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LOCALIZATION AND THE FLUTTER AND DIVERGENCE INSTABILITIES OF DYNAMICAL SYSTEMS

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Abstract. The loss of (Liapunov) stability of a dynamical system can be performed in two ways: by a divergence or by a flutter. The classical setting of the localization leads to a nongeneric case, when this classification is impossible. By introducing dissipative terms into the constitutive equation, the stability investigation can be performed as an investigation of a dynamical system. That means to study the real parts of the eigenvalues of differential operators defined by the fundamental equations of the continuum. Such an investigation was performed in an one-dimensional case. The results show that the classical static localization condition means a divergence instability. We also found a condition for the flutter type of the loss of stability in the field of velocities.

Key words: solid mechanics, material instability, dynamical systems.

1. INTRODUCTION

Material instability problems like the localization of plastic deformation use different stability definitions. Most of them are generalizations of the Hill's concept [1] or the Drucker postulate [2]. When a solid body is considered as a dynamical system [3] and a state of the body is a solution of it, the stability of this state means the stability of the solution. In this case, the obvious stability definition is the one of the theories of dynamical systems, the so-called Liapunov stability [4]. This is a kinematic definition, quite similar to the one used by Eringen [5].

The loss of material stability is in close connection with the singular state of the acoustic tensor [6]. Then there is a change in the nature of the acceleration wave-speeds. One of the possibilities is that one of them is zero, the other is the appearance of a complex conjugate pair. In the case of a zero wave-speed, there is a stationary discontinuity, and when the squares of the two wave-speeds are complex conjugates, it is called a flutter [7, 8]. A similar classification is known for the ways of the stability loss of a solution of the dynamical systems [3]. It is called divergence when the linearized part of the differential equation representing the system has a zero eigenvalue. When there is a pair of pure imaginary eigenvalues, it is the flutter. Now the question is: how these two interpretations relate to each other.

In this paper, the solid body is considered as a dynamical system. Our aim is to investigate and classify plastic localization as a loss of stability of it.

2. BASIC EQUATIONS OF THE LOCALIZATION PROBLEMS

Denoting the position of a material point in the reference and the current configurations by X_J and x_i, the position vectors are $\mathbf{R} = X_J\mathbf{G}_J$, and $\mathbf{r} = x_j\mathbf{g}_j$. As usual, the Cartesian tensor notation and the implied summation of the repeated subscripts are used. The deformation gradient $\mathbf{F}$ is

$$F_{jJ} = \frac{\partial x_j}{\partial X_J}.$$

The equation of motion without volume force is

$$S_{jK,K} = \rho \frac{d^2 u_j}{dt^2}, \tag{1}$$

where S_{jK} is the first Piola–Kirchhoff stress tensor, $S_{jK,K}$ is the divergence of it and $\mathbf{u} = \mathbf{R} - \mathbf{r}$ is the displacement. The classical setting of the equations of material instability problems [9, 10] uses a simplified rate constitutive equation in the form

$$\dot{S}_{jK} = K_{jKlM}\dot{F}_{lM}, \tag{2}$$

where F_{lM} is the deformation gradient and K_{jKlM} is the fourth-order tangent modulus tensor. By substituting (2) into the rate form of (1), the motion of the continuum can be described by

$$\rho \frac{d^2 v_j}{dt^2} = \left(K_{jKlM} v_{l,M}\right)_{,K}. \tag{3}$$

The coefficients K_{jKlM} are considered here as piecewise constants. Now two kinds of questions can be asked: one on the existence of strain localization and the other on the stability of the material.

To answer the first one means to search for the condition of the existence of a thin band in the material, in which the rate field quantities differ from the uniform values outside [10, 11]. By denoting $(\ )^b$ and $(\ )^o$, the values inside and outside the band are

$$(\dot{F}_{lM})^b = (\dot{F}_{lM})^o + q_l n_M, \tag{4}$$

n_M are the coordinates of the vector showing the orientation of the band and q_l is the amplitude of the jump on the band. The rate of stress equilibrium implies

$$n_K((\dot{S}_{jK})^b - (\dot{S}_{jK})^o) = 0,$$

that is, with (2) and (4)

$$(n_K(K_{jKlM})n_M)q_l = 0. \tag{5}$$

There are nonzero amplitudes in (5), if and only if

$$\det\left[n_K(K_{jKlM})n_M\right] = 0. \tag{6}$$

For the second question, the stability of a state of a material should be investigated. In dynamics, a state of a system is said to be stable if its motion remains in an arbitrary small neighbourhood of it by applying sufficiently small perturbations [4]. The same concept of stability is used by [5] for continua. Thus for dynamic stability, the role of perturbations and the role of the propagation of disturbances is essential. It means that in stability investigations one should concentrate on the wave propagation.

Equation (3) has a wave solution in the form

$$v_j = q_j \exp(i(n_K X_K - ct)), \tag{7}$$

where n_K shows the direction of the wave front and $i = \sqrt{(-1)}$. In Eq. (7) the wave speed c determines the stability. When $c^2 > 0$, Eq. (7) is stable, when $c^2 < 0$, it is unstable [10]. By substituting Eq. (7) into Eq. (3)

$$-\rho c^2 q_j \exp(i(n_K X_K - ct)) = (K_{jKlM})n_M n_K q_l \exp(i(n_K X_K - ct))$$

is obtained. Hence,

$$\left((K_{jKlM})n_M n_K - \rho c^2 \delta_{jl}\right) q_l = 0.$$

Thus the condition of the existence of a wave solution of nonzero amplitudes reads

$$\det\left[(K_{jKlM})n_M n_K - \rho c^2 \delta_{jl}\right] = 0, \tag{8}$$

that is, the stability depends on the eigenvalues of

$$\left[(K_{jKlM})n_M n_K\right].$$

When all of them are real, the material is in a stable state. When there is at least one pair of complex or imaginary eigenvalues, there is an unstable state. The loss of stability is connected with the appearance of nonzero imaginary parts of the eigenvalues. Unfortunately, at localization this appearance is a very strange one. Then the appearing imaginary eigenvalue is zero ($\pm 0i$). Thus the classical setting of the localization problem results in a coexistent flutter and divergence. Such a situation cannot be typical, because the flutter and divergence are the two possible distinct ways of the loss of stability of a dynamical system [12] and the coexistence is a degenerate case.

In the following section, by using the basics of the theory of dynamical systems, a possibility of decoupling them from each other is treated.

3. RATE DEPENDENT MATERIALS

Let us introduce the notations of the theory of dynamical systems into this localization problem [13]. For further simplification, small displacements are assumed. In its abstract form, Eq. (3) reads

$$\frac{d^2\mathbf{v}}{dt^2} = f(\mathbf{v}). \tag{9}$$

Here, $\mathbf{v} = (v_1, v_2, v_3)$ is a vector of the coordinates of the velocity field satisfying the boundary conditions, and $f(\mathbf{v})$ is a differential operator defined by the left hand side of Eq. (3). Equation (9) defines an infinite dimensional dynamical system. The stability of a state of the continuum means the Liapunov stability of a solution $\mathbf{v}(t)$ of Eq. (9), that is, by perturbing the system, the velocity field $\mathbf{v}'(t)$ is sufficiently close to the unperturbed one $\mathbf{v}(t)$. The stability investigation of some solution of equations like (9) starts with a transformation into a first order equation by introducing new variables $w = [w_j^1, w_j^2]$, where $w_j^1 = v_j$, $w_j^2 = \dot{v}_j$ $(j = 1, 2, 3)$, and with the linearization at a solution (at $v = 0$ for the sake of simplicity)

$$\frac{dw}{dt} = Dfw.$$

The eigenvalues of the linear operator Df show the stability properties. Unfortunately, an equation like (9) cannot give strict results for stability, because the set of eigenvalues consists of pairs $\pm\sqrt{\alpha}$ and when $\alpha > 0$ there is instability, and when $\alpha < 0$ the real part of the eigenvalues is zero. For conservative or linear systems, this implies stability but nonlinearities can ruin it. Moreover, Eq. (9) is not structurally stable in the sense of [14], that is, any small perturbation can cause qualitative changes of the solutions. To achieve structural stability, as the simplest possibility for small strains, a strain rate dependent material is used instead of Eq. (2). In a general form [15, 16],

$$\sigma_{jk} = K^1_{jklm}\epsilon_{lm} + K^2_{jklm}\dot{\epsilon}_{lm}, \tag{10}$$

where the coefficients K^1_{jklm}, K^2_{jklm} are considered to be piecewise constants. Then the equation of motion is

$$\rho \ddot{v}_j = K_{njkl} v_{k,ln} + L_{njkl} \dot{v}_{k,ln},$$

where $K^1_{njkl} = \frac{1}{2}(K_{njkl} + K_{njlk})$ and $K^2_{njkl} = \frac{1}{2}(L_{njkl} + L_{njlk})$.

Introducing new variables

$$w^1_j = v_j, \quad w^2_j = \dot{v}_j \quad (j = 1, 2, 3), \tag{11}$$

the equation of motion is

$$\dot{w}^1_j = w^2_j,$$
$$\dot{w}^2_j = \frac{1}{\rho}(K_{njkl} w^1_{k,ln} + L_{njkl} w^2_{k,ln}).$$

By introducing linear differential operators

$$\hat{K}_{jk} v_k = K_{njkl} \frac{\partial^2}{\partial x_n \partial x_l} v_k, \quad \hat{L}_{jk} v_k = L_{njkl} \frac{\partial^2}{\partial x_n \partial x_l} v_k$$

and

$$\mathbf{L}\left[w^1_j, w^2_j\right] := \left[w^2_j, \frac{1}{\rho}\left(\hat{K}_{jk} w^1_k + \hat{L}_{jk} w^2_k\right)\right]$$

the equation of motion is

$$\frac{d}{dt}[w^1_j, w^2_j] = \mathbf{L}\left[w^1_j, w^2_j\right]. \tag{12}$$

The Liapunov stability depends on the real part of the eigenvalues λ of the linear operator $\mathbf{L}$. The eigenvalue equation is

$$\mathbf{L}\left[w^1_j, w^2_j\right] = \lambda\left[w^1_j, w^2_j\right]. \tag{13}$$

When an eigenvector

$$\left[\bar{w}^1_j, \bar{w}^2_j\right]$$

is substituted into the equation of motion (12)

$$\frac{d}{dt}[\bar{w}^1_j, \bar{w}^2_j] = \bar{\lambda}\left[\bar{w}^1_j, \bar{w}^2_j\right]$$

is obtained with the eigenvalue $\bar{\lambda}$. The solution of it is

$$\left[\bar{w}^1_j, \bar{w}^2_j\right] e^{\bar{\lambda} t}. \tag{14}$$

Having all the eigenvalues and eigenvectors, a solution of the equation of motion can be given as a linear combination of functions Eq. (14), thus the stability requires negative real parts for all eigenvalues. From Eq. (13)

$$w^2_j = \lambda w^1_j,$$
$$\frac{1}{\rho}(\hat{K}_{jk} w^1_k + \hat{L}_{jk} w^2_k) = \lambda w^2_j,$$

then by substituting the first group of equations into the second and using Eq. (11)

$$\left(\rho\lambda^2 v_j - \hat{K}_{jk}v_k - \lambda\hat{L}_{jk}v_k\right) = 0 \tag{15}$$

is obtained, which is a system of the second order partial differential equations with boundary conditions. Thus the state of the material is stable, when for all values of λ, at which there exist nontrivial solutions of Eq. (15), $Re\lambda < 0$.

Note that by omitting the dissipation in Eq. (10) ($K^2_{ijklm} = 0$) there are only λ^2 terms in Eq. (15), thus having solved the eigenvalue equation, real or pure imaginary solutions can be obtained. In the case $\lambda^2 > 0$, there is instability ($\pm\lambda$), otherwise the nongeneric pure imaginary situation as stated at the beginning of this part.

4. ONE-DIMENSIONAL PROBLEM

The application of the stability condition of the previous section needs to have the solution of the boundary value problem Eq. (15). In a general three axial case, it cannot be done analytically. This section deals with a one-dimensional simplified problem to show how the dynamical system theory of Section 3 works. In a one-dimensional case from Eq. (10)

$$\dot{\sigma} = K\dot{\epsilon} + L\ddot{\epsilon},$$

and the equation of motion (12) is

$$v_{tt} = \frac{K}{\rho}v_{xx} + \frac{L}{\rho}v_{txx}, \tag{16}$$

where subscripts denote derivatives with respect to x and t. Let us study the stability of a stationary plastic state of a rod of length l loaded uniaxially. Introducing notations (11) $w_1 = v, w_2 = \dot{v}$, the equation of motion in the operator form is

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{K}{\rho}\frac{\partial^2}{\partial x^2} & \frac{L}{\rho}\frac{\partial^2}{\partial x^2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

and the boundary conditions are $w_1(0) = w_2(0) = w_1(l) = w_2(l) = 0$. Thus the eigenvalue problem (13) is

$$\begin{bmatrix} 0 & 1 \\ \frac{K}{\rho}\frac{\partial^2}{\partial x^2} & \frac{L}{\rho}\frac{\partial^2}{\partial x^2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \lambda \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

or

$$w_2 = \lambda w_1, \tag{17}$$

$$\frac{K}{\rho}\frac{\partial^2 w_1}{\partial x^2} + \frac{L}{\rho}\frac{\partial^2 w_2}{\partial x^2} = \lambda w_2. \tag{18}$$

Substituting Eq. (17) into Eq. (18)

$$\frac{K}{\rho}\frac{\partial^2 w_1}{\partial x^2}+\lambda\frac{L}{\rho}\frac{\partial^2 w_1}{\partial x^2}=\lambda^2 w_1,$$

with the original notations

$$\frac{K}{\rho}\frac{\partial^2 v}{\partial x^2}+\lambda\frac{L}{\rho}\frac{\partial^2 v}{\partial x^2}=\lambda^2 v \tag{19}$$

is obtained. In the one-dimensional case, Eq. (19) is the form of Eq. (15). The stability investigation means to study the signs of the real parts of the possible λ values, at which Eq. (19) has a nonzero v solution. Searching for the solution in the usual form $v(x,t)=C(t)e^{i\alpha x}$,

$$\lambda^2+b\alpha^2\lambda+a\alpha^2=0 \tag{20}$$

is obtained, where $a=\frac{K}{\rho}$, $b=\frac{L}{\rho}$ have positive values. By using the real solutions, coefficient α is determined by the boundary conditions,

$$\begin{aligned} v(0,t)&=A(t)\cos 0+B(t)\sin 0=0,\\ v(l,t)&=A(t)\cos\alpha l+B(t)\sin\alpha l=0. \end{aligned} \tag{21}$$

Then (21) can be nontrivially solved for A and B, if

$$\alpha_k=\frac{k\pi}{l},\quad k=1,2,\ldots.$$

To study the stability of the system, one should look at the real parts of the solutions $\lambda_{k1},\lambda_{k2}$ of (20)

$$\lambda_{k1,2}=\frac{-b\alpha_k^2\pm\sqrt{b^2\alpha_k^4-4a\alpha_k^2}}{2} \tag{22}$$

In the generic case, no eigenvalue has a zero real part, which means structural stability. Moreover, when $a,b>0$, the real parts of all the eigenvalues $\lambda_{k1},\lambda_{k2}$ are negative, thus the plastic state is Liapunov stable. In the case of the so-called divergence instability or static bifurcation [12] one of the eigenvalues is zero. Then Eq. (20) should have zero solutions, that is,

$$a\alpha_k^2=0.$$

Divergence instability means that the loss of Liapunov stability is connected with the loss of the uniqueness of the solution. While $\alpha_k\neq 0$, the condition of the divergence instability is

$$a=\frac{K}{\rho}=0$$

being obviously identical to the condition of static localization.

The other possibility of the loss of stability is that the pairs of complex eigenvalues cross the imaginary axes. In such a case, Eq. (20) has pairs of imaginary roots. The condition of this so-called dynamic bifurcation or flutter instability [12] is

$$b = 0 \quad \text{or} \quad \frac{L}{\rho} = 0.$$

In this case, the uniqueness persists at the loss of Liapunov stability, but an oscillatory behaviour appears in the velocity field.

Summarizing this part, let us look at what happens on loading a ribbon type body, at which the uniaxial case is a suitable approach. On the stress-strain diagram, the tangent modulus K and consequently a of Eq. (22) decrease (Fig. 1). At the beginning, a is large, thus all the eigenvalues $\lambda_{k1,2}$ are complex values with negative real parts. That means stable wave solutions of Eq. (16). When a decreases, it reaches for some k

$$a = \frac{b^2 \alpha_k^2}{4}$$

and then the k^{th} wave dies out. The last wave disappears at

$$a = \frac{b^2 \pi^2}{4l^2}$$

After that there are no waves, but the material remains stable. The stability is lost at $a = 0$. The divergence instability in this case is equivalent to the stationary discontinuity mode [8], because the loss of stability happens when there is no wave solution. Figure 2 shows the stability chart in the plane of the material parameters L and K.

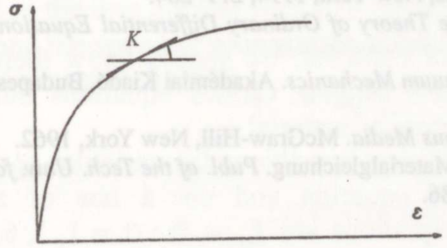

Fig. 1.

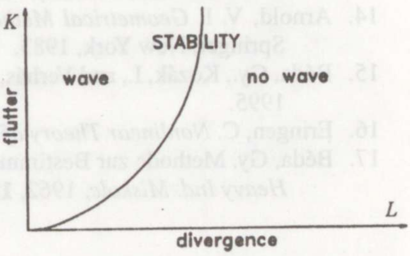

Fig. 2.

In conclusion, in case of a wave of an infinite length, the appearance of the wave and the flutter stability boundary are the same. This result is similar to the one in [17].

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DÜNAAMILISTE SÜSTEEMIDE EBASTABIILSUSE LOKALISEERUMINE, FLATTER JA DIVERGENTNE EBASTABIILSUS

Péter B. BÉDA

Dünaamiliste süsteemide stabiilsuse kadu (Ljapunovi mõttes) võib toimuda kahel viisil: divergentse ebastabiilsuse või flatteri kaudu.

Klassikaline lokaliseerumisülesande püstitus viib erijuhule, kus selline klassifikatsioon ei ole võimalik. Viies olekuvõrranditesse dissipatsiooni arvestavad liikmed, võib stabiilsuse uuringut käsitleda dünaamilise süsteemi uuringuna, s.t. tuleb uurida pideva keskkonna fundamentaalvõrrandite poolt määratud diferentsiaaloperaatorite omaväärtuste reaalosi. Selline uuring on teostatud ühemõõtmelise juhu jaoks. Tulemused näitavad, et klassikaline lokalisatsioonitingimus tähendab divergentset ebastabiilsust. Samuti on leitud flatteri tüüpi stabiilsuse kao tingimused kiiruste väljas.