Proceedings of the Estonian Academy of Sciences,
2019, **68**, 3, 319–323
https://doi.org/10.3176/proc.2019.3.08
Available online at www.eap.ee/proceedings

TOPOLOGICAL
ALGEBRAS

About pushouts in the category $\mathscr{S}(B)$ of Segal topological algebras

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Received 26 March 2019, accepted 28 May 2019, available online 13 August 2019

Abstract. In this paper, we answer positively the open question, posed in [2], about the existence of pushouts in the category $\mathscr{S}(B)$ of Segal topological algebras.

Key words: mathematics, topological algebras, Segal topological algebras, category, pushout.

A *topological algebra* is throughout this paper a topological linear space over the field $\mathbb{K}$ (where $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$), in which is defined a separately continuous associative multiplication.

In [1], the study of general Segal topological algebras started. We begin with recalling the definitions from [1].

A topological algebra (A, τ_A) is a left (right or two-sided) *Segal topological algebra* in a topological algebra (B, τ_B) via an algebra homomorphism $f : A \to B$, if
(1) $\mathrm{cl}_B(f(A)) = B$;
(2) $\tau_A \supseteq \{f^{-1}(U) : U \in \tau_B\}$;
(3) $f(A)$ is a left (respectively, right or two-sided) ideal of B.

In what follows, a Segal topological algebra will be denoted shortly by a triple (A, f, B).

From now on, we will fix a topological algebra (B, τ_B), which we will not change for this paper.

Let us remind to the readers also the definition of the category $\mathscr{S}(B)$ of Segal topological algebras, introduced in [2].

The set $\mathrm{Ob}(\mathscr{S}(B))$ of objects of the category $\mathscr{S}(B)$ consists of all Segal topological algebras in the same topological algebra B, i.e., all Segal algebras in the form of triples $(A, f, B), (C, g, B), \ldots$.

The set $\mathrm{Mor}((A, f, B), (C, g, B))$ of morphisms between Segal topological algebras (A, f, B) and (C, g, B) consists of all continuous algebra homomorphisms $\alpha : A \to C$, satisfying $g(\alpha(a)) = (1_B \circ f)(a) = f(a)$ for every $a \in A$.

In [2] we showed that $\mathscr{S}(B)$ is really a category, which had, among other categorical constructions, also pullbacks. The existence of pushouts was an open problem posed in [2].

The present paper answers this open question positively, using some facts from category theory and results obtained in [2] and [3].

Let us start with the definition of a pushout in the context of Segal topological algebras (in the category $\mathscr{S}(B)$).

Definition 1. *Let* $(A,f,B),(C,g,B),(D,j,B) \in$ Ob $(\mathscr{S}(B))$ *with* $\gamma \in \mathrm{Mor}((D,j,B),(A,f,B))$ *and* $\delta \in \mathrm{Mor}((D,j,B),(C,g,B))$. *An object* (Q,k,B) *of the category* $\mathscr{S}(B)$, *together with morphisms* $\rho \in \mathrm{Mor}((A,f,B),(Q,k,B))$ *and* $\sigma \in \mathrm{Mor}((C,g,B),(Q,k,B))$, *is called the* **pushout** *of morphisms* γ *and* δ, *if*

(1) $\rho \circ \gamma = \sigma \circ \delta$

$$\begin{tikzcd}
Q \arrow[rrr, "k"] \arrow[dddd, dashed, "\nu"'] & & & B \arrow[dddd, "1_B"] \\
& A \arrow[ul, "\rho"'] \arrow[r, "f"] \arrow[dddl, dotted, "\lambda"] & B \arrow[ur, "1_B"] \arrow[dddr, "1_B"] & \\
& D \arrow[u, "\gamma"] \arrow[r, "j"] \arrow[d, "\delta"] & B \arrow[u, "1_B"] \arrow[d, "1_B"] & \\
& C \arrow[uuul, "\sigma"] \arrow[r, "g"] \arrow[dl, dotted, "\mu"] & B \arrow[uuur, "1_B"] \arrow[dr, "1_B"] & \\
R \arrow[rrr, "l"] & & & B
\end{tikzcd};$$

(2) *for every object* (R,l,B) *of the category* $\mathscr{S}(B)$ *and such morphisms* $\lambda \in \mathrm{Mor}((A,f,B),(R,l,B))$, $\mu \in \mathrm{Mor}((C,g,B),(R,l,B))$ *that* $\lambda \circ \gamma = \mu \circ \delta$, *there exists unique morphism* $\nu \in \mathrm{Mor}((Q,k,B),(R,l,B))$ *such that* $\nu \circ \rho = \lambda$ *and* $\nu \circ \sigma = \mu$.

Let $\mathscr{C}$ be any category with the following two properties:

(P1) for any pair of objects $A,B \in \mathrm{Ob}(\mathscr{C})$, the coproduct of A and B exists in $\mathscr{C}$;

(P2) for any pair of objects $C,D \in \mathrm{Ob}(\mathscr{C})$ and any pair of morphisms $\alpha,\beta \in \mathrm{Mor}(C,D)$, the coequalizer of α and β exists in $\mathscr{C}$.

In category theory it is known[1] that, under these two conditions, for any morphisms $\gamma \in \mathrm{Mor}(E,F)$, $\delta \in \mathrm{Mor}(E,G)$, with $E,F,G \in \mathrm{Ob}(\mathscr{C})$, the pushout of γ and δ exists and is constructable in the following way:

(1) Construct the coproduct $(F \sqcup G, i_F, i_G)$ of F and G with injections $i_F : F \to F \sqcup G$ and $i_G : G \to F \sqcup G$. Then $i_F \circ \gamma, i_G \circ \delta \in \mathrm{Mor}(E, F \sqcup G)$.

(2) Construct the coequalizer (Q,λ) of maps $i_F \circ \gamma$ and $i_G \circ \delta$, where $\lambda \in \mathrm{Mor}(F \sqcup G, Q)$.

(3) The triple $(Q, \lambda \circ i_F, \lambda \circ i_G)$ is then the pushout of γ and δ.

Now we continue with the definitions and descriptions of coproduct and coequalizer in the category $\mathscr{S}(B)$. The material about coproducts comes from [3] and the material about coequalizers comes from [2].

Definition 2. *The* **coproduct** *of* $(A,f,B),(C,g,B) \in \mathrm{Ob}(\mathscr{S}(B))$ *is a triple* $((A \sqcup C,h,B),\alpha,\beta)$, *where* $(A \sqcup C,h,B) \in \mathrm{Ob}(\mathscr{S}(B))$, $\alpha \in \mathrm{Mor}((A,f,B),(A \sqcup C,h,B)), \beta \in \mathrm{Mor}((C,g,B),(A \sqcup C,h,B))$ *such that for every* $(X,j,B) \in \mathrm{Ob}(\mathscr{S}(B))$ *and every pair of morphisms* $\gamma \in \mathrm{Mor}((A,f,B),(X,j,B))$ *and* $\delta \in \mathrm{Mor}((C,g,B),(X,j,B))$ *there exists a unique morphism* $\theta \in \mathrm{Mor}((A \sqcup C,h,B),(X,j,B))$ *such that* $\theta \circ \alpha = \gamma$ *and* $\theta \circ \beta = \delta$

[1] It could be obtained as the dual claim of Corollary 5.8 in [4], p. 82, for example.

$$
\begin{tikzcd}
 & & & B \\
 & & B \arrow[ur, "1_B"] \arrow[dll, "1_B"] & \\
B \arrow[urrr, dashed, leftrightarrow, "1_B"] & & A \arrow[u, "f"] \arrow[d, "\gamma"] \arrow[dl, "\alpha"] & \\
 & A \sqcup C \arrow[ul, "h"] \arrow[r, dashed, "\theta"] & X \arrow[uuur, "v"] & \\
 & & C \arrow[u, "\delta"] \arrow[ul, "\beta"] \arrow[d, "g"] & \\
 & & B \arrow[uuull, "1_B"] \arrow[uuuuur, "1_B"] &
\end{tikzcd}
.
$$

In the following result, we need the notion of a tensor algebra, which is explained in more details in [3].

Suppose that $(A, f, B), (C, g, B) \in \mathrm{Ob}(\mathscr{S}(B))$, let T be the tensor algebra of A and C and define a map $h_T : T \to B$ as follows:

$$h_T(t) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} \prod_{l=1}^{N_i} \tilde{h_T}(t_{i,j,l})$$

for every element

$$t = \bigoplus_{i=1}^{n} \sum_{j=1}^{k_i} t_{i,j,1} \otimes \cdots \otimes t_{i,j,N_i}$$

of T, where

$$\tilde{h_T}(t_{i,j,l}) = \begin{cases} f(t_{i,j,l}), & \text{if } t_{i,j,l} \in A \\ g(t_{i,j,l}), & \text{if } t_{i,j,l} \in C \end{cases}.$$

On algebra T we consider the topology $\tau_{h_T} = \{h_T^{-1}(U) : U \in \tau_B\}$, where τ_B denotes the topology of B. Then (T, τ_{h_T}) becomes a topological algebra and h_T becomes a continuous algebra homomorphism in the topology τ_{h_T} (see [3] for details).

Lemma 1. *Let $(A, f, B), (C, g, B) \in \mathrm{Ob}(\mathscr{S}(B))$ and let T be the tensor algebra of A and C. Define the map $h_T : T \to B$ as in (0.1) and equip T with the topology τ_{h_T}. Let I be the two-sided ideal of T, generated by the set*

$$\{a_1 \otimes a_2 - a_1 a_2, c_1 \otimes c_2 - c_1 c_2 : a_1, a_2 \in A, c_1, c_2 \in C\}$$

and $A \sqcup C = T/I$ be equipped with the quotient topology. Let $\kappa_I : T \to T/I$ be the quotient map. Then the triple $(A \sqcup C, h, B)$, where $h(\kappa_I(t)) = h_T(t)$ for every $t \in T$ and every $\kappa_I(t) \in A \sqcup C$, is an object of the category $\mathscr{S}(B)$.

Proof. For the proof, see the proof of Lemma 2.2 in [3]. □

The next Proposition describes the coproducts of two elements in the category $\mathscr{S}(B)$.

Proposition 1. *For any $(A, f, B), (C, g, B) \in \mathrm{Ob}(\mathscr{S}(B))$, their coproduct in $\mathscr{S}(B)$ exists and is the triple $((A \sqcup C, h, B), \alpha, \beta)$, where $(A \sqcup C, h, B)$ is the object of $\mathscr{S}(B)$, described in Lemma 1, $\alpha : A \to A \sqcup C$ and $\beta : C \to A \sqcup C$ are morphisms, defined by $\alpha(a) = \kappa_I(a), \beta(c) = \kappa_I(c)$ for all $a \in A$ and $c \in C$, where κ_I is the quotient map, defined in Lemma 1.*

Proof. For the proof, see the proof of Proposition 3.2 in [3]. □

Now we move on to the coequalizers.

Definition 3. *Let* $(A,f,B),(C,g,B) \in \mathrm{Ob}(\mathscr{S}(B))$. *The* **coequalizer** *of morphisms* $\alpha,\beta \in \mathrm{Mor}((A,f,B),(C,g,B))$ *is a pair* $((Q,k,B),\lambda)$ *such that*

(1) $(Q,k,B) \in \mathrm{Ob}(\mathscr{S}(B))$ *and* $\lambda \in \mathrm{Mor}((C,g,B),(Q,k,B))$ *with* $\lambda(\alpha(a)) = \lambda(\beta(a))$ *for every* $a \in A$

$$\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] & B \arrow[d, "1_B"] & \\
& C \arrow[r, "g"] \arrow[d, "\lambda"] \arrow[ddl, "\mu"'] & B \arrow[d, "1_B"'] \arrow[ddr, "1_B"] & \\
& Q \arrow[r, "k"] \arrow[dl, dashed, "\nu"] & B \arrow[dr, "1_B"'] & \\
R \arrow[rrr, "l"] & & & B
\end{tikzcd};$$

(2) *for any pair* $((R,l,B),\mu)$, *where* $(R,l,B) \in \mathrm{Ob}(\mathscr{S}(B))$ *and* $\mu \in \mathrm{Mor}((C,g,B),(R,l,B))$ *with* $\mu(\alpha(a)) = \mu(\beta(a))$ *for every* $a \in A$, *there exists unique* $\nu \in \mathrm{Mor}((Q,k,B),(R,l,B))$ *with* $\nu \circ \lambda = \mu$.

Next Proposition describes the coequalizers in the category $\mathscr{S}(B)$.

Proposition 2. *Let* $(A,f,B),(C,g,B) \in \mathrm{Ob}(\mathscr{S}(B))$ *and I be the smallest two-sided ideal of C, generated by the set*

$$M = \{\alpha(a) - \beta(a) : a \in A\}.$$

Then the coequalizer of morphisms $\alpha,\beta \in \mathrm{Mor}((A,f,B),(C,g,B))$ *is the pair* $((C/I,\tilde{g},B),p)$, *where* $\tilde{g} : C/I \to B$ *is defined by* $\tilde{g}([c]) = g(c)$ *for each* $[c] \in C/I$, $p : C \to C/I$ *is the canonical projection and* C/I *is equipped with the quotient topology*

$$\tau_{C/I} = \{V \subseteq C/I : p^{-1}(V) \in \tau_C\}.$$

Proof. For the proof, see the proof of Theorem 10 in [2]. □

Thus, the conditions P1) and P2) are fulfilled for the category $\mathscr{S}(B)$. Hence, we can state the main result of this paper.

Theorem 1. *The pushouts exist always in the category* $\mathscr{S}(B)$.

To illustrate the situation, we give the following commutative diagram,

$$\begin{tikzcd}
& & & B \arrow[ddlll, "1_B"'] & & \\
& & & A \arrow[u, "f"] \arrow[dl, "\alpha"] & & \\
B & & A \sqcup C \arrow[ll, "h"] \arrow[d, "\lambda"] & & D \arrow[ul, "\gamma"'] \arrow[dl, "\delta"] \arrow[r, "j"] & B \arrow[uull, "1_B"'] \arrow[ddll, "1_B"] \\
& B \arrow[ul, "1_B"] & Q \arrow[l, "k"] & C \arrow[ul, "\beta"'] \arrow[d, "g"] & & \\
& & & B \arrow[uulll, "1_B"] & &
\end{tikzcd},$$

which describes the pushout of morphisms γ and δ, if one compares this diagram with diagrams given in the Definitions 1–3 and takes $\rho = \lambda \circ \alpha$ and $\sigma = \lambda \circ \beta$ in the diagram of Definition 1.

CONCLUSION

In this paper we showed that the pushouts always exist in the category $\mathscr{S}(B)$.

ACKNOWLEDGEMENTS

The research was supported by the institutional research funding IUT20-57 of the Estonian Ministry of Education and Research. The author wishes to thank the referees for pointing out some misprints in the first version of this paper. The publication costs of this article were covered by the Estonian Academy of Sciences.

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Väljatõukajatest Segali topoloogiliste algebrate kategoorias $\mathscr{S}(B)$

Mart Abel

Olgu B topoloogiline algebra. On näidatud, et Segali topoloogiliste algebrate kategoorias $\mathscr{S}(B)$ leiduvad kõik väljatõukajad.