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TOPOLOGICAL ALGEBRAS

About the cocompleteness of the category $\mathscr{S}(B)$ of Segal topological algebras

Mart Abel

School of Digital Technologies, Tallinn University, Narva mnt. 25, 10120 Tallinn, Estonia
Institute of Mathematics and Statistics, University of Tartu, Liivi 2, 50409 Tartu, Estonia; mart.abel@tlu.ee, mart.abel@ut.ee

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Abstract. In this paper, we show that the category $\mathscr{S}(B)$ of Segal topological algebras is cocomplete, i.e., that the colimits of all direct systems in the category $\mathscr{S}(B)$ exist.

Key words: mathematics, topological algebras, Segal topological algebras, category, colimit, cocompleteness.

In this setting, a *topological algebra* is a topological linear space over the field $\mathbb{K}$ of real or complex numbers, in which there is defined a separately continuous associative multiplication. Let A be a topological algebra, equipped with the topology τ_A. Denote by θ_A the zero element of A and by 1_A the identity map on A, i.e., the map $1_A : A \to A$, which is defined by $1_A(a) = a$ for every $a \in A$. The existence of a unital element in A is not assumed.

Let us remind the notions introduced in [1].

A topological algebra (A, τ_A) is a left (right or two-sided) *Segal topological algebra* in a topological algebra (B, τ_B) via an algebra homomorphism $f : A \to B$, if

(1) $\mathrm{cl}_B(f(A)) = B$;

(2) $\tau_A \supseteq \{f^{-1}(U) : U \in \tau_B\}$, i.e., f is continuous;

(3) $f(A)$ is a left (respectively, right or two-sided) ideal of B.

In what follows, a Segal topological algebra will be denoted shortly by a triple (A, f, B).

Remark. In order to shorten the text of the paper, in what follows, we will write “an ideal” instead of “a left (right or two-sided) ideal”, keeping in mind, that everything in this paper works for left ideals, for right ideals and for two-sided ideals, using the definitions of left, right or two-sided Segal topological algebras, respectively.

For the rest of the paper, fix any topological algebra (B, τ_B). This allows us to recall from [2] the definition of the category $\mathscr{S}(B)$ of Segal topological algebras, the objects of which depend only on B.

The objects of the category $\mathscr{S}(B)$ are all Segal topological algebras in the topological algebra B, i.e., all Segal algebras in the form of triples $(A, f, B), (C, g, B), \ldots$.

The morphisms between Segal topological algebras (A,f,B) and (C,g,B) are all continuous algebra homomorphisms $\alpha : A \to C$, satisfying $g(\alpha(a)) = (1_B \circ f)(a) = f(a)$ for every $a \in A$, i.e., making the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle 1_B} \\ C & \xrightarrow{g} & B \end{array}$$

commutative.

For the classical definitions of the direct system and the colimit (also called direct limit or inductive limit) of the direct system, we follow the definitions from [5], pp. 237–238. As usual, for any category $\mathscr{C}$, we denote the collection of all objects of $\mathscr{C}$ by $\mathrm{Ob}(\mathscr{C})$, and for any $A,B \in \mathrm{Ob}(\mathscr{C})$, we denote by $\mathrm{Mor}(A,B)$ the collection of all morphisms from the object A to the object B.

Given a partially ordered set $(I,\preceq)$ and a category $\mathscr{C}$, a *direct system in* $\mathscr{C}$ is an ordered pair $((A_i)_{i\in I};(\psi_k^j)_{j\preceq k})$ such that $(A_i)_{i\in I}$ is an indexed family of objects in $\mathscr{C}$ and $(\psi_k^j : A_j \to A_k)_{j\preceq k}$ is an indexed family of morphisms of the category $\mathscr{C}$ for which $\psi_j^j = 1_{A_j}$ for every $j \in I$ and $\psi_k^i = \psi_k^j \circ \psi_j^i$ for every $i,j,k \in I$ with $i \preceq j \preceq k$. The last condition means that the diagram

$$\begin{tikzcd} A_i \arrow[rr, "\psi_k^i"] \arrow[dr, "\psi_j^i"'] & & A_k \\ & A_j \arrow[ur, "\psi_k^j"'] & \end{tikzcd}$$

commutes whenever $i \preceq j \preceq k$.

The *colimit* of a direct system $((A_i)_{i\in I};(\psi_k^j)_{j\preceq k})$ in $\mathscr{C}$ is the pair $(\varinjlim A_i;(\alpha_i)_{i\in I})$, where $\varinjlim A_i$ is an object of $\mathscr{C}$ and $(\alpha_j : A_j \to \varinjlim A_i)_{j\in I}$ is a collection of morphisms in $\mathscr{C}$ such that

(i) $\alpha_k \circ \psi_k^j = \alpha_j$ whenever $j \preceq k$;

(ii) for every $X \in \mathrm{Ob}(\mathscr{C})$ and morphisms $(\beta_i : A_i \to X)_{i\in I}$, which satisfy $\beta_k \psi_k^j = \beta_j$ whenever $j \preceq k$, there exists a unique morphism $\theta : \varinjlim A_i \to X$, making the diagram

$$\begin{tikzcd} \varinjlim A_i \arrow[rr, dashed, "\theta"] & & X \\ & A_j \arrow[ul, "\alpha_j"] \arrow[ur, "\beta_j"'] \arrow[d, "\psi_k^j"] & \\ & A_k \arrow[uul, "\alpha_k"] \arrow[uur, "\beta_k"'] & \end{tikzcd}$$

commutative.

As our goal is to use similar diagrams for the case of the category of Segal topological algebras, then we have to modify the condition (ii) in order to avoid crossing arrows in more complicated diagrams.

It is easy to see that condition (ii) is equivalent to the following condition, which we will be using in case of the category $\mathscr{S}(B)$:

(ii′) for any $X \in \mathrm{Ob}(\mathscr{C})$ and morphisms $(\beta_i : A_i \to X)_{i\in I}$, which satisfy $\beta_k \psi_k^j = \beta_j$ whenever $j \preceq k$, there exists a unique morphism $\theta : \varinjlim A_i \to X$, making the diagrams

$$\begin{tikzcd} \varprojlim A_i \arrow[rr, dashed, "\theta"] & & X \\ & A_j \arrow[ul, "\alpha_j"] \arrow[ur, "\beta_j"'] & \end{tikzcd} \qquad \begin{tikzcd} \varprojlim A_i \arrow[r, dashed, "\theta"] & X \\ A_k \arrow[u, "\alpha_k"] & A_j \arrow[l, "\psi_k^j"] \arrow[u, "\beta_j"'] \end{tikzcd}$$

commutative.

A category is called *cocomplete* if the colimits of all directed systems in this category exist.

Now, we are ready to formulate the definitions of direct system and colimit in the context of the category $\mathscr{S}(B)$.

Definition 1. *Given a partially ordered set $(I,\preceq)$, a* **direct system in** $\mathscr{S}(\mathbf{B})$ *is an ordered pair $(((A_i,f_i,B))_{i\in I};(\psi_k^j)_{j\preceq k})$, where $((A_i,f_i,B))_{i\in I}$ is an indexed family of objects of $\mathscr{S}(B)$ and $(\psi_k^j : A_j \to A_k)_{j\preceq k}$ is an indexed family of morphisms in $\mathscr{S}(B)$ such that $\psi_j^j = 1_{A_j}$ for each $j \in I$ and $\psi_k^i = \psi_k^j \circ \psi_j^i$ for each $i,j,k \in I$ with $i \preceq j \preceq k$. The last condition means that the diagram*

$$\begin{tikzcd} & & & B \arrow[dddd, "1_B"] \\ & & A_i \arrow[ur, "f_i"'] \arrow[dl, "\psi_k^i"'] \arrow[dd, "\psi_j^i"] & \\ B \arrow[urrr, "1_B"] \arrow[ddrrr, "1_B"'] & A_k \arrow[l, "f_k"'] & & \\ & & A_j \arrow[ul, "\psi_k^j"] \arrow[dr, "f_j"] & \\ & & & B \end{tikzcd}$$

commutes.

In case of a direct system $(((A_i,f_i,B))_{i\in I};(\psi_k^j)_{j\preceq k})$, we see that, for every $j,k \in I$ with $j \preceq k$, we obtain a commutative diagram

$$\begin{tikzcd} A_j \arrow[r, "f_j"] \arrow[d, "\psi_k^j"] & B \arrow[d, "1_B"] \\ A_k \arrow[r, "f_k"] & B \end{tikzcd},$$

from where we obtain that $(f_k \circ \psi_k^j)(a) = (1_B \circ f_j)(a) = f_j(a)$ for every $a \in A_j$. Hence, $f_k \circ \psi_k^j = f_j$ and $f_j(A_j) = f_k(\psi_k^j(A_j)) \subseteq f_k(A_k)$.

Denoting $B_i = f_i(A_i)$ for every $i \in I$, we obtain an indexed family $(B_i)_{i\in I}$ of dense ideals of B such that $B_j \subseteq B_k$ whenever $j \preceq k$.

Definition 2. *The* **colimit** *of a direct system $(((A_i,f_i,B))_{i\in I};(\psi_k^j)_{j\preceq k})$ in $\mathscr{S}(B)$ is the pair $((\varinjlim A_i,f,B);(\alpha_i)_{i\in I})$, where $(\varinjlim A_i,f,B)$ is an object of $\mathscr{S}(B)$ and $(\alpha_j : A_j \to \varinjlim A_i)_{j\in I}$ is a collection of morphisms in $\mathscr{C}$ such that*

(i) $\alpha_k \circ \psi_k^j = \alpha_j$ *whenever* $j \preceq k$;

(ii) *for any* $(X,g,B) \in \mathrm{Ob}(\mathscr{S}(B))$ *and morphisms* $(\beta_i : A_i \to X)_{i\in I}$*, which satisfy* $\beta_k \circ \psi_k^j = \beta_j$ *whenever* $j \preceq k$*, there exists a unique morphism* $\theta : \varinjlim A_i \to X$ *making the diagrams*

$$
\begin{tikzcd}
\varinjlim A_i \arrow[rr, dashed, "\theta"] \arrow[ddd, "f"'] & & X \arrow[ddd, "g"] \\
& A_j \arrow[ul, "\alpha_j"'] \arrow[ur, "\beta_j"] \arrow[d, "f_j"] & \\
& B \arrow[dl, "1_B"'] \arrow[dr, "1_B"] & \\
B \arrow[rr, "1_B"'] & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& \varinjlim A_i \arrow[r, dashed, "\theta"] \arrow[dddl, "f"'] & X \arrow[dddr, "g"] & \\
& A_k \arrow[u, "\alpha_k"'] \arrow[d, "f_j"] & A_j \arrow[l, "\psi_k^j"] \arrow[u, "\beta_j"'] \arrow[d, "f_j"] & \\
& B \arrow[r, "1_B"'] \arrow[dl, "1_B"] & B \arrow[dr, "1_B"'] & \\
B \arrow[rrr, "1_B"'] & & & B
\end{tikzcd}
$$

commutative.

From category theory it is known that a category has all colimits if and only if this category has all coproducts and all coequalizers (see, for example, the dual statement of Theorem 12.3 from [4], p. 211).

In [2], we proved in Theorem 10, that the coequalizers in $\mathscr{S}(B)$ always exist.

In [3], we proved in Proposition 1, that the coproduct of any family of objects of $\mathscr{S}(B)$ exists.

Based on these facts, we can state the following result.

Theorem 1. *In the category* $\mathscr{S}(B)$ *exist colimits of all direct systems, i.e., the category* $\mathscr{S}(B)$ *is cocomplete.*

CONCLUSION

In this paper we showed that the category $\mathscr{S}(B)$ of Segal topological algebras is cocomplete, i.e., in this category, the colimits of all direct systems of objects exist.

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Segali topoloogiliste algebrate kategooria $\mathscr{S}(B)$ kotäielikkusest

Mart Abel

Olgu B topoloogiline algebra. On näidatud, et Segali topoloogiliste algebrate kategooria $\mathscr{S}(B)$ on kotäielik: selles kategoorias leiduvad kõik kopiirid.