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TOPOLOGICAL ALGEBRAS

About the limits of inverse systems in the category $\mathscr{S}(B)$ of Segal topological algebras

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Abstract. In this paper, we give a necessary condition for the existence of limits of all inverse systems and a sufficient condition for the existence of limits for all countable inverse systems in the category $\mathscr{S}(B)$ of Segal topological algebras.

Key words: mathematics, topological algebras, Segal topological algebras, category, inverse system, limit.

1. INTRODUCTION

For us, a *topological algebra* is a topological linear space over the field $\mathbb{K}$ of real or complex numbers, in which there is defined a separately continuous associative multiplication. For a topological algebra A we denote by τ_A its topology, by θ_A its zero element, and by 1_A the identity map, i.e., $1_A : A \to A$ is defined by $1_A(a) = a$ for every $a \in A$. We do not suppose that our algebras are unital.

The notion of a Segal topological algebra was first introduced in [1].

A topological algebra (A, τ_A) is a left (right or two-sided) *Segal topological algebra* in a topological algebra (B, τ_B) via an algebra homomorphism $f : A \to B$, if

(1) $\mathrm{cl}_B(f(A)) = B$;

(2) $\tau_A \supseteq \{f^{-1}(U) : U \in \tau_B\}$, i.e., f is continuous;

(3) $f(A)$ is a left (respectively, right or two-sided) ideal of B.

In what follows, a Segal topological algebra will be denoted shortly by a triple (A, f, B).

Remark. In order to shorten the text of the paper, in what follows, we will write “an ideal” instead of “a left (right or two-sided) ideal”, keeping in mind that everything in this paper works for left ideals, for right ideals, and for two-sided ideals, using the definitions of left, right, or two-sided Segal topological algebras, respectively.

From now on, we will fix a topological algebra (B, τ_B), which we will not change for this paper. This allows us to move to the definition of the category $\mathscr{S}(B)$ of Segal topological algebras (this definition depends on the fixed topological algebra (B, τ_B)), which was first defined in [4].

The set $\mathrm{Ob}(\mathscr{S}(B))$ of objects of the category $\mathscr{S}(B)$ consists of all Segal topological algebras in the topological algebra B, i.e., all Segal algebras in the form of triples $(A,f,B),(C,g,B),\dots$.

The set $\mathrm{Mor}((A,f,B),(C,g,B))$ of morphisms between Segal topological algebras (A,f,B) and (C,g,B) consists of all continuous algebra homomorphisms $\alpha : A \to C$, satisfying $g(\alpha(a)) = (1_B \circ f)(a) = f(a)$ for every $a \in A$, i.e., the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[d, "1_B"] \\
C \arrow[r, "g"] & B
\end{tikzcd}$$

is commutative.

2. INVERSE SYSTEMS AND LIMITS IN $\mathscr{S}(B)$

Let us first recall the classical definitions of an inverse system and the limit (also called inverse limit or projective limit) of an inverse system (see, e.g., [5], pp. 230–231).

Given a partially ordered set $(I,\preceq)$ and a category $\mathscr{C}$, an *inverse system in* $\mathscr{C}$ is an ordered pair $((A_i)_{i\in I};(\psi_j^k)_{j\preceq k})$ such that $(A_i)_{i\in I}$ is an indexed family of objects in $\mathscr{C}$ and $(\psi_j^k : A_k \to A_j)_{j\preceq k}$ is an indexed family of morphisms of the category $\mathscr{C}$ for which $\psi_j^j = 1_{A_j}$ for every $j \in I$ and $\psi_i^k = \psi_i^j \circ \psi_j^k$ for every $i,j,k \in I$ with $i \preceq j \preceq k$. The last condition means that the diagram

$$\begin{tikzcd}
A_k \arrow[rr, "\psi_i^k"] \arrow[dr, "\psi_j^k"'] & & A_i \\
& A_j \arrow[ur, "\psi_i^j"'] &
\end{tikzcd}$$

commutes whenever $i \preceq j \preceq k$.

The *limit* of an inverse system $((A_i)_{i\in I};(\psi_j^k)_{j\preceq k})$ in $\mathscr{C}$ is the pair $(\varprojlim A_i;(\alpha_i)_{i\in I})$, where $\varprojlim A_i$ is an object of $\mathscr{C}$ and $\alpha_j : \varprojlim A_i \to A_j$ is a morphism in $\mathscr{C}$ for every $j \in I$ such that

(i) $\psi_j^k \circ \alpha_k = \alpha_j$ whenever $j \preceq k$;

(ii) for every $X \in \mathrm{Ob}(\mathscr{C})$ and morphisms $\beta_j : X \to A_j$, which satisfy $\psi_j^k \beta_k = \beta_j$ whenever $j \preceq k$, there exists a unique morphism $\theta : X \to \varprojlim A_i$ making the diagram

$$\begin{tikzcd}
\varprojlim A_i \arrow[dr, "\alpha_j"] \arrow[ddr, "\alpha_k"'] & & X \arrow[ll, dashed, "\theta"'] \arrow[dl, "\beta_j"'] \arrow[ddl, "\beta_k"] \\
& A_j & \\
& A_k \arrow[u, "\psi_j^k"] &
\end{tikzcd}$$

commutative.

It is easy to see that condition (ii) is equivalent to the following condition:

(ii') for any $X \in \mathrm{Ob}(\mathscr{C})$ and morphisms $\beta_j : X \to A_j$, which satisfy $\psi_j^k \beta_k = \beta_j$ whenever $j \preceq k$, there exists a unique morphism $\theta : X \to \varprojlim A_i$ making the diagrams

$$\begin{tikzcd} \varprojlim A_i \arrow[dr, "\alpha_j"'] & & X \arrow[ll, dashed, "\theta"'] \arrow[dl, "\beta_j"] \\ & A_j & \end{tikzcd} \qquad \begin{tikzcd} \varprojlim A_i \arrow[d, "\alpha_j"] & X \arrow[l, dashed, "\theta"'] \arrow[d, "\beta_k"] \\ A_j & A_k \arrow[l, "\psi_j^k"] \end{tikzcd}$$

commutative.

Since the diagrams, involving Segal topological algebras, will have more arrows, we will use condition (ii') instead of (ii) in order to avoid "crossing arrows" in the case of diagrams with Segal topological algebras.

A category is said to be *finitely complete* if all finite limits (i.e., limits $\varprojlim A_i$, where I is a finite set) in this category exist. Now we are ready to formulate the definitions of inverse system and limit of an inverse system in the context of the category $\mathscr{S}(B)$.

Definition 1. *Given a partially ordered set $(I, \preceq)$, an* **inverse system in** $\mathscr{S}(B)$ *is an ordered pair $(((A_i, f_i, B))_{i \in I}; (\psi_j^k)_{j \preceq k})$, where $((A_i, f_i, B))_{i \in I}$ is an indexed family of objects of $\mathscr{S}(B)$ and $(\psi_j^k : A_k \to A_j)_{j \preceq k}$ is an indexed family of morphisms in $\mathscr{S}(B)$ such that $\psi_j^j = 1_{A_j}$ for each $j \in I$ and $\psi_i^k = \psi_i^j \circ \psi_j^k$ for each $i, j, k \in I$ with $i \preceq j \preceq k$. The last condition means that the diagram*

$$\begin{tikzcd} & & & B \\ & & A_i \arrow[ur, "f_i"] & \\ B \arrow[urrr, "1_B"] \arrow[drrr, "1_B"'] & A_k \arrow[l, "f_k"'] \arrow[ur, "\psi_i^k"] \arrow[dr, "\psi_j^k"'] & & \\ & & A_j \arrow[uu, "\psi_i^j"'] \arrow[dr, "f_j"] & \\ & & & B \arrow[uuuu, "1_B"'] \end{tikzcd}$$

commutes.

Notice that in the case of an inverse system $(((A_i, f_i, B))_{i \in I}; (\psi_j^k)_{j \preceq k})$ we see that for every $j, k \in I$ with $j \preceq k$, we obtain a commutative diagram

$$\begin{tikzcd} A_k \arrow[r, "f_k"] \arrow[d, "\psi_j^k"] & B \arrow[d, "1_B"] \\ A_j \arrow[r, "f_j"] & B \end{tikzcd},$$

from where we obtain that $(f_j \circ \psi_j^k)(a) = (1_B \circ f_k)(a) = f_k(a)$ for every $a \in A_k$. Hence, $f_j \circ \psi_j^k = f_k$ and $f_k(A_k) = f_j(\psi_j^k(A_k)) \subseteq f_j(A_j)$.

Denoting $B_i = f_i(A_i)$ for every $i \in I$, we obtain an indexed family $(B_i)_{i \in I}$ of dense ideals of B such that $B_k \subseteq B_j$ whenever $j \preceq k$.

Definition 2. *The* **limit** *of an inverse system $(((A_i,f_i,B))_{i\in I};(\psi_j^k)_{j\preceq k})$ in $\mathscr{S}(B)$ is the pair $((\varprojlim A_i,f,B);(\alpha_i)_{i\in I})$, where $(\varprojlim A_i,f,B)$ is an object of $\mathscr{S}(B)$ and $\alpha_j:\varprojlim A_i\to A_j$ is a morphism in $\mathscr{S}(B)$ for each $j\in I$ such that*

(i) $\psi_j^k\circ\alpha_k=\alpha_j$ *whenever* $j\preceq k$;

(ii) *for any $(X,g,B)\in \mathrm{Ob}(\mathscr{S}(B))$ and morphisms $\beta_j:X\to A_j$, which satisfy $\psi_j^k\circ\beta_k=\beta_j$ whenever $j\preceq k$, there exists a unique morphism $\theta:X\to\varprojlim A_i$ making the diagrams*

$$\begin{tikzcd}
\varprojlim A_i \arrow[dr, "\alpha_j"] \arrow[ddd, "f"'] & & X \arrow[ll, dashed, "\theta"'] \arrow[dl, "\beta_j"'] \arrow[ddd, "g"] \\
& A_j \arrow[d, "f_j"] & \\
& B & \\
B \arrow[ur, "1_B"] & & B \arrow[ul, "1_B"'] \arrow[ll, "1_B"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& \varprojlim A_i \arrow[d, "\alpha_j"] \arrow[ddl, "f"'] & X \arrow[l, dashed, "\theta"'] \arrow[d, "\beta_k"'] \arrow[ddr, "g"] & \\
& A_j \arrow[d, "f_j"] & A_k \arrow[l, "\psi_j^k"] \arrow[d, "f_k"] & \\
& B & B \arrow[l, "1_B"] & \\
B \arrow[ur, "1_B"] & & & B \arrow[ul, "1_B"'] \arrow[lll, "1_B"]
\end{tikzcd}$$

commutative.

Looking at the definition of a Segal topological algebra and the definition of a limit of an inverse system in $\mathscr{S}(B)$, we obtain the following Lemma.

Lemma 1. *Let B be a topological algebra. If the limits of all inverse systems in the category $\mathscr{S}(B)$ exist, then the intersection of all dense ideals of B has to be dense in B.*

Proof. Let $(B_i)_{i\in I}$ be the collection of all dense ideals of B. It is known that $(B_i,1_{B_i},B)$, where $1_{B_i}:B_i\to B$ is an inclusion defined by$1_{B_i}(b)=b$ for every $b\in B_i$, is a Segal topological algebra for each dense ideal B_i of B.

Define a partial order $\preceq$ on the set I as follows: $j\preceq k$ if and only if $B_k\subseteq B_j$. Take $A_i=B_i, f_i=1_{B_i}$, and $\psi_j^k=1_{B_k}$ for every $j,k\in I$ with $j\preceq k$. Then the collection $(((B_i,1_{B_i},B))_{i\in I};(\psi_j^k)_{j\preceq k})$ becomes an inverse system in $\mathscr{S}(B)$, because the inclusion maps are morphisms in $\mathscr{S}(B)$ and satisfy all the conditions of the inverse system in $\mathscr{S}(B)$.

Now, if all limits in the category $\mathscr{S}(B)$ exist, there should exist also the limit $((\varprojlim B_i,f,B);(\alpha_i)_{i\in I})$ of the inverse system $(((B_i,1_{B_i},B))_{i\in I};(\psi_j^k)_{j\preceq k})$. It means that $1_{B_i}\circ\alpha_i=1_B\circ f$ for each $i\in I$, by the commutativity of the diagrams in condition (ii) of the limit.

Hence, $f(a)=(1_B\circ f)(a)=(1_{B_i}\circ\alpha_i)(a)\in 1_{B_i}(B_i)=B_i$ for any $a\in\varprojlim B_i$ and each $i\in I$, which means that $f(\varprojlim B_i)\subseteq B_i$ for each $i\in I$. Therefore,

$$f(\varprojlim B_i)\subseteq\bigcap_{i\in I}B_i.$$

On the other hand, $(\varprojlim B_i,f,B)\in\mathrm{Ob}(\mathscr{S}(B))$, which means that $f(\varprojlim B_i)$ is dense in B. This means that the intersection $\cap_{i\in I}B_i$ of all dense ideals of B has to be dense in B. $\square$

In order to describe the structure of the limits of inverse systems in the category $\mathscr{S}(B)$, we have to recall some facts about the direct products of topological algebras (see [3], pp. 26–28; for the algebraic part, one can see also [5], p. 53).

Let I be any set of indices and $(A_i,\tau_i)_{i\in I}$ a collection of topological algebras. The *direct product* of the collection $(A_i,\tau_i)_{i\in I}$ is the set

$$\prod_{i\in I}A_i=\{(a_i)_{i\in I}:a_i\in A_i \text{ for every } i\in I\},$$

on which the algebraic operations are defined pointwise, i.e.,

$$(a_i)_{i\in I}+(b_i)_{i\in I}=(a_i+b_i)_{i\in I},\ \ \lambda(a_i)_{i\in I}=(\lambda a_i)_{i\in I},\ \text{and}\ (a_i)_{i\in I}(b_i)_{i\in I}=(a_ib_i)_{i\in I}$$

for every

$$(a_i)_{i\in I},(b_i)_{i\in I}\in\prod_{i\in I}A_i,\ \ \lambda\in\mathbb{K},$$

so that the direct product becomes also an algebra over $\mathbb{K}$. Actually, the direct product of topological algebras is a topological algebra,[1] when equipped with the *product topology*, the base of which is the collection

$$\mathscr{B}=\Big\{\prod_{i\in I}U_i:\ \text{there are}\ n\in\mathbb{Z}^+,i_1,\dots,i_n\in I\ \text{such that}\ U_i\in\tau_i\ \text{for}\ i\in\{i_1,\dots,i_n\},\ \text{and}\ U_i=A_i,\ \text{otherwise}\Big\},$$

where

$$\prod_{i\in I}U_i=\{(u_i)_{i\in I}:u_i\in U_i\ \text{for every}\ i\in I\}.$$

In what follows, we need the notion of a thread in the context of an inverse system of Segal topological algebras. For the purely algebraic notion of a thread in the context of an inverse system of left R-modules over some ring R, see the proof of Proposition 5.17 in [5], p. 232.

Definition 3. *Let $(((A_i,f_i,B))_{i\in I};(\psi_j^k)_{j\preceq k})$ be an inverse system of Segal topological algebras. A* **thread** *for the inverse system $(((A_i,f_i,B))_{i\in I};(\psi_j^k)_{j\preceq k})$ is any element $(a_i)_{i\in I}$ of the direct product $\prod_{i\in I}A_i$ of the family $(A_i)_{i\in I}$ of topological algebras such that $\psi_j^k(a_k)=a_j$ whenever $j\preceq k$.*

Consider the set

$$C=\{(c_i)_{i\in I}\in\prod_{i\in I}A_i:\psi_j^k(c_k)=c_j\ \text{ whenever }\ j\preceq k\}$$

of all threads for the inverse system $(((A_i,f_i,B))_{i\in I};(\psi_j^k)_{j\preceq k})$. As the element $(\theta_{A_i})_{i\in I}\in C$, we get $C\neq\emptyset$. It is easy to check that the set C is an algebra with respect to the algebraic operations defined on the direct product of algebra $(A_i)_{i\in I}$. Indeed, take any $a=(a_i)_{i\in I},b=(b_i)_{i\in I}\in C$, and $\lambda\in\mathbb{K}$. Then $\psi_j^k(a_k)=a_j$ and $\psi_j^k(b_k)=b_j$ for each $j,k\in I$ with $j\preceq k$. Moreover,

$$a+b=(a_i+b_i)_{i\in I},\ \lambda a=(\lambda a_i)_{i\in I},\ \text{and}\ \ ab=(a_ib_i)_{i\in I}.$$

Now, take any $j,k\in I$ with $j\preceq k$. Then

$$\psi_j^k(a_k+b_k)=\psi_j^k(a_k)+\psi_j^k(b_k)=a_j+b_j,\ \psi_j^k(\lambda a_k)=\lambda\psi_j^k(a_k)=\lambda a_j,\ \text{and}\ \psi_j^k(a_kb_k)=\psi_j^k(a_k)\psi_j^k(b_k)=a_jb_j,$$

because ψ_j^k is an algebra homomorphism. Therefore, $a+b,\lambda a,ab\in C$ and C is an algebra.

Consider on C the subspace topology inherited from the direct product $\prod_{i\in I}A_i$. Then C becomes a topological algebra.

Lemma 2. *Let $((\varprojlim A_i,f,B);(\alpha_i)_{i\in I})$ be the limit of an inverse system $(((A_i,f_i,B))_{i\in I};(\psi_j^k)_{j\preceq k})$ of Segal topological algebras. Then $(\alpha_i(a))_{i\in I}$, with $a\in\varprojlim A_i$, is a thread for $(((A_i,f_i,B))_{i\in I};(\psi_j^k)_{j\preceq k})$, i.e., $(\alpha_i(a))_{i\in I}\in C$ for each $a\in\varprojlim A_i$.*

Proof. Let $((\varprojlim A_i,f,B);(\alpha_i)_{i\in I})$ be the limit of an inverse system $(((A_i,f_i,B))_{i\in I};(\psi_j^k)_{j\preceq k})$ of Segal topological algebras. By the definition of the limit, we have $\psi_j^k\circ\alpha_k=\alpha_j$ for each $j,k\in I$ with $j\preceq k$.

Now, $\psi_j^k(\alpha_k(a))=(\psi_j^k\circ\alpha_k)(a)=\alpha_j(a)$ for every $a\in\varprojlim A_i$ and all $j,k\in I$ with $j\preceq k$. Hence, $(\alpha_i(a))_{i\in I}\in C$ for every $a\in\varprojlim A_i$. □

[1] For the proof of this fact, see [3], pp. 27–28.

3. LIMITS OF COUNTABLE INVERSE SYSTEMS

Unfortunatley we are yet not able to continue with the general case of the partially ordered set I. At the moment we are only able to follow in case I is countable.

Take $I = \mathbb{N}$ and consider *a countable inverse system*

$$(((A_n, f_n, B))_{n\in\mathbb{N}}; (\psi_j^k)_{j\leqslant k})$$

in $\mathscr{S}(B)$. Let $B_n = f_n(A_n)$ for each $n \in \mathbb{N}$. Then we obtain a descending family $(B_n)_{n\in\mathbb{N}}$ of dense ideals of B, which means that $B_k \subseteq B_l$ whenever $l \leqslant k$. It is easy to check that the intersection $\cap_{n\in\mathbb{N}} B_n$ of ideals of B is not empty (because every ideal contains the zero element of the algebra) and an ideal of B (even when the family is not descending).

Take any thread

$$(a_n)_{n\in\mathbb{N}} \in C = \{(c_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n : \psi_j^k(c_k) = c_j \ \text{ whenever } \ j \leqslant k\}.$$

Let $m, n \in \mathbb{N}$ with $m \leqslant n$. Then $f_m(a_m) = f_m(\psi_m^n(a_n)) = f_n(a_n)$. Hence, we can define a map $f : C \to B$ by

$$f((a_n)_{n\in\mathbb{N}}) = f_j(a_j) = (f_j \circ \mathrm{pr}_{A_j})((a_n)_{n\in\mathbb{N}}),$$

where $\mathrm{pr}_{A_j} : C \to A_j$ is the projection defined by $\mathrm{pr}_{A_j}((a_n)_{n\in\mathbb{N}}) = a_j$ and $j \in \mathbb{N}$ could be chosen arbitrarily. As all maps $(f_n)_{n\in\mathbb{N}}$ and $(\mathrm{pr}_{A_n})_{n\in\mathbb{N}}$ are continuous algebra homomorphisms, f is also a continuous algebra homomorphism.

Take any $m \in \mathbb{N}$ and $(a_n)_{n\in\mathbb{N}} \in C$. Then $f((a_n)_{n\in\mathbb{N}}) = f_m(a_m) \in B_m$. As it holds for every $m \in \mathbb{N}$, we get

$$f((a_n)_{n\in\mathbb{N}}) \in \bigcap_{n\in\mathbb{N}} B_n.$$

Thus, $f(C) \subseteq \cap_{n\in\mathbb{N}} B_n$.

Fix any $m \in \mathbb{N}$, let $\mathbb{N}_m = \{i \in \mathbb{N} : i \leqslant m\}$, and take any $b_0 \in \cap_{n\in\mathbb{N}} B_n$. Then $b_0 \in B_m$, which means that there exists $\overline{a}_m \in A_m$ such that $f_m(\overline{a}_m) = b_0$. For every $k \in \mathbb{N}_m$, define $\overline{a}_k = \psi_k^m(\overline{a}_m)$. Then we obtain a tuple $(\overline{a}_i)_{i\in\mathbb{N}_m}$ such that $f_i(\overline{a}_i) = b_0$ for each $i \in \mathbb{N}_m$. As it holds for each $m \in \mathbb{N}$, there exists $(\overline{a}_n)_{n\in\mathbb{N}} \in C$ such that $f((\overline{a}_n)_{n\in\mathbb{N}}) = b_0$. Since b_0 was an arbitrary element of $\cap_{n\in\mathbb{N}} B_n$, we get $\cap_{n\in\mathbb{N}} B_n \subseteq f(C)$. Thus, we have shown that

$$f(C) = \bigcap_{n\in\mathbb{N}} B_n.$$

Our aim is to prove that the triple (C, f, B) is an object of $\mathscr{S}(B)$. We have already noticed that f is continuous. Moreover, we know that $f(C) = \cap_{n\in\mathbb{N}} B_n$. We already noticed that the intersection of any family of ideals of B is an ideal of B, which means that $\cap_{n\in\mathbb{N}} B_n$ is an ideal of B. Unfortunately, the intersection of a family of dense ideals of an algebra is not always dense in the algebra. Therefore, we have to put some sufficient restrictions on an algebra B, which would guarantee that the intersection $\cap_{n\in\mathbb{N}} B_n$ is dense in B for every descending family $(B_n)_{n\in\mathbb{N}}$ of dense ideals of B.

In case we could somehow assure that every member of the family $(B_n)_{n\in\mathbb{N}}$ is open, we could restrict ourselves to the class of *Baire spaces*, which are topological spaces where the intersection of each countable collection of dense open subsets is dense. But in the case of the Baire space, the condition that the elements of the family $(B_n)_{n\in\mathbb{N}}$ are descending is not used. Hence, some kind of modification for the definition of a Baire space, where the “openness condition” is replaced by the “descendingness condition”, would be appropriate here. Moreover, we will not need the condition for any descending family of dense subsets; we need it only for any descending family of dense ideals. Hence, we obtain the following definition.

Definition 4. *We say that a topological algebra B is* **a Baire-like algebra for descending dense ideals** *in case the intersection of all elements of any countable descending family of dense ideals of B is dense in B.*

This definition resembles the definition of an Artinian ring. A ring R is called *Artinian ring* if it satisfies the descending chain condition on ideals, i.e., every descending chain $I_1 \supseteq I_2 \supseteq \ldots$ of ideals of R eventually stabilizes. It means that there exists $m \in \mathbb{N}$ such that $I_k = I_m$ for every $k \geqslant m$. Notice that every algebra is also a ring and every ideal of the algebra is also its ideal when we consider the algebra as a ring. Since the term "Artin algebra" has already a different meaning in algebra, we will not use a similar term for algebras. A corollary, which describes some classes of Baire-like algebras for descending dense ideals, follows from the definitions.

Corollary 1. *If a topological algebra B satisfies any of the following conditions:*

(a) *B, considered as a ring, is an Artinian ring,*

(b) *B satisfies the descending chain condition on ideals,*

(c) *B has only a finite number of different dense ideals,*

(d) *all dense ideals of B are open in B, and B, considered as a topological space, is a Baire space,*

(e) *B has the smallest dense ideal*[2]*, i.e., a dense ideal, which is a subset of every other dense ideal of B,*

then B is a Baire-like algebra for descending dense ideals.

Now, we are ready to state the following Proposition.

Proposition 1. *Let B be a topological algebra,*

$$(((A_n, f_n, B))_{n\in\mathbb{N}}; (\psi_j^k)_{j\leqslant k})$$

an inverse system in $\mathscr{S}(B)$,

$$C = \{(c_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n : \psi_j^k(c_k) = c_j \ \textit{ whenever } \ j \leqslant k\},$$

and $f : C \to B$ a map, defined by $f((a_n)_{n\in\mathbb{N}}) = f_j(a_j)$, where $j \in \mathbb{N}$ could be chosen arbitrarily. If either

(a) *B is a Baire-like algebra for descending dense ideals;*

or

(b) *B is a Baire space and all maps $(f_n)_{n\in\mathbb{N}}$ are open,*

then the triple (C, f, B) is an object of $\mathscr{S}(B)$.

Proof. As we showed earlier, f is continuous and

$$f(C) = \bigcap_{n\in\mathbb{N}} B_n = \bigcap_{n\in\mathbb{N}} f_n(A_n)$$

is an ideal of B.

(a) If B is a Baire-like algebra for descending dense ideals, then $f(C)$ is also dense in B.

(b) If all maps $(f_n)_{n\in\mathbb{N}}$ are open, then also all subsets $B_n = f_n(A_n)$ are open in B. If B is a Baire space, then the intersection of a countable collection of open dense subsets of B is dense in B.

Hence, (C, f, B) is an object of $\mathscr{S}(B)$ in both cases. □

In order to prove our last result, we need to use the following Lemma.

Lemma 3. *Let I be a set of indices, $(A_i, \tau_i)_{i\in I}$ a family of topological algebras, X a topological algebra, and $(\beta_i : X \to A_i)_{i\in I}$ a family of continuous algebra homomorphisms. Equip the direct product of topological algebras $(A_i, \tau_i)_{i\in I}$ with the product topology. Then the map*

$$\gamma : X \to \prod_{i\in I} A_i, \ \textit{defined by } \gamma(x) = (\beta_i(x))_{i\in I},$$

is a continuous algebra homomorphism.

[2] This class of topological algebras appeared also in the description of initial objects in $\mathscr{S}(B)$ in [2] (Lemma 3, Lemma 13, Proposition 2 and Open question 4).

Proof. See the Proof of Lemma 1 in [3], p. 28. □

Now, we are ready to state our main result of this paper.

Theorem 1. *If B is a Baire-like algebra for descending dense ideals, then the (inverse) limits for all countable inverse systems in $\mathscr{S}(B)$ exist.*

Proof. Let B be a Baire-like algebra for descending ideals, $(((A_n, f_n, B))_{n\in\mathbb{N}}; (\psi_j^k)_{j\leqslant k})$ any countable inverse system in $\mathscr{S}(B)$, C as in Proposition 1, and $f : C \to B$ a map, defined by $f((a_n)_{n\in\mathbb{N}}) = f_j(a_j)$, where $j \in \mathbb{N}$ could be chosen arbitrarily. Then $(C, f, B) \in \mathrm{Ob}(\mathscr{S}(B))$, by Proposition 1. Define the maps $\alpha_i : C \to A_i$ by $\alpha_i((c_n)_{n\in\mathbb{N}}) = c_i$ for each $i \in \mathbb{N}$. We claim that the pair $((C, f, B); (\alpha_i)_{i\in\mathbb{N}})$ is the inverse limit of the inverse system $(((A_n, f_n, B))_{n\in\mathbb{N}}; (\psi_j^k)_{j\leqslant k})$.

Take any $j, k \in \mathbb{N}$ with $j \leqslant k$. Then

$$(\psi_j^k \circ \alpha_k)((c_n)_{n\in\mathbb{N}}) = \psi_j^k(\alpha_k((c_n)_{n\in\mathbb{N}})) = \psi_j^k(c_k) = c_j = \alpha_j((c_n)_{n\in\mathbb{N}})$$

for every $(c_n)_{n\in\mathbb{N}} \in C$. Hence, $\psi_j^k \circ \alpha_k = \alpha_j$ for every $j \leqslant k$ and the first condition of the limit is fulfilled.

Take any $(X, g, B) \in \mathrm{Ob}(\mathscr{S}(B))$ and morphisms $\beta_j : X \to A_j$, which satisfy $\psi_j^k \circ \beta_k = \beta_j$ whenever $j \preceq k$. Then $(\beta_n : X \to A_n)_{n\in\mathbb{N}}$ is a family of continuous algebra homomorphisms. Define the map $\theta : X \to \prod_{n\in\mathbb{N}} A_n$ by $\theta(x) = (\beta_n(x))_{n\in\mathbb{N}}$. Then θ is a continuous algebra homomorphism, by Lemma 3.

Take any $(a_n)_{n\in\mathbb{N}} \in \theta(X)$. Then there exists $x \in X$ such that $\beta_n(x) = a_n$ for each $n \in \mathbb{N}$. Notice that

$$\psi_j^k(a_k) = \psi_j^k(\beta_k(x)) = (\psi_j^k \circ \beta_k)(x) = \beta_j(x) = a_j$$

for every $j, k \in \mathbb{N}$ with $j \leqslant k$. Hence, $(a_n)_{n\in\mathbb{N}} \in C$, which means that $\theta(X) \subseteq C$.

By the definition of θ, it is clear that $\alpha_j \circ \theta = \beta_j$ for each $j \in \mathbb{N}$. As $a_j \in \mathrm{Mor}((C, f, B), (A_j, f_j, B))$ and $\beta_j \in \mathrm{Mor}((X, g, B), (A_j, f_j, B))$, we get $f_j \circ \alpha_j = 1_B \circ f$ and $f_j \circ \beta_j = 1_B \circ g$ for each $j \in \mathbb{N}$. Thus,

$$1_B \circ f \circ \theta = (1_B \circ f) \circ \theta = (f_j \circ \alpha_j) \circ \theta = f_j \circ (\alpha_j \circ \theta) = f_j \circ \beta_j = 1_B \circ g$$

for each $j \in \mathbb{N}$. Hence, the diagram

$$\begin{tikzcd}
C \arrow[dr, "\alpha_j"] \arrow[ddd, "f"'] & & X \arrow[ll, dashed, "\theta"'] \arrow[dl, "\beta_j"'] \arrow[ddd, "g"] \\
& A_j \arrow[d, "f_j"] & \\
& B & \\
B \arrow[ur, "1_B"] & & B \arrow[ul, "1_B"'] \arrow[ll, "1_B"]
\end{tikzcd}$$

commutes.

Similarly, $\alpha_j \circ \theta = \beta_j = \psi_j^k \circ \beta_k$ and

$$1_B \circ f \circ \theta = (1_B \circ f) \circ \theta = (f_j \circ \alpha_j) \circ \theta = f_j \circ (\alpha_j \circ \theta) = f_j \circ (\psi_j^k \circ \beta_k)$$

$$= (f_j \circ \psi_j^k) \circ \beta_k = (1_B \circ f_k) \circ \beta_k = 1_B \circ (f_k \circ \beta_k) = 1_B \circ (1_B \circ g)$$

for each $j, k \in \mathbb{N}$ with $j \leqslant k$. Thus, the diagram

$$\begin{tikzcd}
& C \arrow[d, "\alpha_j"] \arrow[dddl, "f"'] & X \arrow[l, dashed, "\theta"'] \arrow[d, "\beta_k"'] \arrow[dddr, "g"] & \\
& A_j \arrow[d, "f_j"] & A_k \arrow[l, "\psi_j^k"] \arrow[d, "f_k"] & \\
& B & B \arrow[l, "1_B"] & \\
B \arrow[uur, "1_B"'] & & & B \arrow[uul, "1_B"] \arrow[lll, "1_B"]
\end{tikzcd}$$

also commutes.

Suppose that there exists a morphism $\omega : X \to C$ that makes those two diagrams commute. Take any $x \in X$ and let $(d_n)_{n\in\mathbb{N}} = \omega(x)$. From the commutativity of the first diagram it follows that

$$\beta_j(x) = (\alpha_j \circ \omega)(x) = \alpha_j(\omega(x)) = \alpha_j((d_n)_{n\in\mathbb{N}}) = d_j.$$

Hence, $\omega(x) = (d_n)_{n\in\mathbb{N}} = (\beta_n(x))_{n\in\mathbb{N}} = \theta(x)$. As it is so for every $x \in X$, we get $\omega = \theta$ and $\theta : X \to C$ is the unique morphism making those two diagrams commute.

With that we have shown that (C, f, B) is the limit of the inverse system $(((A_n, f_n, B))_{n\in\mathbb{N}}; (\psi_j^k)_{j<k})$. As this holds for any inverse system in $\mathscr{S}(B)$, the limit of any inverse system in $\mathscr{S}(B)$ exist. $\square$

By Theorem 1 we also see that all finite limits in the category $\mathscr{S}(B)$ exist when B is a Baire-like algebra for descending dense ideals. Hence, we obtain the following corollary.

Corollary 2. *Let B be a Baire-like algebra for descending dense ideals. Then the category $\mathscr{S}(B)$ is finitely complete.*

4. CONCLUSIONS

Let B be a topological algebra. In this paper we showed that if limits of all inverse systems in the category $\mathscr{S}(B)$ of Segal topological algebras exist, then the intersection of all dense left (right or two-sided) ideals of B must be dense in B. We also showed that if B is a Baire-like algebra for descending dense ideals, then the limits of all countable inverse systems in $\mathscr{S}(B)$ exist and $\mathscr{S}(B)$ is a finitely complete category.

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REFERENCES

1. Abel, M. Generalisation of Segal algebras for arbitrary topological algebras. *Period. Math. Hung.*, 2018, **77**(1), 58–68.
2. Abel, M. Initial, terminal and zero objects in the category $\mathscr{S}(B)$ of Segal topological algebras. In *Proceedings of the ICTAA 2018; Math. Stud. (Tartu)*, 2018, **7**, 7–24.

3. Abel, M. About products in the category $\mathscr{S}(B)$ of Segal topological algebras. In *Proceedings of the ICTAA 2018; Math. Stud. (Tartu)*, 2018, **7**, 25–32.
4. Abel, M. About some categories of Segal topological algebras. *Poincare J. Anal. Appl.*, 2019 (1), 1–14.
5. Rotman, J. J. *An Introduction to Homological Algebra.* Second Edition. Springer, New York, 2009.

Projektiivsete süsteemide piiridest Segali topoloogiliste algebrate kategoorias $\mathscr{S}(B)$

Mart Abel

Olgu B topoloogiline algebra. Käesolevas artiklis on näidatud, et kui Segali topoloogiliste algebrate kategoorias $\mathscr{S}(B)$ leiduvad kõigi projektiivsete süsteemide piirid, siis peab algebra B kõigi tihedate vasakpoolsete (parempoolsete või kahepoolsete) ideaalide ühisosa olema tihe algebras B. Samuti on näidatud, et kui B on Baire'i-sarnane algebra kahanevate tihedate ideaalide jaoks, siis leiduvad kõigi loenduvate projektiivsete süsteemide jaoks kategoorias $\mathscr{S}(B)$ piirid ja kategooria $\mathscr{S}(B)$ osutub lõplikult täielikuks kategooriaks.