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THE TRANSFER MATRIX AND THE BOUNDARY ELEMENT METHOD

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Abstract. In this paper, the transfer matrix has been used for formulating finite element continuity and equilibrium equations. In particular, the joint and support continuity and equilibrium equations are considered. Examples of a solution with the boundary element method for a statically indeterminate frame are described.

Key words: transfer matrix, boundary element method.

1. INTRODUCTION

Finite element structural analysis is based on the displacement (stiffness) method, on the force (flexibility) method and on the boundary element method. The force method has been applied in [1–8] and the boundary element method in [9–13]. In the boundary element method, compatibility equations and equilibrium equations for finite elements and joint points (boundary element is a point) are considered. In the boundary element method, the field matrix is evaluated from the fundamental solution of an infinite beam [12]. It is also possible to evaluate the field matrix from the fundamental solution of a semi-infinite beam. In this case, the field matrix is called a transfer matrix [5, 14–17].

By means of the field (transfer) matrix, the forces and the displacements at one end of the finite element are transferred to the opposite end. There are three compatibility equations and three equilibrium equations for a beam-column finite element. For the structure, the joint and support continuity and equilibrium equations with the finite element continuity and equilibrium equations are described. These equations give a large unsymmetrical system of algebraic equations. To solve the system, it can be maintained in its original size or can be condensed [1, 7, 18]. The method described can be called EST method (element, system and transfer).

2. ELEMENT FORCES AND DEFORMATIONS

Figure 1 shows the beam finite element. The reference frame for the element is the local coordinate system x, y, z. The X, Y, Z denote the global reference system. The longitudinal axis x is the union of geometric centroids of the section. The element displacements and forces in the local coordinate system are expressed by the vectors $\mathbf{d}$, $\mathbf{v}$, $\Re$, $\mathbf{s}$, and $\mathbf{z}$.

$$\mathbf{d} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} \mathbf{d} \\ \varphi_y \end{bmatrix}, \quad \Re = \begin{bmatrix} N \\ Q_z \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} \Re \\ M_y \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} \mathbf{v} \\ \mathbf{s} \end{bmatrix}.$$

The transformation matrices $\mathbf{\Theta}$ and $\mathbf{T}$ are

$$\mathbf{\Theta} = \begin{bmatrix} \cos(x, X) & -\cos(x, Z) \\ \cos(x, Z) & \cos(x, X) \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} \mathbf{\Theta} & 0 \\ 0 & 1 \end{bmatrix}.$$

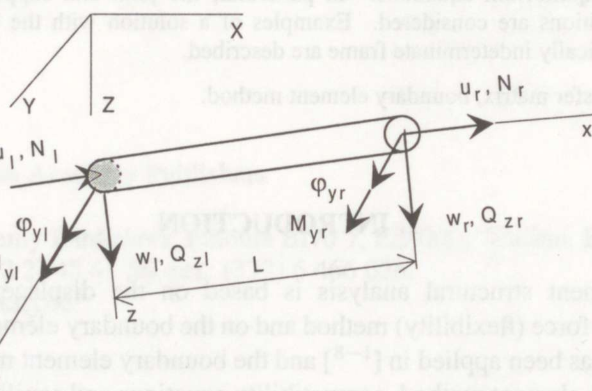

Fig. 1. Local and global coordinates.

3. ELEMENT COMPATIBILITY AND EQUILIBRIUM EQUATIONS

For statically determinate structures, we consider only the equilibrium equations $\frac{dN}{dx} = -q_x$, $\frac{dM}{dx} = Q_z$, $\frac{dQ}{dx} = -q_z$. From the transfer matrix method [14–16], we obtain the finite element equilibrium equation

$$\mathbf{s}_{r-\epsilon} - \mathbf{U}^*_{x=L}\mathbf{s}_{l+\epsilon} = \mathbf{s}^o_{x=L}, \tag{1}$$

where $\mathbf{U}^*$ is the transfer matrix for forces

$$\mathbf{U}^* = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -x & -1 \end{bmatrix}, \tag{2}$$

and the load vector $\mathbf{s}^o$ is

$$\mathbf{s}^o = \begin{bmatrix} -\sum F_x \frac{(x-a_{F_x})_+^0}{0!} - \sum q_x \frac{(x-a_{q_x})_+^1}{1!} \\ -\sum F_z \frac{(x-a_{F_z})_+^0}{0!} - \sum q_z \frac{(x-a_{q_z})_+^1}{1!} \\ -\sum M_y \frac{(x-a_M)_+^0}{0!} - \sum F_z \frac{(x-a_{F_z})_+^1}{1!} - \sum q_z \frac{(x-a_{q_z})_+^2}{2!} \end{bmatrix}, \tag{3}$$

here

$$(x-a_{F_z})_+ = \begin{cases} 0, & \text{when} \quad (x-a_{F_z}) < 0 \\ x-a_{F_z}, & \text{when} \quad (x-a_{F_z}) \geq 0 \end{cases}, \tag{4}$$

and a_{F_x}, a_{q_x}, a_{F_z}, a_{q_z}, and a_M are coordinates of the forces in the local coordinate system (Fig. 2).

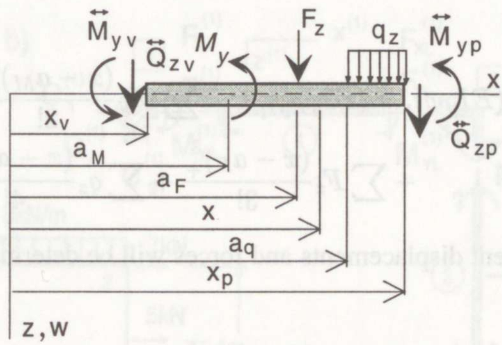

Fig. 2. Element loads.

For statically indeterminate structures, we consider the compatibility and equilibrium equations

$$\mathbf{z}_{r-\epsilon} - \mathbf{U}_{x=L}\mathbf{z}_{l+\epsilon} = \mathbf{z}^o_{x=L}, \tag{5}$$

where $\mathbf{U}$ is the transfer matrix

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{0} & \mathbf{U}^* \end{bmatrix} \tag{6}$$

with

$$\mathbf{U}_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{U}_{12} = \begin{bmatrix} -\frac{x}{EA} & 0 & 0 \\ 0 & \frac{x^3}{6EI_y} - \frac{x}{GA_{red}} & +\frac{x^2}{2EI_y} \\ 0 & -\frac{x^2}{2EI_y} & -\frac{x}{EI_y} \end{bmatrix},$$

and the load vector $\mathbf{z}^o$ is

$$\mathbf{z}^o = \begin{bmatrix} \mathbf{u}^o \\ \mathbf{s}^o \end{bmatrix} \tag{7}$$

with

$$\mathbf{u}^o = \begin{bmatrix} -\sum F_x \frac{(x-a_F)_+}{EA} - \sum q_x \frac{(x-a_q)_+^2}{2EA} \\ \sum M_y \frac{(x-a_M)_+^2}{EI_y 2!} + \sum F_z \frac{(x-a_F)_+^3}{EI_y 3!} + \sum q_z \frac{(x-a_q)_+^4}{EI_y 4!} \\ -\sum M_y \frac{(x-a_M)_+}{EI_y 1!} - \sum F_z \frac{(x-a_F)_+^2}{EI_y 2!} - \sum q_z \frac{(x-a_q)_+^3}{EI_y 3!} \end{bmatrix}. \tag{8}$$

The first three equations in the system of equations (5) are compatibility equations and the latter three are equilibrium equations. The transfer matrix $\mathbf{U}$ and the load vector $\mathbf{z}^o$ can be produced from the universal solution (9) for a semi-infinite beam ($\varphi_y = -\frac{dw}{dx}$)

$$EI_y w = (EI_y w)_l - (EI_y \varphi_y)_l \, x + \sum M_y \frac{(x-a_M)_+^2}{2!} + \\ + \sum F_z \frac{(x-a_F)_+^3}{3!} + \sum q_z \frac{(x-a_q)_+^4}{4!}. \tag{9}$$

The beam element displacements and forces will be determined by the transfer matrix

$$\mathbf{z}_x = \mathbf{U}_x \mathbf{z}_l + \mathbf{z}_x^o. \tag{10}$$

4. SYSTEM OF ELEMENTS

For a statically indeterminate system, the finite elements compatibility and equilibrium equations (5), the displacement continuity equations of the joint and support, and the joint and support equilibrium equations were considered. The number of equations m is

$$m = 6n_e + m_s + m_v, \tag{11}$$

where

$$m_v = m_d + m_\Delta, \tag{12}$$

and

$$m_s = m_j + m_b, \tag{13}$$

where n_e, number of finite elements; m_j, number of equilibrium equations for the joint and support points; m_b, number of prescribed boundary forces for elements (hinges); m_d, number of compatibility equations for the joint points and m_Δ, number of prescribed boundary displacements for the supports. The number of unknowns n is

$$n = 12n_e + n_c, \tag{14}$$

where n_c is the number of reaction forces.

4.1. Example

Figure 3 shows a statically indeterminate structure.

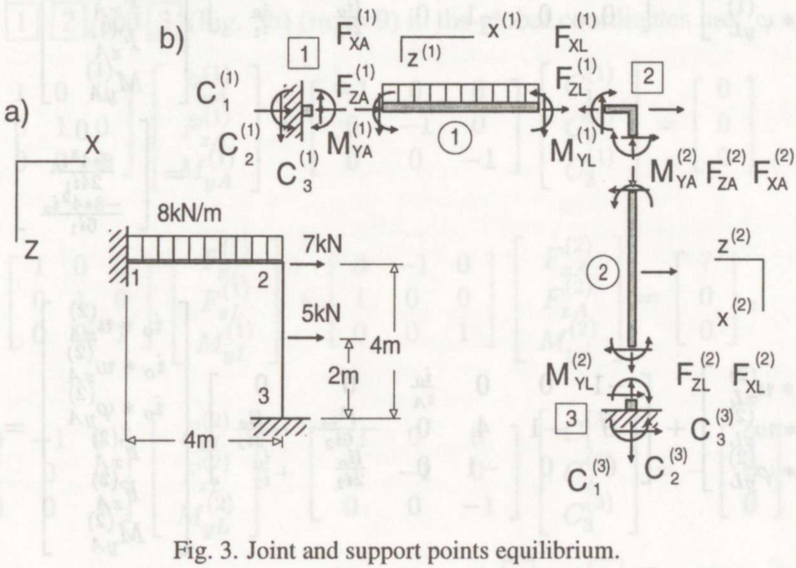

Fig. 3. Joint and support points equilibrium.

Transformation matrices T_1 and T_2 of the beams ① and ② are

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{15}$$

$$T_2 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{16}$$

The equilibrium equations of the beams ① and ② are

$$\begin{bmatrix} F_{xL}^{(1)} \\ F_{zL}^{(1)} \\ M_{yL}^{(1)} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} F_{xA}^{(1)} \\ F_{zA}^{(1)} \\ M_{yA}^{(1)} \end{bmatrix} = - \begin{bmatrix} 0 \\ 8*4 \\ 8\frac{1}{2}*4^2 \end{bmatrix}, \tag{17}$$

$$\begin{bmatrix} F_{xL}^{(2)} \\ F_{zL}^{(2)} \\ M_{yL}^{(2)} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} F_{xA}^{(2)} \\ F_{zA}^{(2)} \\ M_{yA}^{(2)} \end{bmatrix} = - \begin{bmatrix} 0 \\ -5 \\ -5*2 \end{bmatrix}. \tag{18}$$

The compatibility equations (19) and (20) for the beam elements ①, ② are

$$\begin{bmatrix} i_o * u_{xL}^{(1)} \\ i_o * w_{zL}^{(1)} \\ i_o * \varphi_{yL}^{(1)} \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 & \frac{i_o}{i_A} & 0 & 0 \\ 0 & -1 & 4 & 0 & -\frac{l^2 i_o}{6i_1} & -\frac{l i_o}{2i_1} \\ 0 & 0 & -1 & 0 & \frac{l i_o}{2i_1} & \frac{i_o}{i_1} \end{bmatrix} \begin{bmatrix} i_o * u_{xA}^{(1)} \\ i_o * w_{zA}^{(1)} \\ i_o * \varphi_{yA}^{(1)} \\ F_{xA}^{(1)} \\ F_{zA}^{(1)} \\ M_{yA}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{8*4^3 i_o}{24 i_1} \\ \frac{-8*4^2 i_o}{6 i_1} \end{bmatrix}, \tag{19}$$

$$\begin{bmatrix} i_o * u_{xL}^{(2)} \\ i_o * w_{zL}^{(2)} \\ i_o * \varphi_{yL}^{(2)} \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 & \frac{i_o}{i_A} & 0 & 0 \\ 0 & -1 & 4 & 0 & -\frac{l^2 i_o}{6i_2} & \frac{l i_o}{2i_2} \\ 0 & 0 & -1 & 0 & \frac{l i_o}{2i_2} & \frac{i_o}{i_2} \end{bmatrix} \begin{bmatrix} i_o * u_{xA}^{(2)} \\ i_o * w_{zA}^{(2)} \\ i_o * \varphi_{yA}^{(2)} \\ F_{xA}^{(2)} \\ F_{zA}^{(2)} \\ M_{yA}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-5*2^3 i_o}{6*4 i_2} \\ \frac{-5*2^2 i_o}{2*4 i_2} \end{bmatrix}, \tag{20}$$

where $i_o = 0.966 \cdot 10^6\, Pa \cdot m^3$ and $i_A = 1.407 \cdot 10^8\, Pa \cdot m$.

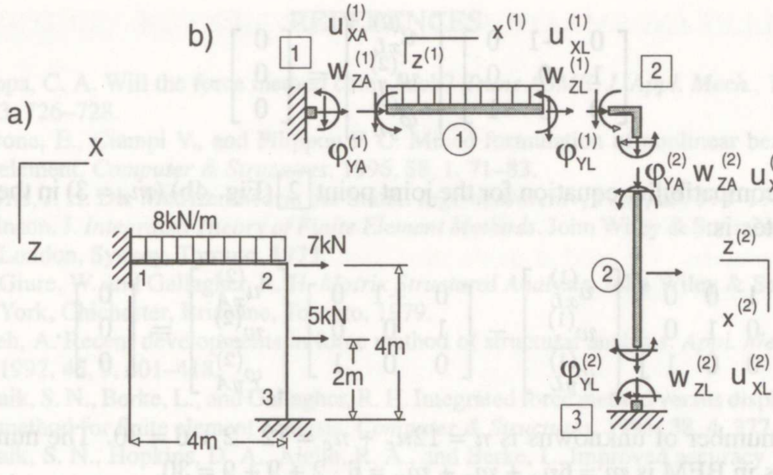

Fig. 4. Schematic of the joint and support points.

The bending stiffnesses of the beam elements are: $i_1 = \frac{EI_y}{L} = i_o$, $i_2 = \frac{EI_y}{L} = i_o$ ($EI_y = 3.864\ MPa \cdot m^4$) and the extension stiffness of the beam element $i_A = \frac{EA}{L}$ ($EA = 562.8\ MPa \cdot m^2$).

The equilibrium equations (21), (22), and (23) for the joint and the support points $\boxed{1}$, $\boxed{2}$, and $\boxed{3}$ (Fig. 3b) ($m_j = 9$) in the global coordinates are:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_{xA}^{(1)} \\ F_{zA}^{(1)} \\ M_{yA}^{(1)} \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_1^{(1)} \\ C_2^{(1)} \\ C_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{21}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_{xL}^{(1)} \\ F_{zL}^{(1)} \\ M_{yL}^{(1)} \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_{xA}^{(2)} \\ F_{zA}^{(2)} \\ M_{yA}^{(2)} \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}, \tag{22}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_{xL}^{(2)} \\ F_{zL}^{(2)} \\ M_{yL}^{(2)} \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} C_1^{(3)} \\ C_2^{(3)} \\ C_3^{(3)} \end{bmatrix} = - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{23}$$

The prescribed displacements of the supports $\boxed{1}$ and $\boxed{3}$ (Fig. 4b) in the global coordinates are ($m_\Delta = 6$):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{xA}^{(1)} \\ w_{zA}^{(1)} \\ \varphi_{yA}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{24}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{xL}^{(2)} \\ w_{zL}^{(2)} \\ \varphi_{yL}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} . \tag{25}$$

The compatibility equation for the joint point $\boxed{2}$ (Fig. 4b) ($m_d = 3$) in the global coordinates is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{xL}^{(1)} \\ w_{zL}^{(1)} \\ \varphi_{yL}^{(1)} \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{xA}^{(2)} \\ w_{zA}^{(2)} \\ \varphi_{yA}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} . \tag{26}$$

The number of unknowns is $n = 12n_e + n_c = 12 \cdot 2 + 6 = 30$. The number of equations in BEM is $m = 6n_e + m_s + m_v = 6 \cdot 2 + 9 + 9 = 30$.

The system of equations (17)–(26) was solved by Crotty [19] block solver with 3×3 blocks. Figure 5 demonstrates the results.

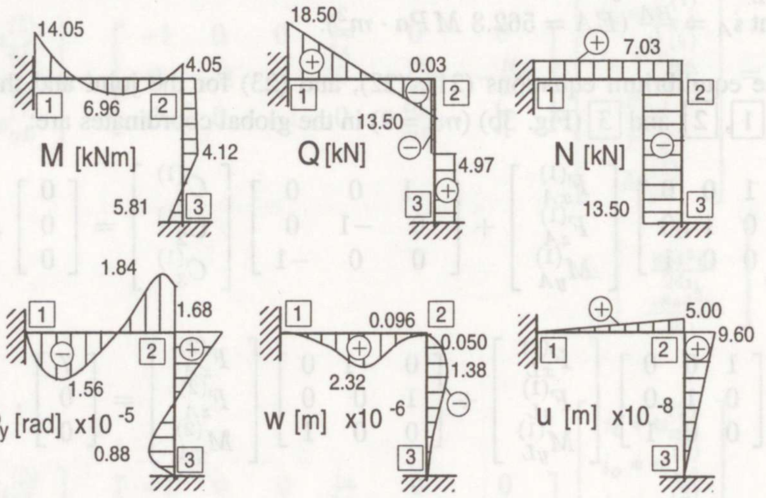

Fig. 5. The forces and the displacements.

5. CONCLUSIONS

The transfer matrix has been used in the boundary element method for formulation of the beam finite elements. In particular, it suits for formulation of the thin-walled beam elements and of the beam finite elements in the 2nd order theory.

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ÜLEKANDEMAATRIKS JA RAJAELEMENTIDE MEETOD

Andres LAHE

On vaadeldud ülekandemaatriksi kasutamist varraste tasakaalu- ja pidevusvõrrandite koostamisel ning käsitletud sõlmpunktide tasakaalu- ja pidevusvõrrandite koostamist. On toodud näide staatiliselt määramatu raami arvutamise kohta rajaelementide meetodiga.