



## About pushouts in the category $\mathcal{S}(B)$ of Segal topological algebras

Mart Abel

School of Digital Technologies, Tallinn University, Narva 29, 10120 Tallinn, Estonia

Institute of Mathematics and Statistics, University of Tartu, Liivi 2, 50409 Tartu, Estonia; [mart.abel@tlu.ee](mailto:mart.abel@tlu.ee), [mart.abel@ut.ee](mailto:mart.abel@ut.ee)

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**Abstract.** In this paper, we answer positively the open question, posed in [2], about the existence of pushouts in the category  $\mathcal{S}(B)$  of Segal topological algebras.

**Key words:** mathematics, topological algebras, Segal topological algebras, category, pushout.

A *topological algebra* is throughout this paper a topological linear space over the field  $\mathbb{K}$  (where  $\mathbb{K}$  stands for either  $\mathbb{R}$  or  $\mathbb{C}$ ), in which is defined a separately continuous associative multiplication.

In [1], the study of general Segal topological algebras started. We begin with recalling the definitions from [1].

A topological algebra  $(A, \tau_A)$  is a left (right or two-sided) *Segal topological algebra* in a topological algebra  $(B, \tau_B)$  via an algebra homomorphism  $f : A \rightarrow B$ , if

- (1)  $\text{cl}_B(f(A)) = B$ ;
- (2)  $\tau_A \supseteq \{f^{-1}(U) : U \in \tau_B\}$ ;
- (3)  $f(A)$  is a left (respectively, right or two-sided) ideal of  $B$ .

In what follows, a Segal topological algebra will be denoted shortly by a triple  $(A, f, B)$ .

From now on, we will fix a topological algebra  $(B, \tau_B)$ , which we will not change for this paper.

Let us remind to the readers also the definition of the category  $\mathcal{S}(B)$  of Segal topological algebras, introduced in [2].

The set  $\text{Ob}(\mathcal{S}(B))$  of objects of the category  $\mathcal{S}(B)$  consists of all Segal topological algebras in the same topological algebra  $B$ , i.e., all Segal algebras in the form of triples  $(A, f, B), (C, g, B), \dots$

The set  $\text{Mor}((A, f, B), (C, g, B))$  of morphisms between Segal topological algebras  $(A, f, B)$  and  $(C, g, B)$  consists of all continuous algebra homomorphisms  $\alpha : A \rightarrow C$ , satisfying  $g(\alpha(a)) = (1_B \circ f)(a) = f(a)$  for every  $a \in A$ .

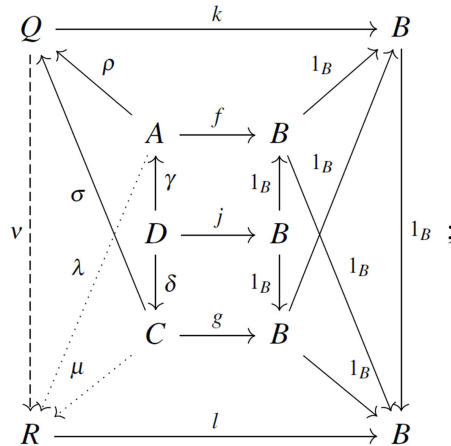
In [2] we showed that  $\mathcal{S}(B)$  is really a category, which had, among other categorical constructions, also pullbacks. The existence of pushouts was an open problem posed in [2].

The present paper answers this open question positively, using some facts from category theory and results obtained in [2] and [3].

Let us start with the definition of a pushout in the context of Segal topological algebras (in the category  $\mathcal{S}(B)$ ).

**Definition 1.** Let  $(A, f, B), (C, g, B), (D, j, B) \in \text{Ob}(\mathcal{S}(B))$  with  $\gamma \in \text{Mor}((D, j, B), (A, f, B))$  and  $\delta \in \text{Mor}((D, j, B), (C, g, B))$ . An object  $(Q, k, B)$  of the category  $\mathcal{S}(B)$ , together with morphisms  $\rho \in \text{Mor}((A, f, B), (Q, k, B))$  and  $\sigma \in \text{Mor}((C, g, B), (Q, k, B))$ , is called the **pushout** of morphisms  $\gamma$  and  $\delta$ , if

(1)  $\rho \circ \gamma = \sigma \circ \delta$



(2) for every object  $(R, l, B)$  of the category  $\mathcal{S}(B)$  and such morphisms  $\lambda \in \text{Mor}((A, f, B), (R, l, B))$ ,  $\mu \in \text{Mor}((C, g, B), (R, l, B))$  that  $\lambda \circ \gamma = \mu \circ \delta$ , there exists unique morphism  $v \in \text{Mor}((Q, k, B), (R, l, B))$  such that  $v \circ \rho = \lambda$  and  $v \circ \sigma = \mu$ .

Let  $\mathcal{C}$  be any category with the following two properties:

- (P1) for any pair of objects  $A, B \in \text{Ob}(\mathcal{C})$ , the coproduct of  $A$  and  $B$  exists in  $\mathcal{C}$ ;
- (P2) for any pair of objects  $C, D \in \text{Ob}(\mathcal{C})$  and any pair of morphisms  $\alpha, \beta \in \text{Mor}(C, D)$ , the coequalizer of  $\alpha$  and  $\beta$  exists in  $\mathcal{C}$ .

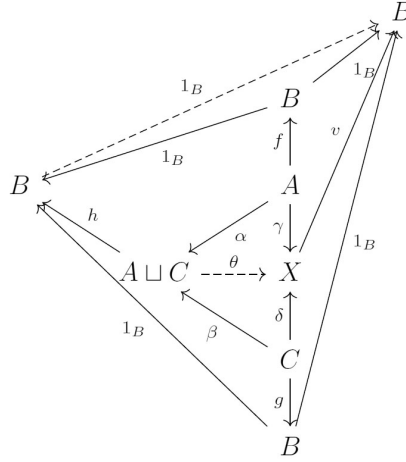
In category theory it is known<sup>1</sup> that, under these two conditions, for any morphisms  $\gamma \in \text{Mor}(E, F)$ ,  $\delta \in \text{Mor}(E, G)$ , with  $E, F, G \in \text{Ob}(\mathcal{C})$ , the pushout of  $\gamma$  and  $\delta$  exists and is constructable in the following way:

- (1) Construct the coproduct  $(F \sqcup G, i_F, i_G)$  of  $F$  and  $G$  with injections  $i_F : F \rightarrow F \sqcup G$  and  $i_G : G \rightarrow F \sqcup G$ . Then  $i_F \circ \gamma, i_G \circ \delta \in \text{Mor}(E, F \sqcup G)$ .
- (2) Construct the coequalizer  $(Q, \lambda)$  of maps  $i_F \circ \gamma$  and  $i_G \circ \delta$ , where  $\lambda \in \text{Mor}(F \sqcup G, Q)$ .
- (3) The triple  $(Q, \lambda \circ i_F, \lambda \circ i_G)$  is then the pushout of  $\gamma$  and  $\delta$ .

Now we continue with the definitions and descriptions of coproduct and coequalizer in the category  $\mathcal{S}(B)$ . The material about coproducts comes from [3] and the material about coequalizers comes from [2].

**Definition 2.** The **coproduct** of  $(A, f, B), (C, g, B) \in \text{Ob}(\mathcal{S}(B))$  is a triple  $((A \sqcup C, h, B), \alpha, \beta)$ , where  $(A \sqcup C, h, B) \in \text{Ob}(\mathcal{S}(B))$ ,  $\alpha \in \text{Mor}((A, f, B), (A \sqcup C, h, B))$ ,  $\beta \in \text{Mor}((C, g, B), (A \sqcup C, h, B))$  such that for every  $(X, j, B) \in \text{Ob}(\mathcal{S}(B))$  and every pair of morphisms  $\gamma \in \text{Mor}((A, f, B), (X, j, B))$  and  $\delta \in \text{Mor}((C, g, B), (X, j, B))$  there exists a unique morphism  $\theta \in \text{Mor}((A \sqcup C, h, B), (X, j, B))$  such that  $\theta \circ \alpha = \gamma$  and  $\theta \circ \beta = \delta$

<sup>1</sup> It could be obtained as the dual claim of Corollary 5.8 in [4], p. 82, for example.



In the following result, we need the notion of a tensor algebra, which is explained in more details in [3]. Suppose that  $(A, f, B), (C, g, B) \in \text{Ob}(\mathcal{S}(B))$ , let  $T$  be the tensor algebra of  $A$  and  $C$  and define a map  $h_T : T \rightarrow B$  as follows:

$$h_T(t) = \sum_{i=1}^n \sum_{j=1}^{k_i} \prod_{l=1}^{N_i} \tilde{h}_T(t_{i,j,l})$$

for every element

$$t = \bigoplus_{i=1}^n \sum_{j=1}^{k_i} t_{i,j,1} \otimes \cdots \otimes t_{i,j,N_i}$$

of  $T$ , where

$$\tilde{h}_T(t_{i,j,l}) = \begin{cases} f(t_{i,j,l}), & \text{if } t_{i,j,l} \in A \\ g(t_{i,j,l}), & \text{if } t_{i,j,l} \in C \end{cases}$$

On algebra  $T$  we consider the topology  $\tau_{h_T} = \{h_T^{-1}(U) : U \in \tau_B\}$ , where  $\tau_B$  denotes the topology of  $B$ . Then  $(T, \tau_{h_T})$  becomes a topological algebra and  $h_T$  becomes a continuous algebra homomorphism in the topology  $\tau_{h_T}$  (see [3] for details).

**Lemma 1.** Let  $(A, f, B), (C, g, B) \in \text{Ob}(\mathcal{S}(B))$  and let  $T$  be the tensor algebra of  $A$  and  $C$ . Define the map  $h_T : T \rightarrow B$  as in (0.1) and equip  $T$  with the topology  $\tau_{h_T}$ . Let  $I$  be the two-sided ideal of  $T$ , generated by the set

$$\{a_1 \otimes a_2 - a_1 a_2, c_1 \otimes c_2 - c_1 c_2 : a_1, a_2 \in A, c_1, c_2 \in C\}$$

and  $A \sqcup C = T/I$  be equipped with the quotient topology. Let  $\kappa_I : T \rightarrow T/I$  be the quotient map. Then the triple  $(A \sqcup C, h, B)$ , where  $h(\kappa_I(t)) = h_T(t)$  for every  $t \in T$  and every  $\kappa_I(t) \in A \sqcup C$ , is an object of the category  $\mathcal{S}(B)$ .

*Proof.* For the proof, see the proof of Lemma 2.2 in [3]. □

The next Proposition describes the coproducts of two elements in the category  $\mathcal{S}(B)$ .

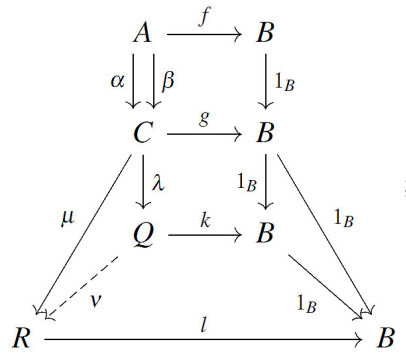
**Proposition 1.** For any  $(A, f, B), (C, g, B) \in \text{Ob}(\mathcal{S}(B))$ , their coproduct in  $\mathcal{S}(B)$  exists and is the triple  $((A \sqcup C, h, B), \alpha, \beta)$ , where  $(A \sqcup C, h, B)$  is the object of  $\mathcal{S}(B)$ , described in Lemma 1,  $\alpha : A \rightarrow A \sqcup C$  and  $\beta : C \rightarrow A \sqcup C$  are morphisms, defined by  $\alpha(a) = \kappa_I(a), \beta(c) = \kappa_I(c)$  for all  $a \in A$  and  $c \in C$ , where  $\kappa_I$  is the quotient map, defined in Lemma 1.

*Proof.* For the proof, see the proof of Proposition 3.2 in [3]. □

Now we move on to the coequalizers.

**Definition 3.** Let  $(A, f, B), (C, g, B) \in \text{Ob}(\mathcal{S}(B))$ . The **coequalizer** of morphisms  $\alpha, \beta \in \text{Mor}((A, f, B), (C, g, B))$  is a pair  $((Q, k, B), \lambda)$  such that

(1)  $(Q, k, B) \in \text{Ob}(\mathcal{S}(B))$  and  $\lambda \in \text{Mor}((C, g, B), (Q, k, B))$  with  $\lambda(\alpha(a)) = \lambda(\beta(a))$  for every  $a \in A$



(2) for any pair  $((R, l, B), \mu)$ , where  $(R, l, B) \in \text{Ob}(\mathcal{S}(B))$  and  $\mu \in \text{Mor}((C, g, B), (R, l, B))$  with  $\mu(\alpha(a)) = \mu(\beta(a))$  for every  $a \in A$ , there exists unique  $v \in \text{Mor}((Q, k, B), (R, l, B))$  with  $v \circ \lambda = \mu$ .

Next Proposition describes the coequalizers in the category  $\mathcal{S}(B)$ .

**Proposition 2.** Let  $(A, f, B), (C, g, B) \in \text{Ob}(\mathcal{S}(B))$  and  $I$  be the smallest two-sided ideal of  $C$ , generated by the set

$$M = \{\alpha(a) - \beta(a) : a \in A\}.$$

Then the coequalizer of morphisms  $\alpha, \beta \in \text{Mor}((A, f, B), (C, g, B))$  is the pair  $((C/I, \tilde{g}, B), p)$ , where  $\tilde{g} : C/I \rightarrow B$  is defined by  $\tilde{g}([c]) = g(c)$  for each  $[c] \in C/I$ ,  $p : C \rightarrow C/I$  is the canonical projection and  $C/I$  is equipped with the quotient topology

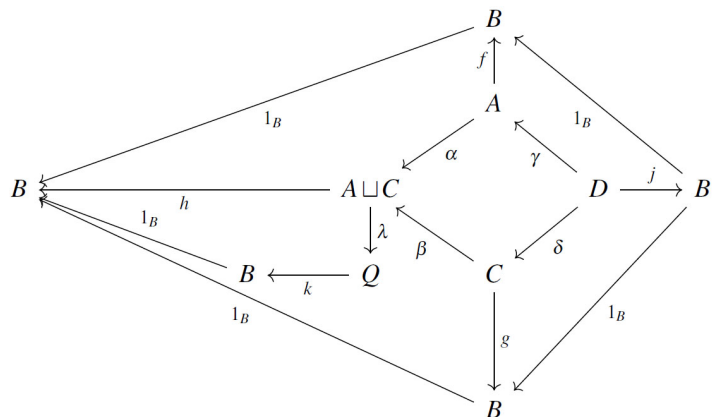
$$\tau_{C/I} = \{V \subseteq C/I : p^{-1}(V) \in \tau_C\}.$$

*Proof.* For the proof, see the proof of Theorem 10 in [2]. □

Thus, the conditions P1) and P2) are fulfilled for the category  $\mathcal{S}(B)$ . Hence, we can state the main result of this paper.

**Theorem 1.** The pushouts exist always in the category  $\mathcal{S}(B)$ .

To illustrate the situation, we give the following commutative diagram,



which describes the pushout of morphisms  $\gamma$  and  $\delta$ , if one compares this diagram with diagrams given in the Definitions 1–3 and takes  $\rho = \lambda \circ \alpha$  and  $\sigma = \lambda \circ \beta$  in the diagram of Definition 1.

## CONCLUSION

In this paper we showed that the pushouts always exist in the category  $\mathcal{S}(B)$ .

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## Väljatõukajatest Segali topoloogiliste algebrate kategoorias $\mathcal{S}(B)$

Mart Abel

Olgu  $B$  topoloogiline algebra. On näidatud, et Segali topoloogiliste algebrate kategoorias  $\mathcal{S}(B)$  leiduvad kõik väljatõukajad.