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CONTROL SYSTEMS

A geometric approach to position tracking control of a nonholonomic planar rigid body: case study of an underwater vehicle

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Abstract. This paper addresses the position tracking control design problem for an autonomous underwater vehicle (AUV). The vehicle dynamics is subjected to a non-holonomic velocity constraint arising due to fluid interactions, resulting in a differential-algebraic equation (DAE) formulation for the equations of motion. A reduced order state-space model in a chained form is derived after solving the constraint force. A hierarchical geometric control law is designed for tracking the position of the centre of mass, via this chained form so that the tracking error is almost globally exponentially stable. Simulations on a planar AUV model have been presented to illustrate the performance of the control law.

Key words: Nonholonomic systems, underwater robotics, geometric control.

1. INTRODUCTION

Nonholonomic mechanical systems are those which velocities are restricted to a distribution (i.e. a subbundle of the tangent bundle of their configuration space), which is not integrable (i.e. not involutive) [25]. The mathematical theory of nonholonomic systems was introduced already in 1928 in [38], within a differential geometric framework. This theory was developed further in 1937 by Vagner via the Schouten–Vagner curvature tensor, republished in [17]. More recently, this mathematical framework has been studied in the variational context where trajectories of nonholonomic systems are considered as geodesics of the Schouten–Vranceanu connection [24]. Nonholonomic systems in the control paradigm were studied by Brockett in 1982 [8] within the framework of a singular Riemannian metric. More recently, there has been a significant body of work such as [2,3,5] in the area of optimal control, time-varying stabilization and motion planning of nonholonomic systems. In [10], the equations of motion have been studied in a differential-algebraic framework on manifolds, motivated by the fact that a large class of complex mechanical systems can be understood as individual components evolving on free configuration manifolds, whose dynamics are related with one another due to inter-connection, contact or relative motion constraints. The equations of motion have been derived in the state-space form after solving the algebraic variable, which denotes the reaction force that is natural to the system, responsible for satisfying the constraint.

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One of the techniques in geometric control theory that has gained wide popularity in recent times is the extension of classical proportional-derivative (PD) controllers to systems on manifolds. This approach offers the advantage that the control law is globally defined, and is almost globally exponentially stable. Moreover, other desirable properties of PD control, such as robustness, transient performance etc., are globally extended. An important factor that allows these advantages to be used in geometric controllers is that the control design does not employ local coordinate charts. While using coordinates simplifies the design process (by working in a Euclidean space with well-known techniques), a disadvantage is that the system is only defined within the domain of the chart, and encounters singularities at the chart-boundary. This makes the control system only locally valid. In order to work intrinsically on the manifold, a coordinate invariant (geometric) definition of proportional and derivative errors is necessary. One way of achieving this is to exploit certain structures on the manifold, natural to mechanical systems, such as a Riemannian geometric or Lie group structure. This idea was first developed in [20] for stabilization of fully-actuated mechanical systems, and later extended for tracking control in [10] and [16]. Some other works in this direction are [11,12,29,28] for Lie groups and [37,27] for output tracking.

The aforementioned methods, however, can not be applied to nonholonomic systems due to the well-known Brockett condition, which states that such systems can not be asymptotically stabilized by static feedback [9]. Due to this, several methods have been developed for nonholonomic systems, where time-varying controls are employed, such as [40,18], where accessible nonholonomic systems have been stabilized via Lie-bracket approximation techniques. Another class of systems for which tracking controllers have been designed, are those, which can be transformed into chained forms, with a cascade control structure, such as [34,33,31]. In fact, this method has also been applied to underwater vehicles in [39]. Some other techniques have been reviewed in [30].

In this paper, we study the dynamics of an underwater robotic vehicle, whose velocity is constrained to Lie in a distribution that is not integrable. This constraint arises due to the hydrodynamic interaction with the ambient fluid (see [39]), which allows the vehicle to translate only in the longitudinal direction of its body frame. The equations of motion of this nonholonomic system are expressed as an index-2[1] differential-algebraic control system, where the algebraic variable represents a reaction force, applied on the robot by the ambient fluid, in a direction perpendicular to the admissible direction of motion. The equations of motion are solved by expressing the algebraic variable in terms of the position and velocity, and thereby eliminating the constraint, yielding a state space model. This model is shown to be in a chained form, where the control has a hierarchical structure. The tracking control problem is solved by employing a geometric controller for the inner loop on the unit circle, cascaded with a dynamic-inversion controller ([36]) for the outer loop. This approach is advantageous over [39] because the control law is designed and stability analysis are carried out directly on the nonlinear manifold without employing local coordinates, thereby avoiding resulting singularities. The main contribution in this paper is to combine differential-algebraic equation methods with geometric controllers. Output trajectories, which are consistent with the constraint, are chosen, and the tracking controller has been numerically simulated for point stabilization and time-varying tracking cases. It is demonstrated via examples that the tracking has appreciable transient performance, almost global stability, and robustness to parameter perturbations.

The paper is organized as follows. In section 2, the theory of nonholonomic systems and the output tracking problem has been recalled. In section 3, the dynamics of the nonholonomic robot has been derived. The state space model has been derived in section 4, followed by the control design in section 5, and simulations in section 6, followed by concluding remarks.

2. OVERVIEW OF NONHOLONOMIC CONTROL SYSTEMS

In this section we briefly review the dynamical structure of nonholonomic constrained systems as well as their control design. The reader is referred to [4,13] for further details. In general, such systems arise

[1] The index of a differential-algebraic system is the number of derivatives required to resolve the constraint output.

when the dynamics are subjected to velocity constraints due to contact and constrained relative motion. The constrained dynamics are described by a set of differential-algebraic equations (DAE)

$$\begin{aligned} \ddot{q} &= F(q,\dot{q}) + \sum_{j=1}^{p} \lambda_j A_j^T(q) + \sum_{j=1}^{m} C_j(q) u_j, \\ 0 &= A(q)\dot{q}, \end{aligned} \tag{1}$$

where $q \in \mathbb{R}^n$ is a vector of coordinates on the configuration manifold M, $F(q,\dot{q})$ represents forces such as gyroscopic, friction and potential forces, and the external (control) forces are parameterized by u_j along m independent control vector fields $C_j(q)$. The velocity constraint is determined by a set of p independent, smooth differential 1-forms which are represented as $A_j(q) \in \mathbb{R}^{1\times n}$, $A(q) = [A_1^T(q), \dots, A_p^T(q)]^T$ (with respect to the basis dq). The velocity is constrained to Lie in the kernel of these 1-forms, i.e. in *ker* $A(q)$. The Lagrange multipliers $\lambda_j(t)$ represent a parameterization of the reaction forces which are intrinsic to the system, whose evolution guarantees that the constraint is satisfied. The reaction forces are orthogonal to the constrained velocity subspace along the vector fields which are represented as $A_j^T(q) \in \mathbb{R}^{n\times 1}$ (with respect to the basis $\partial/\partial q$). The reader is referred to [1] for the analysis of such DAE systems and [7] for their numerical treatment.

In certain cases, it may be possible to find p smooth functions $\phi_j(q)$ satisfying

$$A_j(q) = d\phi_j(q),\ j = 1,\dots,p. \tag{2}$$

Such constraints are called *holonomic*, and the system can be regarded as restricted to a submanifold which is the level set $S = \{q|\phi_j(q) = c_j,\ j = 1,\dots p\}$, where c_j are determined by the initial conditions. The velocity constraint ensures that the motion is tangential to S, therefore reducing the problem under consideration to an unconstrained system on S. We term S as the *integral submanifold* of $\{A_j(q),\ 1 \le j \le p\}$. In case such functions do not exist, we call the constraints *nonholonomic*. The integrability of the set of 1-forms $\{A_j(q)\}$ can be checked by the Frobenius theorem [6]. Therefore, in order for the system under consideration to be nonholonomic, the following assumption needs to be satisfied.

Assumption 1. The differential system $\{A_j(q)\}$ is not completely integrable.

Some examples of nonholonomic systems are knife edge dynamics [21], rolling disc [22], unicycle [32], nonholonomic car [14], and wheeled mobile robots [35].

2.1. Output tracking of nonholonomic systems

We briefly review the general principles involved in designing output tracking control laws for nonholonomic systems. The reader is referred to [4,13,32] for details about control design for various mechanical systems with nonholonomic constraints.

Consider the problem of tracking a set of independent output configurations $y = h(q) = [h_1(q),\dots,h_k(q)]^T$, which satisfy the following two conditions in Assumption 2 below.

Assumption 2.

1. The trajectory is consistent with the constraints in (3), i.e. given smooth $y(t)$, there exists a trajectory $q(t)$ such that

$$\begin{aligned} y(0) &= h(q(0)), \\ A_j(q(t))\dot{q}(t) &= 0,\ j = 1,\dots,p,\ \forall\, t. \end{aligned} \tag{3}$$

2. $y(t)$ corresponds uniquely to an admissible trajectory $q(t)$ and vice-versa.

In general, the dimension of the configuration space may be higher than the output space. However, local invertibility may still be guaranteed due to velocity constraints which restrict the space of admissible trajectories.

We additionally assume that the set of admissible velocities, i.e. $ker\ \{A_j(q)\}$, can be spanned by $n-p$ independent vector fields $\{G_j(q)\}$, in which case the velocity can be parameterized as

$$\dot{q} = \sum_{j=1}^{n-p} G_j(q) v_j. \tag{4}$$

Given a reference trajectory $y_r(t)$ we first consider the kinematic control problem of determining $v(t) = [v_1(t), \dots, v_{n-p}(t)]$ such that $y(t)$ asymptotically tracks $y_r(t)$. We refer the reader to [4] for details about kinematic controllability and control design. Subsequently, we determine the control force $u(t)$, required to ensure that the velocity $\dot{q}(t) \to \sum_{j=1}^{p} G_j(q) v_j(t)$ asymptotically. Such a control law $u(t)$ exists under the following assumption:

Assumption 3. The control vector fields, when projected onto the constrained velocity subspace, span the entire subspace, i.e.

$$\text{rank}\ (G^T(q)C(q)) = \text{rank}\ (G(q)),\ \forall q \in M, \tag{5}$$

where $G(q) = [G_1(q), \dots, G_{n-p}(q)]$, $C(q) = [C_1(q), \dots, C_m(q)]$.

3. NONHOLONOMIC ROBOT DYNAMICS

The simplified dynamical model of the planar AUV is represented by the following DAE system [10].

$$\begin{aligned} \dot{q} &= R\eta, \\ \dot{\eta} &= f(q,\eta) + \lambda R^T A^T(q) + B\bar{u}, \\ 0 &= A(q)\dot{q}, \end{aligned} \tag{6}$$

where in (6)

- the configuration $q = [q_1, q_2, q_3]^T$ is a vector consisting of two coordinates of the centre of mass position, and the orientation angle of the longitudinal axis;
- η is the velocity in the body-fixed frame;
- the rotation matrix from the body-fixed frame to the inertial frame (alternatively called global fixed-frame) is represented as

$$R = \begin{bmatrix} \cos q_3 & -\sin q_3 & 0 \\ \sin q_3 & \cos q_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \tag{7}$$

- $f(q,\eta)$ are forces represented in the body frame;
- the control inputs are allocated as $\bar{u} = [\bar{u}_1, \bar{u}_2]^T$,

$$\bar{u}_1 = u_1 \cos u_2,\ \bar{u}_2 = u_1 \sin u_2 \tag{8}$$

(note that the controls are well defined when $u_1 \neq 0$ i.e. when the robot is under a non-trivial thrust);

- the control coefficient matrix is defined as

$$B = [B_1, B_2] = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & \alpha_3 \end{bmatrix} \tag{9}$$

α_1, α_2;

- the velocity of the centre of mass $(\dot{q}_1, \dot{q}_2)$ is constrained to be along the longitudinal axis. Hence, $A(q)$ is obtained as

$$A(q) = [-\sin q_3 \;\; \cos q_3 \;\; 0]; \tag{10}$$

- the term $\lambda R^T \begin{bmatrix} -\sin q_3 \\ \cos q_3 \\ 0 \end{bmatrix}$ represents the reaction force (parameterized by $\lambda(t) \in \mathbb{R}$) which acts on the robot due to its interaction with the ambient fluid. The evolution of $\lambda(t)$ is implicitly determined in order to satisfy the velocity constraint.

We now express (6) in the standard form as (1). The first equation in (6) is differentiated to obtain:

$$\ddot{q} = \dot{R}\eta + R\dot{\eta}. \tag{11}$$

By differentiating (7), we obtain

$$\dot{R} = R\hat{\omega}, \quad \hat{\omega} = \begin{bmatrix} 0 & -\dot{q}_3 & 0 \\ \dot{q}_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{12}$$

We substitute $\dot{R}$ from (12) and $\dot{\eta}$ from (6) into (11) to obtain the dynamics in the form of (1) as

$$\begin{aligned} \ddot{q} &= F(q,\dot{q}) + \lambda A^T(q) + C(q)\bar{u}, \\ 0 &= A(q)\dot{q}, \end{aligned} \tag{13}$$

where $F(q,\dot{q})$ and $C(q)$ are defined as

$$F(q,\dot{q}) = R\hat{\omega}R^T\dot{q} + Rf(q,\dot{q}), \; C(q) = [C_1(q), C_2(q)] = [RB_1, RB_2]. \tag{14}$$

We analyse the integrability of the constraints by using the Jacobian of $A(q)$:

$$\frac{\partial A(q)}{\partial q} = \begin{bmatrix} 0 & 0 & -\cos q_3 \\ 0 & 0 & -\sin q_3 \\ 0 & 0 & 0 \end{bmatrix}. \tag{15}$$

If the constraint would be integrable, i.e. $A(q) = d\phi(q)$ for some smooth function ϕ, then $\dfrac{\partial A(q)}{\partial q} = \dfrac{\partial^2 \phi(q)}{\partial q^2}$ would have been symmetric, which is not the case. Note that this is a simplified version of the Frobenius theorem [25]. Hence, the constraint satisfies **Assumption 1** and the system is indeed nonholonomic.

4. REDUCED STATE SPACE REPRESENTATION

In this section we perform kinematic reduction of the constrained system which satisfies the assumptions in section 2.1, in order to obtain a reduced order state space model for control design. Since the algebraic constraint is implicitly satisfied for each q, the velocity $\dot{q}$ must be in *ker* $A(q)$.

Define

$$G(q) = \begin{bmatrix} \cos q_3 & 0 \\ \sin q_3 & 0 \\ 0 & 1 \end{bmatrix}, \tag{16}$$

which spans *ker* $A(q)$, i.e. that satisfies

$$A(q)G(q) = 0. \tag{17}$$

Then, the constrained kinematics can be written as (see (4))

$$\dot{q} = G(q)v, \tag{18}$$

where $v = [v_1, v_2]^T$ is a parameterization of $ker\ A(q)$. In order to obtain the dynamics $\dot{v}$, we project $\ddot{q}$ as obtained in (13) and on differentiating (18), onto $G(q) = ker\ A(q)$ as follows:

$$\begin{aligned} G^T(q)\ddot{q} &= G^T(q)G(q)\dot{v} + G^T(q)\dot{G}(q)v \\ &= G^T(q)[F(q,\dot{q}) + \lambda A^T(q) + C(q)\bar{u}]. \end{aligned} \tag{19}$$

Observe that in (19) $G^T(q)G(q) = I$ and $G^T(q)A(q) = 0$. Therefore,

$$\dot{v} = -G(q)\dot{G}(q)v + G^T(q)[F(q,\dot{q}) + C(q)\bar{u}]. \tag{20}$$

The matrix of control vector fields is obtained after substituting for $G(q)$ from (16) and $C(q)$ from (14) as

$$G^T(q)C(q) = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_3 \end{bmatrix}. \tag{21}$$

Since α_1 and α_2 are non-zero constants, the control matrix is uniformly non-singular. From the above equation, it can be seen that condition (5) in **Assumption 3** is satisfied, thereby allowing us to construct a control law which can track v.

In order to linearize the dynamics in (20) with respect to the control, we define the following regular static state feedback

$$\bar{u} = \begin{bmatrix} \alpha_1^{-1} & 0 \\ 0 & \alpha_3^{-1} \end{bmatrix} [G(q)\dot{G}(q)v - G^T(q)F(q,\dot{q}) + \hat{u}], \tag{22}$$

which, when substituted into (20), results in

$$\dot{v} = \hat{u}. \tag{23}$$

Equations (18) and (23) constitute the reduced explicit state equations of the nonholonomic constrained dynamics, described in terms of the original variables as

$$\begin{aligned} \dot{q}_1 &= v_1 \cos q_3, \\ \dot{q}_2 &= v_1 \sin q_3, \end{aligned} \tag{24}$$

$$\begin{aligned} \dot{q}_3 &= v_2, \\ \dot{v} &= \hat{u}. \end{aligned} \tag{25}$$

5. HIERARCHICAL CONTROL DESIGN

In this section, a position tracking feedback law is designed for the state equations (24)–(25). The tracking outputs $y = [y_1, y_2]^T$ are chosen as

$$y_1 = q_1,\ y_2 = q_2. \tag{26}$$

Let $y_r = [y_{1_r}, y_{2_r}]^T$ denote the reference output trajectory.

Given an output trajectory $y(t)$ such that $\dot{y}(t) \neq (0,0)^T$, a configuration trajectory can be obtained as $q_1(t) = y_1(t)$, $q_2(t) = y_2(t)$, and $q_3(t)$, i.e. the angle made by the axis of the robot is oriented in the direction of $\dot{y}(t)$. Hence, all output trajectories whose velocities do not vanish, are consistent according to the condition (3) in **Assumption 2**. However, when the velocity of the output trajectory passes through the origin, the corresponding configuration trajectory may encounter a discontinuity.

A hierarchical output tracking controller is designed based on the inner-outer loop paradigm which is explained as follows. In the outer loop (24), q_3 and v_1 are assumed as *virtual* control inputs for which feedback laws q_{3_r} and v_{1_r} are computed in order to track q_1 and q_2. Subsequently in the inner loop (25), a feedback control law for $\hat{u}$ is designed in order to ensure that q_3 and v_1 track q_{3_r} and v_{1_r}. As described in [26], hierarchical controllers stabilize the tracking errors provided that the dynamics of the inner loop is significantly faster than that of the outer loop.

5.1. Outer loop tracking

Define the tracking error of the outer loop as

$$e_1 = (y - y_r) \tag{27}$$

and specify the error dynamics by

$$\dot{e}_1 = -k_1 e_1, \tag{28}$$

where K_1 is a positive definite gain matrix. This ensures that e_1 is exponentially stable.

Using (27), (26), (28) and (24) one can write:

$$\begin{bmatrix} v_1 \cos q_3 \\ v_1 \sin q_3 \end{bmatrix} = \dot{y}_r - k_1 e_1. \tag{29}$$

We solve (29) for v_1 and q_3 by computing the magnitude and phase of the right hand side and denote the solutions as

$$\begin{aligned} v_{1_r} &= ||\dot{y}_r - K_1 e_1||_2, \\ q_{3_r} &= \angle(\dot{y}_r - K_1 e_1), \end{aligned} \tag{30}$$

where $\angle$ denotes the angle (or unit-vector evolving on $\mathbb{S}^1$).

5.2. Inner loop tracking

The feedback $\hat{u}$ is computed in order to track q_{3_r} and v_{1_r} in the subsystem (25) rewritten as

$$\begin{aligned} \dot{v}_1 &= \hat{u}_1, \\ \dot{q}_3 &= v_2, \\ \dot{v}_2 &= \hat{u}_2. \end{aligned} \tag{31}$$

Tracking v_1: Define a tracking error

$$e_2 := v_1 - v_{1_r} \tag{32}$$

and specify its stable dynamics by

$$\dot{e}_2 = -k_2 e_2, \tag{33}$$

where k_2 is a positive constant. From (33) we get

$$\hat{u}_1 = -k_2 e_2 + \dot{v}_{1_r}, \tag{34}$$

thereby ensuring exponential stability of e_2.

Tracking q_3 on the unit circle $\mathbb{S}^1$: A nonlinear feedback controller is designed for the angle $q_3 \in \mathbb{S}^1$.

Define $\theta := (q_3 - q_{3_r})$. A Lyapunov function is chosen as

$$V(\theta, \dot{\theta}) = k_3(1 - \cos\theta) + \frac{1}{2}(\dot{\theta} + \sin\theta)^2. \tag{35}$$

The above Lyapunov function is based on the *Morse* function approach as described in [10], which is commonly used in geometric control of mechanical systems. A control law for $\hat{u}_2$ is designed as

$$\hat{u}_2 = \ddot{x}_{5_r} - \dot{\theta}\cos\theta - k_4(\dot{\theta} + \sin\theta) - k_3\sin\theta, \tag{36}$$

where k_3 and k_4 are positive constants. From this, the derivative of V is obtained as

$$\dot{V} = -k_3\sin^2(\theta) - k_4(\dot{\theta} + \sin\theta)^2. \tag{37}$$

We now employ the LaSalle-Yoshizawa theorem [23].

Theorem 5.1 (LaSalle–Yoshizawa). *Let $x = 0$ be an equilibrium point of the system $\dot{x} = \mathscr{F}(x)$ and suppose $\mathscr{F}$ is locally Lipschitz in x, uniformly in t. Let $V(x)$ be a continuously differentiable, positive definite function such that*

$$\lim_{||x|| \to \infty} V(x) = \infty,$$

and

$$\dot{V}(x) \leq -W(x) \leq 0, \ \forall x. \tag{38}$$

Then, all solution trajectories are globally uniformly bounded and satisfy

$$\lim_{t \to \infty} W(X(t)) = 0. \tag{39}$$

In the context of the tracking problem, we consider the (closed-loop) system under the control law $\hat{u}_2$

$$\ddot{\theta} = -\dot{\theta}\cos\theta - k_4(\dot{\theta} + \sin\theta) - k_3\sin\theta, \tag{40}$$

where $V(\theta, \dot{\theta})$ is defined as in (35) and $W(\theta, \dot{\theta}) = k_3\sin^2\theta + k_4(\dot{\theta} + \sin\theta)^2$. It can be observed that

$$W \to 0 \implies \theta \to \{0, \pi\}, \ \dot{\theta} \to 0, \tag{41}$$

i.e. θ converges to 0 or π. In order to ensure that the tracking errors converge to zero, it is necessary for the orientation error θ to converge to zero. Hence, the error equilibrium $(\theta, \dot{\theta}) = (\pi, 0)$ is undesired. We now employ Chetaev theorem [19] in order to show that the error equilibrium $(\pi, 0)$ is unstable, and therefore almost all trajectories converge to the desired error equilibrium $(\theta, \dot{\theta}) = (0, 0)$.

Theorem 5.2 (Chetaev). *Let x_e be an equilibrium point of the system $\dot{x} = \mathscr{F}(x)$. Let $V_1(x)$ be a continuously differentiable function such that $V_1(x_e) = 0$ and $V_1(x) > 0$ for some x with $||x - x_e||$ being arbitrarily small. Define a set $U = \{x \in B_r(x_e) | V_1(x) > 0\}$, where B_r is an open ball of radius r. Suppose that $\dot{V}_1(x) > 0$ in U, then x_e is an unstable equilibrium point.*

Consider a function $V_1 = 2k_3 - V$ which vanishes at $(\pi, 0)$. From the continuity of V_1, it can be observed that in any neighbourhood of $(\pi, 0)$, there exist points, where $V_1 > 0$. Further, in an open neighbourhood of the equilibrium point, excluding it, we observe that $\dot{V}_1 = -\dot{V} > 0$. Hence, Chetaev theorem proves that the undesired error equilibrium $(\pi, 0)$ is locally unstable. From the feedback law $\hat{u}$, one can solve for $\bar{u}$ from (22) and then u from (8).

6. SIMULATION RESULTS

The drift forces and control coefficients for the model (6) were taken as

$$f(q,\eta) = \begin{bmatrix} -\frac{(Y_{\dot v}-m)\eta_2\eta_3+D_u\eta_1}{m-X_{\dot u}} \\ -\frac{(X_{\dot u}-m)\eta_1\eta_3+D_v\eta_2}{m-Y_{\dot v}} \\ -\frac{(X_{\dot u}-Y_{\dot v})\eta_1\eta_2+D_r\eta_3}{I_{zz}-N_{\dot j}} \end{bmatrix}, \tag{42}$$

$$\alpha_1 = \frac{1}{m-X_{\dot u}},\ \alpha_2 = \frac{1}{m-Y_{\dot v}},\ \alpha_3 = \frac{1}{I_{zz}-N_{\dot j}}\frac{l}{2}, \tag{43}$$

where the parameters (in SI units) are $m = 3.04$ is the mass, $l = 0.5$ is the vehicle's length, $X_{\dot u} = -0.3852$, $Y_{\dot v} = -2.5166$, $N_{\dot j} = -0.0091$ are the added mass parameters, $D_u = -0.18$, $D_v = -19.9$, $D_r = -1.8$ are the drag parameters. The above model is based on [15]. We present the simulation results of the closed-loop system governed by the equations (24) and (25), under the control law $\hat{u}$.

In the first simulation, a constant reference trajectory is chosen as $y_r(t) = (5,-5)$. The initial conditions are chosen as $q_1(0) = q_2(0) = q_3(0) = 0$, and $v(0) = 0$. Next, a time-varying reference trajectory is chosen as $y_r(t) = (5\cos(t), 5\sin(t))$. The initial conditions are chosen as $q_1(0) = q_2(0) = q_3(0) = 0$, and $v(0) = 0$. Such a trajectory is chosen as it ensures that the underwater robot goes through all angles during the tracking phase.

Figure 1 shows the evolution of the vehicles configuration during the stabilization maneuver. Figure 2 shows the output error during stabilization, and Fig. 3 shows the corresponding control trajectories. It can be seen that the output converges exponentially, despite the errors in model due to inaccurately prescribed gyroscopic force components. Figures 4, 5, and 6 show the evolution of the configuration, output error, and control, respectively, while tracking a circular trajectory. Here it can be seen that the output error converges exponentially with appreciable transient performance. It can be seen that there is a slight steady state error, due to modelling inaccuracies; however the control action is appreciably robust, which is an intrinsic characteristic of geometric PD controllers.

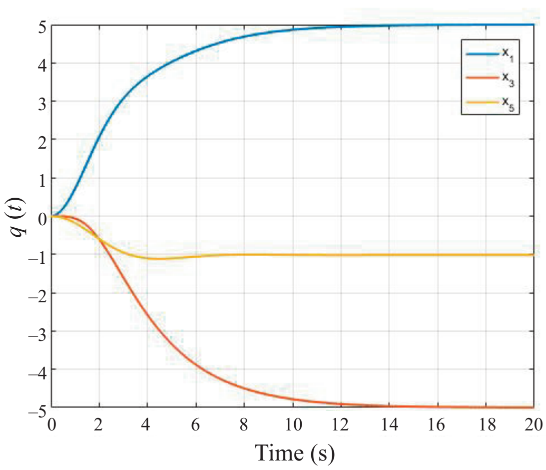

Fig. 1. Configuration $q(t)$ during stabilization maneuver.

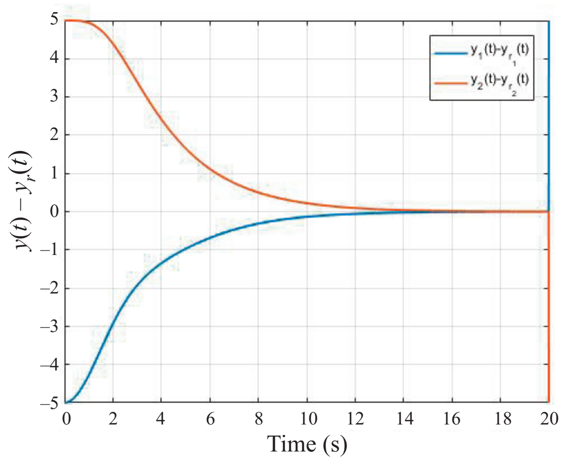

Fig. 2. Output error $y(t) - y_r(t)$ during stabilization maneuver.

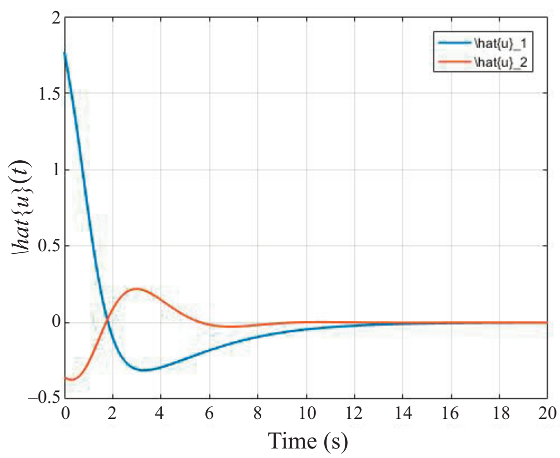

Fig. 3. Control input $\hat{u}$ during stabilization maneuver.

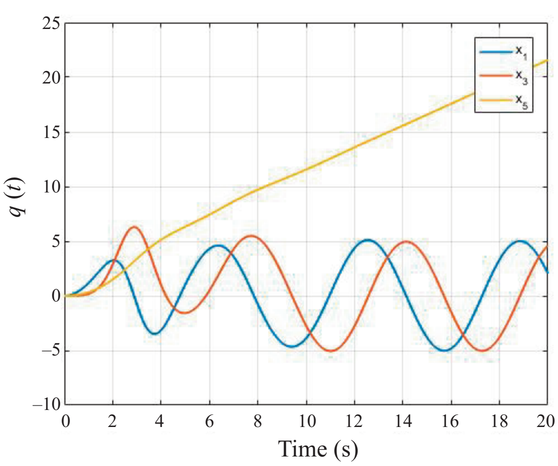

Fig. 4. Configuration $q(t)$ during tracking maneuver.

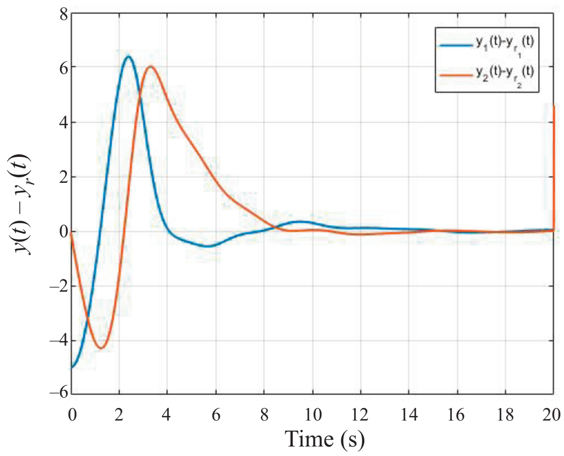

Fig. 5. Output error $y(t) - y_r(t)$ during tracking maneuver.

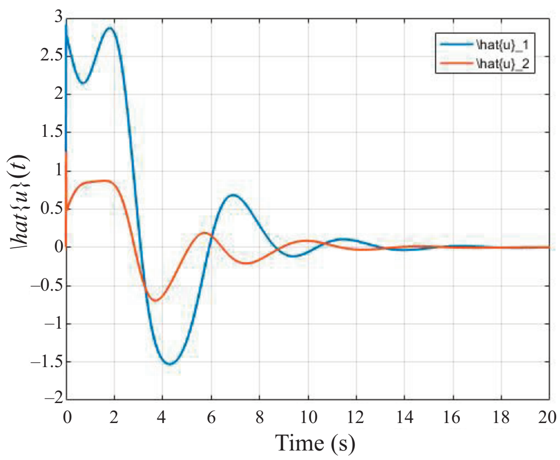

Fig. 6. Control input $\hat{u}$ during tracking maneuver.

7. CONCLUDING REMARKS

An output tracking control law for a general class of nonholonomic systems with outputs, whose trajectories were consistent with the constraints and invertible with respect to state trajectories, was developed. The Euler–Lagrange equations of motion were presented in a differential-algebraic framework, and a constraint elimination based reduction process was described for deriving the state space equations. This approach was demonstrated on a model of an underwater vehicle with nonholonomic velocity constraints. The derived state space model was shown to possess a control structure in a chained form. The hierarchical structure of the state space model was exploited to design a tracking controller for the position of the vehicle. A coordinate invariant geometric controller was employed in the inner loop, with a dynamic inversion controller in the outer loop. Since the controller did not use local coordinates, it was free from singularities due to coordinate charts, and exhibited (almost) global, exponential stability, and robustness to parameter variations. Some avenues for further research include tracking of outputs with internal dynamics, tracking on a general class of manifolds and approximate tracking of non-consistent output trajectories.

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Geomeetriline lähenemine mitteholonoomse allveesõiduki asendi trajektoori järgimiseks

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On käsitletud juhtimisseaduse leidmist autonoomsele mehitamata allveesõidukile, mis kindlustab, et sõiduki massikese järgib etteantud trajektoori. Et sõiduki dünaamika allub vee interaktsioonidest põhjustatud kitsendustele, on liikumisvõrrandid kirjeldatud mittelineaarsete diferentsiaal- ja algebraliste võrrandite süsteemina. Sellest esitusest on esmalt tuletatud madalamat järku klassikalised olekuvõrrandid. Seejärel on hierarhiline juhtimisseadus konstrueeritud selliselt, et vea dünaamika sõiduki keskme ja etteantud trajektoori vahel on peaaegu kõikjal ekponentsiaalselt stabiilne. Simulatsioonid illustreerivad juhtimisseaduse tõhusust ja näitavad, et see kindlustab etteantud trajektoori järgimisel nõutud täpsuse.