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A THEORY OF CONSTITUTIVE EQUATIONS BASED ON WAVEDYNAMICS

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Abstract. We gave the constitutive assumptions by supposing that the constitutive equation has a differential equation form by using the property of the propagation of an acceleration wave. In this way, generally there is a possibility to apply the experimental results of an acceleration wave caused by a simple tension. Surprisingly, the expressions of the small deformation formally appear even in the case of finite deformations.

Key words: compatibility conditions, acceleration wave, Poisson bracket.

1. INTRODUCTION

When we investigate the motions of continua, we should know their constitutive equations. Different authors suggest various equations but these can describe only some given motion of material.

This article presents a theory, which gives conditions for variables and functions to form a real constitutive equation and not a law of a phenomenon. When the constitutive equations are $f_\alpha = 0$, $\alpha = 1,..., 6$, the theory is based on the following constitutive assumptions [1]:

a) f_α is a function of stress, strain and their first partial derivatives and of coordinates x_i and time t.

b) In spite of any physically possible initial and boundary conditions, acceleration wave propagating with the finite velocity can be induced into the body.

c) There exists at least one progressive and one return acceleration wave.

d) f_α is a continuously differentiable function of its variables.

We shall investigate the finite deformation of solids.

2. THE BASIC EQUATIONS OF CONTINUA

In a solid body, the investigation of the propagating wave is based on three groups of equations, the equations of motion are

$$t^{ij}_{;j} + q^{i} = \rho \dot{v}^{i}, \quad t^{ij} = t^{ji} \quad (i, j = 1, 2, 3), \tag{1}$$

the kinematic equations are

$$\dot{A}_{ij} = \ldots, \tag{2}$$

and the constitutive equations are

$$f_{\alpha}(\ldots) = 0 \quad (\alpha = 1, 2, \ldots, 6), \tag{3}$$

containing the tensors of stress t^{ij}, and strain A_{ij}, and the objective derivatives of them, taking into consideration also the initial and boundary conditions.

In (1) ρ is the mass density, v^{i} the velocity of the element of the continuum, and q^{i} is the density of the body force.

3. THE ACCELERATION WAVE

Let the basic functions v^{i}, t^{ij}, A_{ij} remain continuous by crossing the wave front $\varphi(x^{p}, t) = 0$, however, the derivative of them should possess a definite jump. Thus $[v^{i}] = [t^{ij}] = [A_{ij}] = 0$, but the jumps denoted by [ ] do not equal zero for the first derivatives of the previous functions. In φ, x^{p} denotes the spatial coordinates of the element of the continuum and t is the time.

The wave described before is generally called the acceleration wave. Let the unit normal vector of the acceleration wave front be denoted by n_{p} and the speed of propagation according to the moving continuum by C. As it is already known

$$C = c - v^{p} n_{p},$$

where

$$c = -\frac{\frac{\partial \varphi}{\partial t}}{\sqrt{\delta^{pq} \frac{\partial \varphi}{\partial x^{p}} \frac{\partial \varphi}{\partial x^{q}}}}$$

and

$$n_{p} = \frac{\frac{\partial \varphi}{\partial x^{p}}}{\sqrt{\delta^{ij} \frac{\partial \varphi}{\partial x^{i}} \frac{\partial \varphi}{\partial x^{j}}}}.$$

In the following, the starting point is that neither Eq. (2) nor Eq. (3) is known. Substituting some possible forms of Eq. (2) into Eq. (3), assuming that the acceleration wave must not have an infinite speed and assuming also that at least two positive and two negative C exist, one would take conclusions for the constitutive equation.

4. THE COMPATIBILITY CONDITIONS

The investigation under consideration is based on the dynamic, kinematic (see e.g. [2]), and constitutive compatibility conditions.

Let us denote the jump on the wave front by

$$\left[\dot{v}^i\right] = v^i(-c + v^p n_p) = -Cv^i, \qquad \left[t^{ij}_{;j}\right] = \mu^{ij} n_j,$$

and

$$\left[A_{ij;k}\right] = a_{ij} n_k \quad \text{or} \quad \left[\dot{A}_{ij}\right] = -a_{ij} C,$$

where v^i, μ^{ij}, a_{ij} are the generalized wave amplitudes.

The dynamic compatibility condition is

$$\rho C v^i = -\mu^{ij} n_j . \tag{4}$$

Having determined the strain tensor A_{ij}, its derivative and the stress rate can also be obtained. Taking this expression into the form of Eq. (2), the kinematic compatibility condition can be given

$$a_{ij} = \frac{1}{2\rho C^2} (\ldots)_{ijp} \mu^{pq} n_q . \tag{5}$$

The Table summarizes some forms of Eq. (5).

5. CONDITIONS OF THE CONSTITUTIVE EQUATIONS

An arbitrary objective derivative is denoted by a star "*" over the given quantity, for example, $\overset{*}{t}{}^{ij}$ denotes the stress rate. Let us consider now that f_α in Eq. (3) depends also on the derivatives of the basic functions,

$$f_\alpha(\overset{*}{t}{}^{ij}, Q^{ij}, \overset{*}{A}_{ij}, q_{ij}, \ldots) = 0, \tag{6}$$

where $Q^{ij} \equiv B^{ilj}_{pq} t^{pq}_{;l}$ and $q_{ij} \equiv b^{pql}_{ij} t^{pq}_{;l} A_{pq;l}$ are physically objective tensors. Taking f_α before and after the wave front, the constitutive compatibility conditions are obtained, because it is the difference of

Kinematic equation and compatibility condition

Deformation or strain	Derivative	Kinematic equation	Dynamic compatibility condition using kinematic compatibility condition
ε_{ij} small strain	$\dot{\varepsilon}_{ij}$ material derivative	$\dot{\varepsilon}_{ij} = \frac{1}{2}(v_{i;j} + v_{j;i})$	$a_{ij} = \frac{1}{2\rho C^2}(\delta_p n_j + \delta_p n_i)\mu^{pq} n_q$
a_{ij} Euler strain	Lie derivative	$\dot{a}_{ij} = L_v(a)_{ij} - a_{kj}v^k_{;i} - a_{ik}v^k_{;j}$	$a_{ij} = \frac{1}{1\rho C^2}((\delta_{ip} - 2a_{ip})n_j + (\delta_{jp} - 2a_{pj})n_i)\mu^{pq} n_q$
c_{ij} Cauchy deformation	Lie derivative	$\dot{c}_{ij} = -(c_{pj}v^p_{;i} + c_{iq}v^q_{;j})$	$a_{ij} = \frac{1}{\rho C^2}(c_{pj}\delta^q_i + c_{ip}\delta^q_j)n_q n_r \mu^{pr}$
c^{-1}_{ij} Finger deformation	Jaumann derivative	$\dot{c}^{-1}_{ij} = c^{1p}_j v_{i;p} - c^{-1}_{ip}v^p_{;j} + 2c^{-1}_{ip}v^p_j$	$a_{ij} = -\frac{1}{\rho C^2}(c^{-1p}_j \delta_{ki} + c^{-1p}_i \delta_{kj})n_p n_l \mu^{kl}$

them [2]. Assuming that f_α and its derivatives are continuous, the differences F_α satisfy $F_\alpha = 0$. F_α depends on the function $\frac{\partial\varphi}{\partial x^{\hat{k}}} \equiv \varphi_{\hat{k}}$ ($\hat{k} = 1,...,4$), that $\varphi_4 \equiv \frac{\partial\varphi}{\partial t}$ and on $\tilde{v}^i, \tilde{\mu}^{ij}, \tilde{a}_{ij}$. If, for example,

$$\mu^{ij} = \tilde{\mu}^{ij}\sqrt{\delta^{pq}\varphi_p\varphi_q},$$

then the constitutive compatibility conditions consist of six equations. This is a first order nonlinear system of equations for the unknown φ. The system of equations does not explicitly contain φ, but includes $\tilde{\mu}^{ij}$ and $\tilde{a}^{ij}$ amplitudes and is compatible if the Poisson bracket satisfies

$$\left(\frac{\partial F_\alpha}{\partial \varphi_{\hat{p}}}\frac{\partial F_\beta}{\partial x} - \frac{\partial F_\alpha}{\partial x^{\hat{p}}}\frac{\partial F_\beta}{\partial \varphi_{\hat{p}}}\right) = 0\,, \tag{7}$$

and

where

$$\frac{\partial F_\alpha}{\partial \varphi_{\hat{p}}} \equiv \frac{\partial F_\alpha}{\partial t^{*ij}}\frac{\partial\left[t^{*ij}\right]}{\partial \varphi_{\hat{p}}} + \frac{\partial F_\alpha}{\partial Q^{ij}}\frac{\partial\left[Q^{ij}\right]}{\partial \varphi_{\hat{p}}} + \frac{\partial F_\alpha}{\partial t^{*ij}}\frac{\partial\left[t^{*ij}\right]}{\partial \varphi_{\hat{p}}} + \frac{\partial F_\alpha}{\partial A^{*ij}}\frac{\partial\left[A^{*ij}\right]}{\partial \varphi_{\hat{p}}} +$$

$$\frac{\partial F_\alpha}{\partial q^{ij}}\frac{\partial\left[q^{ij}\right]}{\partial \varphi_{\hat{p}}}$$

denoting the derivatives of F_α by $S_{\alpha ij}, P_{\alpha ij}, E_\alpha^{ij}$, and R_α^{ij}

$$\frac{\partial F_\beta}{\partial x^{\hat{p}}} \equiv S_{\alpha ij}\left(\frac{\partial t_0^{*ij}}{\partial x^{\hat{p}}} + \frac{\partial\left[t^{*ij}\right]}{\partial \mu^{kl}}\frac{\partial \mu^{kl}}{\partial x^{\hat{p}}}\right) + P_{\alpha ij}\left(\frac{\partial Q_0^{ij}}{\partial x^{\hat{p}}} + \frac{\partial\left[Q^{ij}\right]}{\partial \mu^{kl}}\frac{\partial \mu^{kl}}{\partial x^{\hat{p}}}\right) + \dots\,.$$

Now one can express Eq. (7) in detail. Let us assume that $t^{*ij} = L_\nu(t)^{ij}$ is the Lie derivative of t^{ij}, and A^{*ij} denotes Lie derivative, too. In Eq. (7), Poisson bracket must be equal to zero if

$$S_{\alpha pq}\mu^{pq} + E_\alpha^{ij}a_{ij} = 0\,, \tag{8a}$$

$$P_{\alpha ij}B_{pq}^{ikj}\mu^{pq} + R_\alpha^{ij}b_{ij}^{pqk}a_{pq} = 0\,, \tag{8b}$$

and

$$S_{\alpha pq}(t^{kl}\delta_r^p + t^{pk}\delta_r^q) - E_\alpha^{ij}(a_{rj}\delta_j^k + a_{ir}\delta_j^k) = 0\,, \tag{8c}$$

where a_{ij} denotes the Euler strain tensor. By varying the deformation tensor and the objective derivatives, expression (8c) changes, too. The form of Eq. (8a) does not change, Eq. (8b) depends on the selection of B and b quantities. Now the equation of the wave propagation can be obtained from Eq. (6) if the "*" Lie derivative is applied,

$$\big[2\rho S_{\alpha pq} C^3 - 2\rho P_{\alpha ij} B^{ilk}_{pq} n_j C^2 + (E^{ij}_{\alpha}(n_q n_j \delta_{ip} + n_q n_i \delta_{pj}) -$$

$$2S_{\alpha uv} n_p n_r (t^{kv} \delta^u_p + t^{ur})) C + R^{ij\,vwl}_{\alpha} b_{ij} n_l n_q (n_w(\delta_{vp} - 2a_{vp}) +$$

$$n_v(\delta_{wp} - 2a_{wp}))\big] \mu^{pq} = 0 \,. \tag{9}$$

The function μ^{pq} is not identically zero, thus the determinant of the matrix in the bracket must be zero. To make the description of the determinant easier, the index function [3]

$$\gamma = \gamma(pq) = \begin{cases} p & \text{if} \quad p = q \\ p+q+1 & \text{if} \quad p \neq q \end{cases} \qquad \text{is used.}$$

Then Eq. (9) is

$$\{\dots\}_{\alpha\gamma} \mu^{\gamma} = 0 \,, \tag{10}$$

that is

$$\det(\{\dots\}_{\alpha\gamma}) = 0 \tag{11}$$

is the wave equation. The expression $\{\dots\}_{\alpha\gamma}$ allows us to introduce a generalized acoustic matrix [4]. Generally, the wave equation is the 18th order expression in C.

For the investigation of the acceleration wave, in the case of a simple tension and small strain, the wave equation is

$$S_1 C^3 + S_2 C^2 + E_1 C + E_2 = 0. \tag{12}$$

In this case, Eq. (9) can be connected to Eq. (12). The quality of the roots of Eq. (12) can be investigated by using Sturm series. When only the real roots are taken into consideration:

– 1 positive and 2 negative roots exist for any

$$S_2 \text{ if } S_1 > 0, E_1 < 0, E_2 < 0,$$

– 2 positive and 1 negative roots exist

$$S_1 > 0, S_2 \neq 0, E_1 < 0, E_2 > 0,$$

or

$$S_1 > 0, S_2 < 0, \ E_1 > 0, E_2 > 0, \ S_2^2 > 3 S_1 E_1,$$

or

$$S_1 > 0, S_2 > 0,\ \ E_1 > 0, E_2 < 0,\ \ S_2^2 > 3S_1E_1,$$

or

$$S_1 > 0, S_2 = 0,\ \ E_1 < 0, E_2 > 0,$$

– 1 positive, 1 zero and 1 negative roots exist
if

$$S_1 > 0, S_2 \neq 0,\ \ E_1 < 0, E_2 = 0,$$

or

$$S_1 > 0, S_2 = 0,\ \ E_1 < 0, E_2 = 0.$$

Using the previous conditions, the (6×6) matrix coefficient of C^3 in Eq. (9) is positive definite, while the matrix coefficient of C^2 can even be (positive or negative) definite, indefinite or zero. Similar conclusions can also be found to the other matrices. It is particularly obvious if

$$\{\ldots\}_{\alpha\gamma}\mu^{\gamma}\mu^{\alpha}$$

is attached to Eq. (9).

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LAINEDÜNAAMIKALE BASEERUVATE OLEKUVÕRRANDITE TEOORIA

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On eeldatud, et olekuvõrrand on esitatav diferentsiaalvõrrandi kujul, kasutades kiirenduslaine leviku omadusi. Üldiselt kehtib see lihtsa tõmbe poolt tekitatud kiirenduslaine puhul saadud katsetulemuste kohta. Üllatuslikult selgus, et ka lõplike deformatsioonide korral esinevad formaalselt väikeste deformatsioonide avaldised.