			
	1	3	5	7
10	2.4	3.9	4.1	4.3
20	4.6	6.0	6.2	6.3
30	11.3	12.3	12.6	12.8
50	16.5	18.7	18.8	18.8

In this case, it is not difficult to solve the equation system with five unknowns in every time step that may be avoided by using the iterational procedure within a given time step. The time interval in the problem was assumed as the shortest vibration time belonging to eigenvectors in the solution, of a value $\Delta t = T_P/10$, as recommended in literature.

This procedure proved to be convenient even for three eigenvectors. However, considering only the first, the so-called fundamental vibration, and computing the time interval as $\Delta t = T_P/10$ (mainly for velocity < 20 m/s), applying a more accurate computation, the displacement is much less than that for the given vibration pattern. Furthermore, the interval $\Delta t = T_P/100$ in this case yields an adequate result.

However, this remark is only theoretical, namely, applying at least five eigenvectors, an interval still less than this critical one was obtained. Adequacy of the iteration procedure depends on the number of eigenvectors considered in the analysis, on the size and velocity of the moving mass. Using only the first eigenvector, corresponding to the fundamental vibration, the procedure is convergent even for a moving load of 800 kN.

As it was shown, to achieve the desired accuracy, in the given problem, it is advisable to have five eigenvectors. In the iteration process, the number of iterations in a given step depends on the load velocity. For a lower velocity, this number is lower. The statements above are illustrated in Table 3.

Table 3

Number of iterations required in a given time step of the iteration process

No. of eigenvalue	Force, kN											
	100			300			500			800		
	Velocity, m/s											
	10	20	50	10	20	50	10	20	50	10	20	50
1	3	4	4	6	6	8	9	10	11	24	28	36
3	5	6	7	12	13	16	53	42	76	–	–	–
5	7	8	9	32	34	34	–	–	–	–	–	–
7	11	11	13	–	–	–	–	–	–	–	–	–

4.3. Additional dynamic displacements along the structure

In the structure, dynamic effects cause dynamic displacements in addition to those static displacements, depending on the velocity and on the structural rigidity. There is a system of the dynamic coefficient as illustrated in Table 4.

Table 4

Percentages of additional dynamic displacements

Velocity, m/s	Place of displacement, L									
	0.1		0.2		0.3		0.4		0.5	
	Type of structure									
	a	**b**	**a**	**b**	**a**	**b**	**a**	**b**	**a**	**b**
5	0.1	–	1.0	–	1.3	0.2	2.1	1.3	2.4	2.0
10	2.1	–	3.0	0.5	3.8	0.8	4.5	1.7	4.1	2.2
20	4.0	1.0	4.8	1.8	5.7	3.0	6.5	4.1	6.2	4.2
30	9.1	5.0	9.9	5.9	10.9	6.6	12.1	6.4	12.6	5.2
50	24.0	11.7	24.3	14.4	23.3	14.4	21.3	13.1	18.8	10.8
Statical displ., mm	3.29	2.02	6.27	3.85	8.62	5.30	10.1	6.23	10.7	6.55

This table shows the percentages of additional dynamic displacements of different structural cross-sections for both types of structures at various velocities for a load of 800 kN .

Table 5

Effect of the moving load mass and of the internal damping

Velocity, m/s	Type **a**		Type **b**	
	internal damping			
	$\gamma = 0$	$\gamma = 0.1$	$\gamma = 0$	$\gamma = 0.1$
5	2.4	5.6	2.0	4.1
10	4.2	7.4	2.2	3.4
20	6.2	11.1	4.2	8.4
30	12.9	17.8	5.2	10.0
50	18.8	22.5	10.8	16.3

The aim of this paper was to develop a computation method taking into consideration the effect of the internal damping of the structure.

Table 5 shows the results with additional dynamic displacement percentages, ignoring the damping effect of the internal damping ($\gamma = 0.1$), omitting the mass of the moving load ($M_1 = 800$ kN). Examinations showed that the omission of the internal damping results in a significant overestimation of the dynamic effect.

4.4. Displacement diagrams

Displacements of three different points of the structure (type **b**) due to a moving load at the velocity of 30 m/s are shown in Fig. 2.

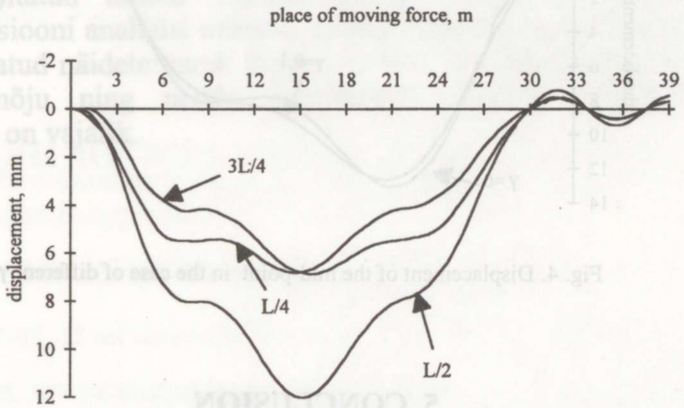

Fig. 2. Displacement at points L/4, L/2, 3L/4 of the examined structure.

Displacements at the mid-point of the structure for various velocities are illustrated in Fig. 3, and the effect of internal damping is shown in Fig. 4.

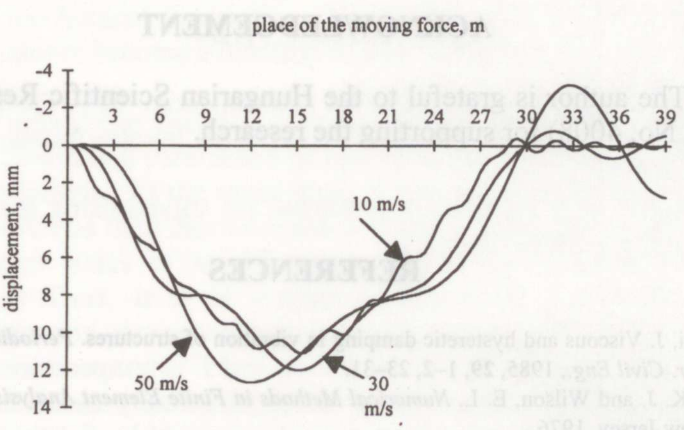

Fig. 3. Displacement of the mid-point at different velocities.

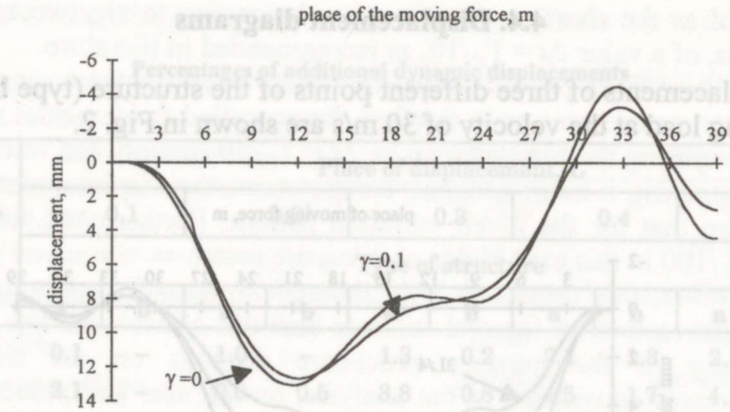

Fig. 4. Displacement of the mid-point in the case of different γ.

5. CONCLUSION

An algorithm was developed for computing additional dynamic displacements of structures, if the effects of the moving mass and internal friction are to be taken into consideration. The algorithm and the numerical method have been tested on realistic problems. It may be stated that the above-mentioned factors have an important effect, therefore they must be considered in the analysis of real structures.

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OTSESE INTEGREERIMISE RAKENDAMINE VÄLIS- JA SISESUMBUVUSE KORRAL

József GYÖRGYI

On esitatud mitme vabadusastmega välis- ja sisesumbuvusega konstruktsiooni analüüsi meetod. Leitud algoritmi ja numbrilist meetodit on katsetatud näidete varal. Ilmnes, et sise- ja välissumbuvus avaldavad olulist mõju ning nende arvestamine reaalsete konstruktsioonide uurimisel on vajalik.