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STRUCTURAL ENGINEERING

An exact solution of truss vibration problems

Andres Lahe*, Andres Braunbrück*, and Aleksander Klauson

Department of Civil Engineering and Architecture, Tallinn University of Technology, Ehitajate tee 5

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Abstract. The Elements by a System of Transfer equations (EST) method offers exact solutions to various vibration problems of trusses, beams, and frames. The method can be regarded as an improved or modified transfer matrix method. Using the EST method, the roundoff errors generated by multiplying transfer arrays are avoided. It is assumed that the bars of trusses are connected by frictionless joints. Longitudinal vibration of a truss bar is described by a differential equation. In a direction perpendicular to the longitudinal axis, no bending can occur. In a transverse direction the rigid bar displacements vary linearly. The rigid bar rotational moment of inertia is taken into account. The transfer equations for the truss bar are presented. The transverse displacements at the joint (node) of an elastic and a rigid bar are equal. The essential boundary conditions at joints for the differential equation are the compatibility conditions of the displacements of truss elements. The natural boundary conditions at joints are the equilibrium equations of longitudinal elastic forces and transverse inertial forces of rigid bars.

Key words: truss bar vibration, transfer equations, essential boundary conditions at joints, natural boundary conditions at joints, master–slave connectivity, transverse inertial forces.

1. INTRODUCTION

In structural engineering, natural frequency analysis finds the natural or resonant frequencies and the mode shape of mechanical structures. Natural frequency analysis assumes that a structure vibrates in the absence of excitation and damping (the so-called free undamped vibration). This kind of analysis of a system is used to keep the natural frequencies away from excitation. For such analysis finite element techniques – the finite element method, boundary element method, transfer matrix method, a.o. methods – can be used. The transfer matrix method (TMM) in the natural frequency analysis for elastomechanical elements was used by Pestel and Leckie [1]. A good literature review about natural frequency analysis with the TMM can be found in He et al. [2]. The drawback of this method is numerical difficulties when the transfer matrix manipulation involves differences of large numbers and roundoff and truncation errors by the progressive multiplication of transfer matrices [3, p. 236]. This paper outlines the enhancement of the Elements by a System of Transfer equations (EST) method to include truss vibrational problems. The EST method [4–6] avoids the progressive multiplication of transfer matrices and the build-up of a sparse system of linear equations with scaling the displacements and rotations. After solving an asymmetric sparse system of linear equations, we unscale the initial parameter vectors (state vectors) of the elements.

* Corresponding authors, Andres.Lahe@ttu.ee, andres@ioc.ee

The equations system of the EST method contains transfer equations (solutions of differential equations), which are the basic equations of the method, and the following boundary conditions:
– compatibility equations of the displacements at nodes;
– joint equilibrium equations at nodes;
– side conditions (bending moment, axial and shear force hinges);
– support conditions (restrictions on support displacements).

Thus, the EST method can be regarded as an improved or modified TMM.

There is a wide range of ways to verify the correctness of the solutions obtained by the EST method [6] for a natural frequency analysis of bars, shafts, Euler–Bernoulli and Timoshenko beams, and frames, whereas data on natural frequency analysis for trusses are scarce.

For natural frequency analysis of truss structures, the EST method uses the *concept of master–slave connections*. The longitudinal vibrations of a truss bar are regarded as *master element* vibrations. The rotary inertia of a rigid bar is taken into account through a *slave element*, which vibrates transversely and is connected to the master bar at truss nodes. Considering this, the support reactions are as shown in Fig. 1b.

The natural frequencies for a truss structure are calculated by the EST method and are compared with true frequencies of natural vibration obtained through constraint equations by Ramsay [7]. Our results coincide exactly with those of [7, fig. 4].

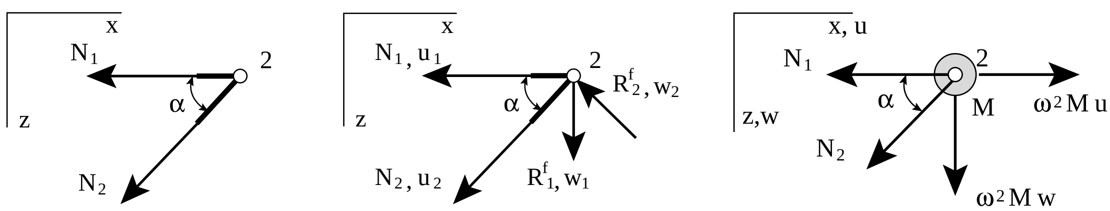
The natural frequencies for a fifteen-bar truss found by the adaptive generalized finite element method (adaptive GFEM) by Arndt [8, p. 243]) differ (e.g. by 48.3% for the frequences ω_1) from the natural frequencies for a fifteen-bar truss obtained by Braun et al. [9], cited in [10, p. 141]. As pointed out in the closure of the NAFEMS Benchmark Challenge problem 5 [7], such inaccuracy might be the result of the failure to take into account the rotary inertia of a rigid bar. To affirm this proposition, we computed the natural frequencies for a fifteen-bar truss decoupling master–slave connections (see Fig. 1a), i.e. the rotary inertia of a rigid bar was not considered. Ten digits in each of the 14 frequencies found coincided with the results obtained by the adaptive GFEM. We believe that we have confirmed the reason of inaccuracy.

The natural frequencies computed by the EST method for a seven-bar truss were compared with the frequencies obtained by the adaptive GFEM in [11, p. 207] and [8, p. 240], and were found to be different. We again computed the frequencies without master–slave connections and the results of the calculations for the six frequencies found matched in 10 digits. We believe that, once again, the reason of the inaccuracy has been proved.

In the finite element method (FEM), the mass of the truss bar is lumped to the nodes. Let the truss structure with massless bars (Fig. 1c) have the total lumped mass M at joint 2. The inertial forces $\omega^2 Mu$ and $\omega^2 Mw$ will appear in the structure vibrating at a frequency ω. To produce a lumped mass matrix in FEM, direct mass lumping and variational mass lumping that produce diagonally (directly) lumped mass (DLM) and consistent mass (CM), respectively, are widely used [12; 13, pp. 12–14].

The usual equilibrium equations for an unloaded two-bar joint (node 2, Fig. 1a) are

$$\Sigma X = 0, \ \ -N_1 - N_2 \cos\alpha = 0; \qquad \Sigma Z = 0, \ \ N_2 \sin\alpha = 0. \tag{1}$$



(a) Reactions of elastic elements at a truss node

(b) Reactions of elastic-rigid elements at a truss node

(c) Reactions of elastic and inertial forces at a truss node with lumped mass

Fig. 1. Reactions of elements at a truss node.

It follows that $N_1 = 0$ and $N_2 = 0$, which means that the elastic forces at the unloaded node are trivially equal to zero. These equations alone are not suitable as natural boundary conditions.

According to the d'Alembert principle, in truss vibration problems both the longitudinal elastic forces N_1, N_2 and the transverse inertial forces R_1^f, R_2^f of a rigid bar are necessary to find natural boundary conditions at a truss bars joint. Therefore, let us write the equations of joint equilibrium (natural boundary conditions) at truss node 2 (Fig. 1b):

$$\Sigma X = 0, \; -N_1 - N_2 \cos\alpha - R_2^f \sin\alpha = 0, \tag{2}$$

$$\Sigma Z = 0, \; N_2 \sin\alpha + R_1^f - R_2^f \cos\alpha = 0. \tag{3}$$

We use similar equilibrium equations as the boundary conditions at a truss bars joint.

The compatibility conditions of truss elements at the hinged joint 2 (Fig. 1b), i.e. the essential boundary conditions, are

$$u_1 - u_2 \cos\alpha - w_2 \sin\alpha = 0, \tag{4}$$

$$u_2 \sin\alpha - w_1 + w_2 \cos\alpha = 0. \tag{5}$$

These equations express the equality of the displacements of the truss bars in global coordinates at joint 2.

2. TRANSFER EQUATIONS OF A TRUSS BAR

2.1. Longitudinal vibration of a truss bar

Let us now apply the local right-handed coordinate system (x,z) to the truss element shown in Fig. 2 [14, p. 384]. Here, sign convention 2 [5, p. 18] is used for the directions of displacements and forces.

The transfer equations for longitudinal vibration of a *truss bar* (cf. axial vibration of a *bar* [15, p. 36]) are

$$u_L = u_A \cos\kappa\ell - \frac{N_A}{EA}\frac{1}{\kappa}\sin\kappa\ell, \tag{6}$$

$$w_L = w_A - \varphi_A \ell, \tag{7}$$

$$\varphi_L = \varphi_A, \tag{8}$$

$$N_L = -u_A EA\kappa \sin\kappa\ell - N_A \cos\kappa\ell, \tag{9}$$

where

u_A, u_L – axial displacements at the beginning A and at the end L of the element, respectively;
w_A, w_L – transverse displacements at A and L;
φ_A, φ_L – rotation at A and L;
N_A, N_L – axial forces at A and L;
$\kappa = \omega\sqrt{m/EA} = \omega\sqrt{\rho/E}$ – wavenumber;
ω – frequency of vibration;
$m = \rho A$ – mass per unit length;
ρ – mass density;
A – cross-sectional area;
E – elastic or Young's modulus;
ℓ – length of the element.

In this paper the *master element* is subjected to longitudinal vibrations of a truss bar and its vibrations can be written as tranfer equations in matrix form

$$\mathbf{Z_L} = \mathbf{U} \cdot \mathbf{Z_A}, \tag{10}$$

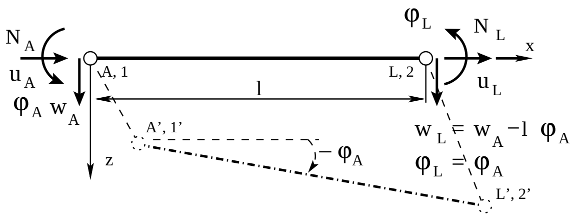


Fig. 2. Truss bar.

where $\mathbf{Z_L}$ and $\mathbf{Z_A}$ are the initial and end parameters (also called state vectors of the output and input end [2]),

$$\mathbf{Z_L} = \begin{bmatrix} u_L \\ w_L \\ \varphi_L \\ \cdots \\ N_L \end{bmatrix}, \qquad \mathbf{Z_A} = \begin{bmatrix} u_A \\ w_A \\ \varphi_A \\ \cdots \\ N_A \end{bmatrix}, \tag{11}$$

and the transfer matrix $\mathbf{U}$ is

$$\mathbf{U} = \begin{bmatrix} \cos\kappa\ell & 0 & 0 & \vdots & -\frac{1}{EA}\frac{1}{\kappa}\sin\kappa\ell \\ 0 & 1 & -\ell & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ \cdots & \cdots & \cdots & \vdots & \cdots \\ -EA\kappa\sin\kappa\ell & 0 & 0 & \vdots & -\cos\kappa\ell \end{bmatrix}. \tag{12}$$

2.2. Transverse vibration of a truss bar

The *slave element* is considered to be a rigid bar that vibrates transversely (Fig. 3). The degrees of freedom at connection nodes can be defined as masters and slaves [16]. Kinematic constraints between the slave and master degrees of freedom represent the equality of transverse displacements: $w_A^f = w_A$ and $w_L^f = w_L$ (see Fig. 3).

The transverse displacement $w(x)$ of the rigid bar of length ℓ varies linearly:

$$w(x) = \frac{\ell - x}{\ell} w_A + \frac{x}{\ell} w_L. \tag{13}$$

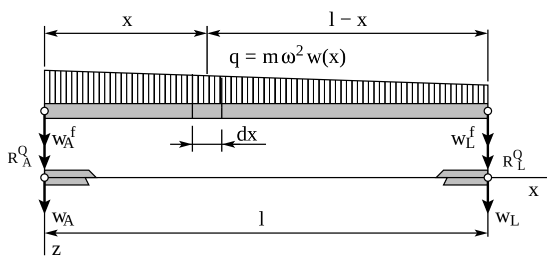


Fig. 3. Rigid bar.

Let the sum of moments about point L be equal to zero:

$$\Sigma M_L = 0; \qquad R_A^Q \ell + \int_0^\ell m\,\omega^2 w(x)\ (\ell - x)\,dx = 0. \tag{14}$$

Then the dynamic support reaction R_A^Q can be written as

$$R_A^Q = -\frac{1}{\ell}\int_0^\ell m\,\omega^2 w(x)\,(\ell - x)\,dx = -m\ell\omega^2\left(\frac{1}{3}w_A + \frac{1}{6}w_L\right). \tag{15}$$

Now we consider the sum of moments about point A to be equal to zero:

$$\Sigma M_A = 0; \qquad R_L^Q \ell + \int_0^\ell m\,\omega^2 w(x)\,x\,dx = 0. \tag{16}$$

Hence the dynamic support reaction R_L^Q can be written as

$$R_L^Q = -\frac{1}{\ell}\int_0^\ell m\,\omega^2 w(x)\,x\,dx = -m\ell\omega^2\left(\frac{1}{6}w_A + \frac{1}{3}w_L\right). \tag{17}$$

The support reactions R_A^Q and R_L^Q can also be found from the rigid bar transverse inertial force F^f and angular momentum M^f (see Fig. 4 and Eqs (20), (21)).

$$F^f = -m\ell\ddot{w}\left(\frac{\ell}{2}\right) = -m\ell\left[-\omega^2 w\left(\frac{\ell}{2}\right)\right] = m\ell\omega^2 w\left(\frac{\ell}{2}\right), \tag{18}$$

$$M^f = -\frac{m\ell^3}{12}\ddot{\varphi} = -\frac{m\ell^3}{12}\left(-\omega^2\varphi\right) = \frac{m\ell^3}{12}\omega^2\varphi, \tag{19}$$

$$\Sigma M_{yL}{=}0\ |\quad R_A^Q\ell{+}F^f\frac{\ell}{2} + M^f = 0, \tag{20}$$

$$\Sigma M_{yA}{=}0\ |\quad M^f - F^f\frac{\ell}{2} - R_L^Q\ell = 0, \tag{21}$$

$$\varphi \approx \tan\varphi = \frac{w_A - w_L}{\ell}. \tag{22}$$

Equations (10), (15), and (17) are the basic equations of the EST method [4,5] for a *truss master-and-slave element*.

$$\widehat{\mathbf{UI}}_{6\times 12}\cdot\widehat{\mathbf{Z}} = 0. \tag{23}$$

Here $\widehat{\mathbf{UI}}_{6\times 12}$ is the augmented transfer matrix.

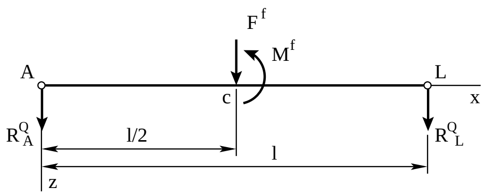


Fig. 4. Inertial force and angular momentum of a rigid bar.

$$
\widehat{\mathbf{UI}} = \begin{array}{c}
\begin{array}{ccccccc} u_A & w_A & \varphi_A & N_A & \vdots & R_A^Q & w_A^f \end{array} \\
\left[\begin{array}{ccccccc}
\cos\kappa\ell & 0 & 0 & -i_0\frac{1}{EA}\,\frac{1}{\kappa}\sin\kappa\ell & \vdots & 0 & 0 \\
0 & 1 & -\ell & 0 & \vdots & 0 & 0 \\
0 & 0 & 1 & 0 & \vdots & 0 & 0 \\
-\frac{1}{i_0}EA\kappa\sin\kappa\ell & 0 & 0 & -\cos\kappa\ell & \vdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & . & \cdots & \cdots \\
0 & 0 & 0 & 0 & \vdots & 1 & \frac{1}{i_0}m\ell\cdot\omega^2(1/3) \\
0 & 0 & 0 & 0 & \vdots & 0 & -\frac{1}{i_0}m\ell\cdot\omega^2(1/6)
\end{array}\right.
\end{array}
$$

$$
\begin{array}{c}
\begin{array}{ccccccc} u_L & w_L & \varphi_L & N_L & \vdots & R_L^Q & w_L^f \end{array} \\
\left.\begin{array}{ccccccc}
-1 & 0 & 0 & 0 & \vdots & 0 & 0 \\
0 & -1 & 0 & 0 & \vdots & 0 & 0 \\
0 & 0 & -1 & 0 & \vdots & 0 & 0 \\
0 & 0 & 0 & -1 & \vdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & . & \cdots & \cdots \\
0 & 0 & 0 & 0 & \vdots & 0 & \frac{1}{i_0}m\ell\cdot\omega^2(1/6) \\
0 & 0 & 0 & 0 & \vdots & -1 & -\frac{1}{i_0}m\ell\cdot\omega^2(1/3)
\end{array}\right]
\end{array} \tag{24}
$$

and

$$
\widehat{\mathbf{Z}} = \begin{bmatrix} \mathbf{Z_A} \\ \mathbf{Z_L} \end{bmatrix}, \tag{25}
$$

where

$$
\mathbf{Z_A} = \begin{bmatrix} u_A \\ w_A \\ \varphi_A \\ \cdots \\ N_A \\ R_A^Q \\ w_A^f \end{bmatrix}, \qquad \mathbf{Z_L} = \begin{bmatrix} u_L \\ w_L \\ \varphi_L \\ \cdots \\ N_L \\ R_L^Q \\ w_L^f \end{bmatrix}. \tag{26}
$$

The basic equation for a *truss master-and-slave element* with a CM or a DLM is

$$
\widehat{\mathbf{UI}}_{8\times 16}\cdot\widehat{\mathbf{Z}} = 0, \tag{27}
$$

where $\widehat{\mathbf{Z}}^{\mathbf{T}} = \left[u_A \;\; w_A \;\; \varphi_A \;\; N_A \;\; R_A^N \;\; R_A^Q \;\; u_A^f \;\; w_A^f \,\vdots\, u_L \;\; w_L \;\; \varphi_L \;\; N_L \;\; R_L^N \;\; R_L^Q \;\; u_L^f \;\; w_L^f\right]$.

Let us now consider how to form a matrix $\widehat{\mathbf{UI}}_{8\times16}$ from $\widehat{\mathbf{UI}}_{6\times12}$ of Eq. (24). We transform the transfer matrix (12) into a *master element* transfer matrix of a massless truss bar. The following limits ($\lambda = \kappa\ell \to 0$)

$$\begin{aligned} &\lim_{\lambda\to 0} U_{11} = \lim_{\lambda\to 0}\cos\lambda = 1, && \lim_{\lambda\to 0} U_{14} = -\tfrac{\ell}{EA}\lim_{\lambda\to 0}\tfrac{\sin\lambda}{\lambda} = -\tfrac{\ell}{EA},\\ &\lim_{\lambda\to 0} U_{41} = \lim_{\lambda\to 0} -\tfrac{EA}{\ell}\lambda\sin\lambda = 0, && \lim_{\lambda\to 0} U_{44} = \lim_{\lambda\to 0} -\cos\lambda = -1 \end{aligned} \tag{28}$$

give the transfer matrix for truss statics problems [14, p. 385]. We replace the converted components in matrix $\widehat{\mathbf{UI}}_{8\times16}$.

The longitudinally vibrating mass of Eq. (12) is taken into account through a slave element (rigid bar): $\lim_{EA\to\infty} U_{44} = \lim_{E\to\infty}\left(EA\,\omega\sqrt{\frac{\rho}{E}}\sin\left(\omega\,\ell\sqrt{\frac{\rho}{E}}\right)\right) = \rho A\ell\omega^2$.

We add into matrix (24), in addition to the transversal reactions R_A^Q and R_L^Q, the two equations of inertial forces (acceleration–reaction forces) R_A^N and R_L^N caused by consistent mass matrix (CMM) oscillations ω:

$$\begin{bmatrix} R_A^N \\ R_L^N \end{bmatrix} + \frac{\rho\cdot A\cdot\ell\cdot\omega^2}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u_A^f \\ u_L^f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} R_A^Q \\ R_L^Q \end{bmatrix} + \frac{\rho\cdot A\cdot\ell\cdot\omega^2}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} w_A^f \\ w_L^f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{29}$$

In the truss master-and-slave element with diagonally (directly) lumped mass matrix (DLMM) the inertial forces (acceleration–reaction forces) R_A^N and R_L^N from matrix $\widehat{\mathbf{UI}}_{8\times16}$ were replaced by the inertial forces below:

$$\begin{bmatrix} R_A^N \\ R_L^N \end{bmatrix} + \frac{\rho\cdot A\cdot\ell\cdot\omega^2}{6}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} u_A^f \\ u_L^f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} R_A^Q \\ R_L^Q \end{bmatrix} + \frac{\rho\cdot A\cdot\ell\cdot\omega^2}{6}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} w_A^f \\ w_L^f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{30}$$

2.3. Modal analysis of a truss

In a modal analysis, for the system of equations (31) the load vector is set to zero. To get the nontrivial solution $\mathbf{\Phi}_i$ of the homogeneous system (31), we will choose a free variable in accordance with the natural frequency ω_i of a truss structure.

$$\mathbf{spA}\,(\omega_i)\cdot\mathbf{\Phi}_i = 0, \tag{31}$$

where the components of vector $\mathbf{\Phi}_i$ are $\Phi_{1,i}$, $\Phi_{2,i}$, ..., $\Phi_{N,i}$. Here N is the number of components of the state vectors and support reactions.

The work of support reactions for a conservative system is zero ($C_i\cdot\Delta_i = 0$). Here the support displacement $\Delta_i = 0$, or the support reaction $C_i = 0$ (side conditions enforce zero support reactions at a sliding or roller support). In a master–slave element model the rigid body (slave element) motion is coupled with the elastic element (master element) deformation at nodes except for the truss support nodes [16]. At a support node in the constraint direction of an elastic truss we rule this connection out. We also restrain the rigid body (slave element) displacement in that direction. In joint equilibrium equations (boundary conditions) in constraint direction we do not include inertial force ($R_i^{N\vee Q} = 0$).

The linear system (31) is homogeneous. To obtain a non-trivial solution, we pick a value ω_i that will make the determinant of the system singular:

$$\det(\mathbf{spA}\,(\omega_i)) = 0, \tag{32}$$

where ω_i denotes different natural frequencies, characteristic or normal values of the truss structure, and are found here numerically by the bisection method. These values are conventionally arranged in sequence from smallest to largest ($\omega_1 < \omega_2 < \ldots\omega_n$).

For all the frequencies picked out from (32), the given boundary conditions and transfer equations are met. Equation (32) for bars, shafts, and beams gives the expression for frequency equation or characteristic equation [6].

The truss master-and-slave element with a CM was tested with a fifteen-bar truss presented in [9; 10, p.141], whose eigenvalues were computed by the Jacobi method with a CMM. All 14 natural frequencies found by the EST method coincide exactly with those found by Braun et al. [9], cited in [10, p. 141]. A GNU Octave script compiled and solved the linear equations involving sparse matrices (`Compressed Column Sparse (rows = 244, cols = 244, nnz = 698 [1.2%])`) for a fifteen-bar truss relatively quickly.

The basic equation (27) for a truss master-and-slave element is used in our next examples to find the natural frequencies with CMM and DLMM lumping.

The results of the calculation are presented in Tables 1 and 2. The density of the sparse matrix **spA** in the linear system of equations (44) for finding boundary values **Z** with CMM lumping is 2.6% (`Compressed Column Sparse (rows = 116, cols = 116, nnz = 350 [2.6%])`) and with DLMM lumping 2.4% (`Compressed Column Sparse (rows = 116, cols = 116, nnz = 322 [2.4%])`).

Example 2.1 (free vibrations of a truss structure). Find the natural frequencies of the truss structure shown in Fig. 5.

Length ℓ and height h of a cantilever truss structure are 1.0 m, cross-sectional area of a bar is $A_r = A_d = 3.14\times10^{-4}\,\mathrm{m}^2$, elastic or Young's modulus $E = 210\,\mathrm{GPa}$, and mass density $\rho = 7.80\times10^3\,\mathrm{kg/m}^3$ (cf. [7, fig. 1]).
Solution. The system of EST method equations is

$$\mathbf{spA}\cdot\mathbf{Z} = 0, \tag{33}$$

where **Z** is the vector of unknowns that includes support reactions C_1, C_2, C_3, and C_4:

$$\mathbf{Z} = \begin{bmatrix} Z(1)\\ Z(2)\\ Z(3)\\ Z(4)\\ Z(5)\\ Z(6)\\ \cdots\\ Z(25)\\ Z(26)\\ Z(27)\\ Z(28) \end{bmatrix} = \begin{bmatrix} u_A^{(1)}\\ w_A^{(1)}\\ \varphi_A^{(1)}\\ N_A^{(1)}\\ R_A^{Q(1)}\\ w_A^{f(1)}\\ \cdots\\ C_1\\ C_2\\ C_3\\ C_4 \end{bmatrix}. \tag{34}$$

The components of the state vector $Z(i)$ (i = 1, 2, ..., 28) in the index notation are shown in Fig. 6.

In the system of EST method equations (33), the first 12 equations represent the basic equation of the EST method for a truss Eq. (23) (see Fig. 7). The next four equations coupling transverse displacements (displacements indices are shown in Fig. 6) express master–slave connections (see Fig. 7):

$$\begin{aligned} w_A^{(1)} &= w_A^{f(1)}, & Z(2) &= Z(6),\\ w_L^{(1)} &= w_L^{f(1)}, & Z(8) &= Z(12),\\ w_A^{(2)} &= w_A^{f(2)}, & Z(14) &= Z(18),\\ w_L^{(2)} &= w_L^{f(2)}, & Z(20) &= Z(24). \end{aligned} \tag{35}$$

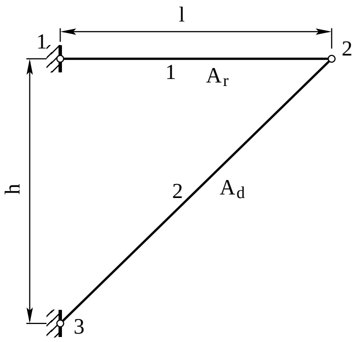


Fig. 5. Truss structure.

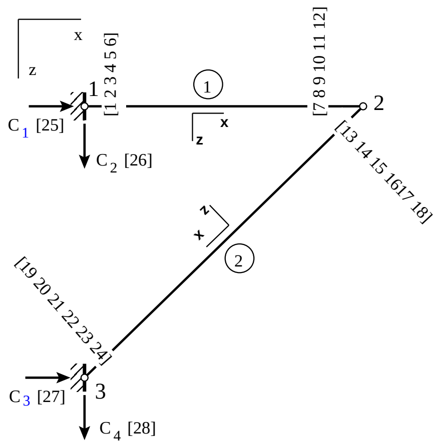


Fig. 6. Indices of state vector components.

The following 12 equations are those of displacements compatibility, joint equilibrium at nodes, and the support conditions (restrictions on support displacements).

The equation of compatibility (see Fig. 7) at node 2:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_L^{(1)} \\ w_L^{(1)} \end{bmatrix} - \begin{bmatrix} -0.70711 & -0.70711 \\ 0.70711 & -0.70711 \end{bmatrix} \begin{bmatrix} u_A^{(2)} \\ w_A^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{36}$$

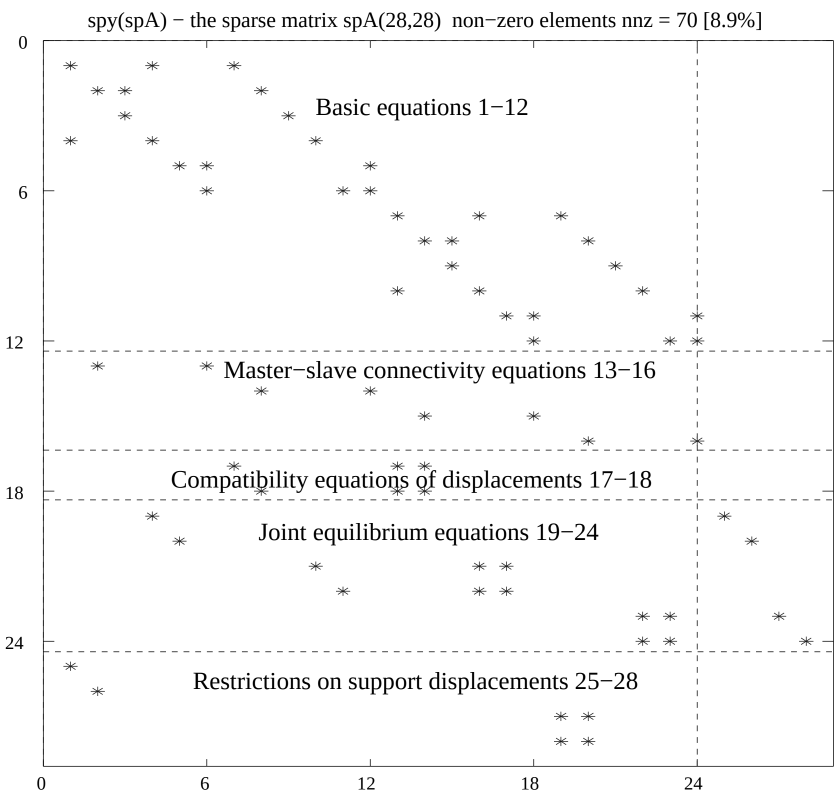


Fig. 7. Sparsity pattern of matrix spA of the cantilever truss structure.

Equations of joint equilibrium (see Fig. 7) at
node 1:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} N_A^{(1)} \\ R_A^{Q(1)} \end{bmatrix} - \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{37}$$

node 2:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} N_L^{(1)} \\ R_L^{Q(1)} \end{bmatrix} + \begin{bmatrix} -0.70711 & -0.70711 \\ 0.70711 & -0.70711 \end{bmatrix} \begin{bmatrix} N_A^{(2)} \\ R_A^{Q(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{38}$$

node 3:

$$\begin{bmatrix} -0.70711 & -0.70711 \\ 0.70711 & -0.70711 \end{bmatrix} \begin{bmatrix} N_L^{(2)} \\ R_L^{Q(2)} \end{bmatrix} - \begin{bmatrix} C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{39}$$

Equations of support conditions (restrictions on support displacements) (see Fig. 7) at
node 1:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_A^{(1)} \\ w_A^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{40}$$

node 3:

$$\begin{bmatrix} -0.70711 & -0.70711 \\ 0.70711 & -0.70711 \end{bmatrix} \begin{bmatrix} u_L^{(2)} \\ w_L^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{41}$$

The sparsity pattern of the coefficient matrix spA of the system of equations (33) is shown in Fig. 7.

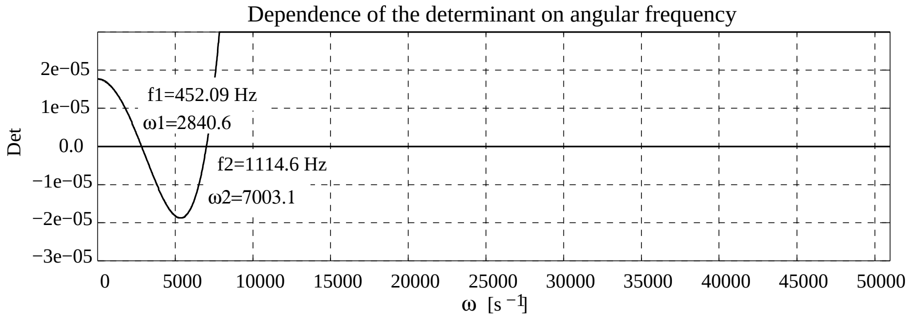
Next we find the frequencies when the determinant of the coefficient matrix of equations (33) is zero, i.e. $\det(\mathbf{spA}) = 0$. The first six natural frequencies of the truss structure (Fig. 5) are shown in Table 1. With CM and DLM, the natural frequencies are calculated by the EST method [6, p. 205].

Figure 8 b shows the dependence of the determinant of the coefficient matrix (33) on the angular frequency ω of the cantilever truss structure. Figures 8 a and 8 c depict the dependence of the determinant on the angular frequency calculated by the EST method with CM and DLM.

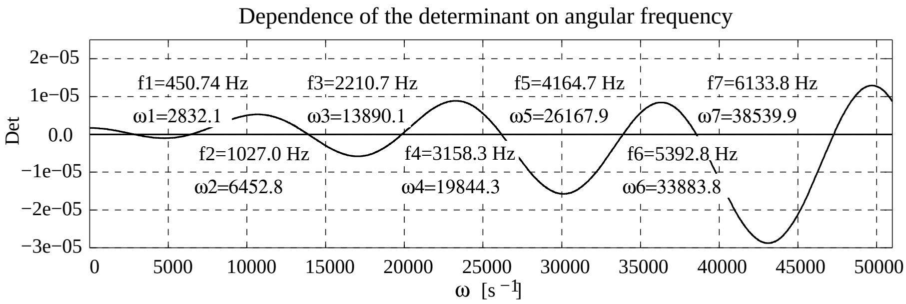
The natural frequencies calculated by the EST method with a GNU Octave script coincide with the exact ones in [7].

Table 1. Natural frequencies of the truss structure

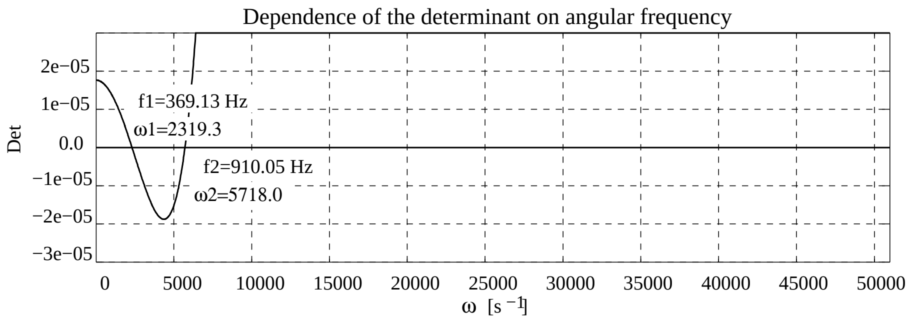
Natural frequency	EST method					
	DLM lumping frequency f [Hz]	Difference from DLM [%]	DM frequency f [Hz]	Difference from CM [%]	CM lumping frequency f [Hz]	DM frequency $\omega\,[\mathrm{s}^{-1}]$
1	369.13	18.11	450.74	– 0.30	452.09	2 832.1
2	910.05	11.39	1 027.00	– 8.53	1 114.60	6 452.8
3			2 210.70			13 890.1
4			4 164.70			19 844.3
5			580.00			26 167.9
6			5 392.80			33 883.8



(a) Consistent mass (CM) $\rho A\,(\ell_1+\ell_2)/3$



(b) Distributed mass (DM) in the longitudinal direction of the bar



(c) Directly lumped mass (DLM) $\rho A\,(\ell_1+\ell_2)/2$

Fig. 8. Natural frequencies of the truss structure.

2.4. Mode displacements of the truss structure

For the nontrivial solution $\boldsymbol{\Phi}_i$ of the homogeneous system (42), we will choose a free variable in accordance with the natural frequency ω_i of a truss structure.

$$\mathbf{spA}\,(\omega_i)\cdot\boldsymbol{\Phi}_i = 0, \tag{42}$$

where the components of vector $\boldsymbol{\Phi}_i$ are $\Phi_{1,i}$, $\Phi_{2,i}$, ..., $\Phi_{N,i}$. Here N is the number of components of the state vectors and support reactions (in this example, N = 28).

The next step is to find a vector $\boldsymbol{\Phi}_i$ that matches ω_i. If we scale the free variable $\Phi_{j,i}$ by any real number, the scaled nontrivial solution $\boldsymbol{\Phi}_i$ is still a solution to Eq. (42).

Next we extract the state vectors input ends (initial parameters) $\mathbf{Z_A}$ (11) of elements from vector $\boldsymbol{\Phi}_i$. The mode shape displacements of the elements are computed with the transfer equations (10).

In a similar way, the mode shape displacements of bars, shafts, and beams of the Euler–Bernoulli and Timoshenko beam theories were computed in [6]. The results obtained in all of these cases are consistent with the results found in the literature. For the mode shape of frames, longitudinal and transversal displacements were determined. Due to the difference (ca 400-fold) of the values of the axial rigidity EA and the flexural rigidity EI of the rigid portal frames observed, the longitudinal and transversal displacements were significantly different. The boundary conditions guarantee the compatibility of the displacements in global coordinates.

To express the mode–shape longitudinal and transversal displacements of the truss structure bar, the transfer equations (10) are used. To find the true input ends (initial parameters) of the state vectors, we implement the dynamic support reactions $R_L^{f(1)} = -m\ell_1\omega^2\frac{1}{3}w_L^{(1)}$ (17) and $R_A^{f(2)} = -m\ell_2\omega^2\frac{1}{3}w_A^{(2)}$ (15) at node 2 (38). In other words, at node 2, there is variational mass lumping in local coordinates.

For the free scaled variables of the nontrivial solution $\mathbf{\Phi}_i$ of the homogeneous system (42), we give the following values: $\Phi_{18,1} = 1.0$, $\Phi_{12,2} = 0.24668$, $\Phi_{12,3} = 0.5$, $\Phi_{18,4} = 0.5$, $\Phi_{12,5} = 0.5$, $\Phi_{18,7} = 0.2$, $\Phi_{18,7} = 0.2$, ..., where $\Phi_{12,i}$ and $\Phi_{18,i}$ are the displacements $w_L^{f(1)}$ and $w_A^{f(2)}$ of slave elements 1 and 2, respectively (see Fig. 6). The extracted state vectors (initial parameters) of the transfer equations (10) for computing ith mode shape displacements of bar 1 are $\mathbf{Z_A^{(1)}} = [\Phi_{1,i}\ \Phi_{2,i}\ \Phi_{3,i}\ \Phi_{4,i}]^T$, and of bar 2, $\mathbf{Z_A^{(2)}} = [\Phi_{13,i}\ \Phi_{14,i}\ \Phi_{15,i}\ \Phi_{16,i}]^T$ (see Fig. 6).

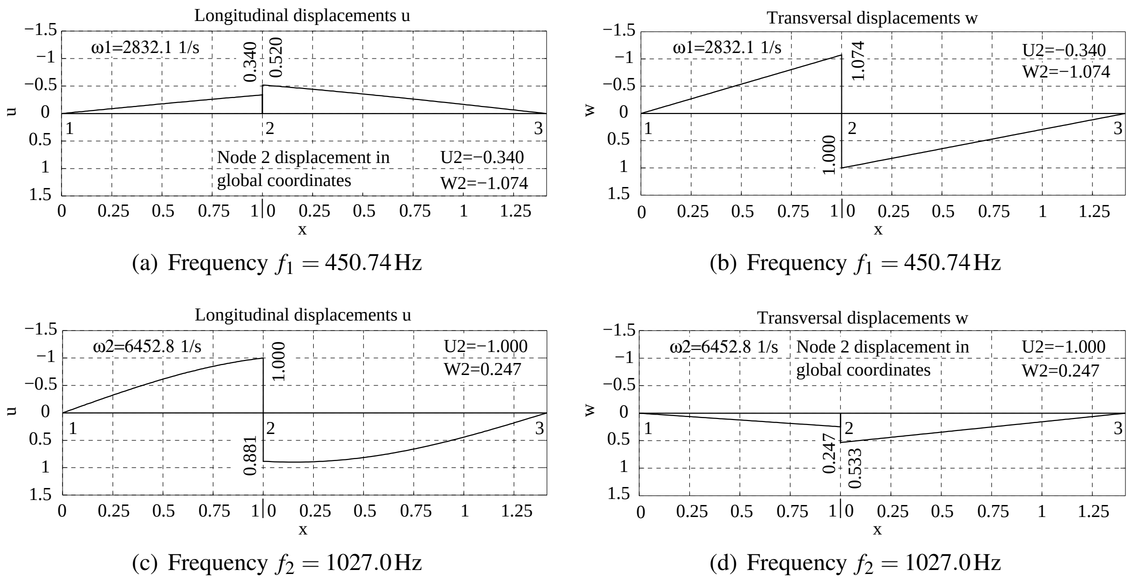
To check the compatibility of the displacements at node 2, we transform the displacements of bar 2 from the local coordinate system to the global coordinate system (Fig. 6).

$$\begin{bmatrix} U \\ W \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\cos\beta \\ \cos\beta & \cos\alpha \end{bmatrix} = \begin{bmatrix} -0.70711 & -0.70711 \\ 0.70711 & -0.70711 \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}. \tag{43}$$

Figure 9 depicts the displacements of the first five natural modes of the truss structure.

The displacements U2 and W2 at node 2 of both bars in global coordinates are equal.

The first three natural mode longitudinal displacements (Fig. 9 a, c, and e) are comparable to those in [7, fig. 5].



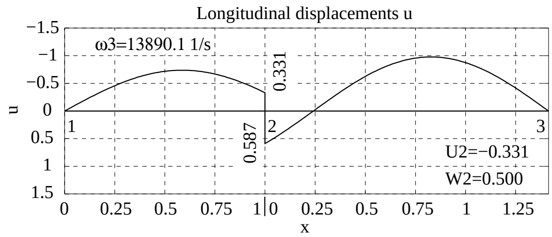
(a) Frequency $f_1 = 450.74\,$Hz

(b) Frequency $f_1 = 450.74\,$Hz

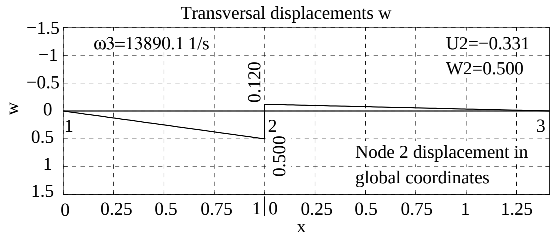
(c) Frequency $f_2 = 1027.0\,$Hz

(d) Frequency $f_2 = 1027.0\,$Hz

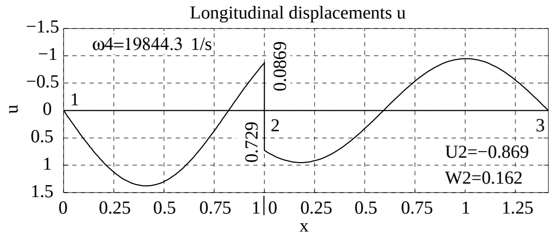
Fig. 9. Natural modes of a truss structure. (*Continued on the next page.*)



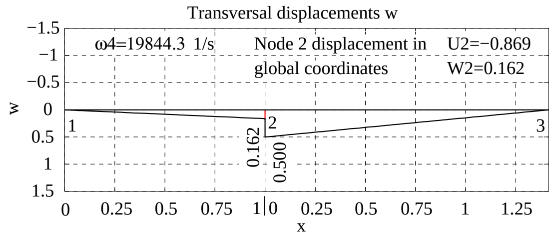
(e) Frequency $f_3 = 2210.7\,\mathrm{Hz}$



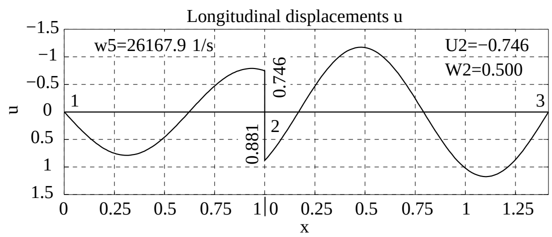
(f) Frequency $f_3 = 2210.7\,\mathrm{Hz}$



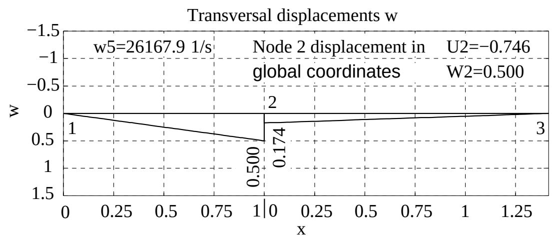
(g) Frequency $f_4 = 3158.3\,\mathrm{Hz}$



(h) Frequency $f_4 = 3158.3\,\mathrm{Hz}$



(i) Frequency $f_5 = 4164.7\,\mathrm{Hz}$



(j) Frequency $f_5 = 4164.7\,\mathrm{Hz}$

Fig. 9. *Continued.*

Example 2.2 (free vibrations of a seven-bar truss). Find the natural frequencies of the seven-bar truss shown in Fig. 10.

The truss span $\ell = 4.0$ m and height $h = 2.0$ m, panel length $d = 2.0$ m, cross-sectional area of bar is $A = 1.0\times10^{-3}$ m^2, elastic or Young's modulus $E = 210\,\mathrm{GPa}$, and mass density $\rho = 8.0\times10^3$ kg/m^3 (cf. [11, p. 207]).

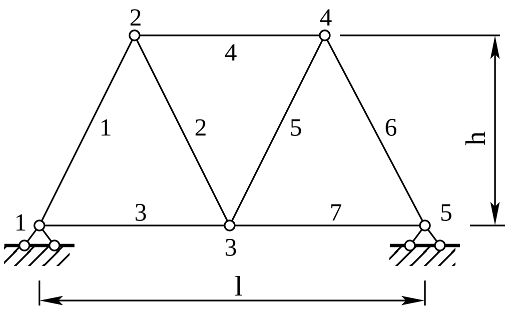


Fig. 10. Seven-bar truss.

Solution. To calculate the natural frequencies of the seven-bar truss, the EST method is used. The system of the EST method equations is

$$\mathbf{spA} \cdot \mathbf{Z} = 0, \tag{44}$$

where $\mathbf{Z}$ is the vector of unknowns that includes support reactions C_1, C_2, C_3, and C_4:

$$\mathbf{Z} = \begin{bmatrix} Z(1) \\ Z(2) \\ Z(3) \\ Z(4) \\ Z(5) \\ Z(6) \\ \cdots \\ Z(85) \\ Z(86) \\ Z(87) \\ Z(88) \end{bmatrix} = \begin{bmatrix} u_A^{(1)} \\ w_A^{(1)} \\ \varphi_A^{(1)} \\ N_A^{(1)} \\ R_A^{f(1)} \\ w_A^{f(1)} \\ \cdots \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}. \tag{45}$$

The components of the state vector $Z(i)$ (i = 1, 2, ..., 88) in the index notation are shown in Fig. 11.

In the system of equations (44), the first 42 equations represent the basic equation (23) of the EST method for a truss. The next 14 equations coupling transverse displacements (Fig. 11) express master–slave connectivity. The following 18 equations describe the compatibility of displacements. The next ten equations are joint equilibrium equations. The last four equations are support conditions – restrictions on support displacements.

The sparsity pattern of the coefficient matrix spA of the system of equations (44) is shown in Fig. 12.

With the GNU Octave script we find the frequencies when the determinant of the coefficient matrix of equations (44) is zero. The first eleven natural frequencies of the seven-bar truss (Fig. 10) are given in Table 2. In addition to the frequencies presuming distributed mass (DM), the frequencies of the truss determined with CM and DLM are presented.

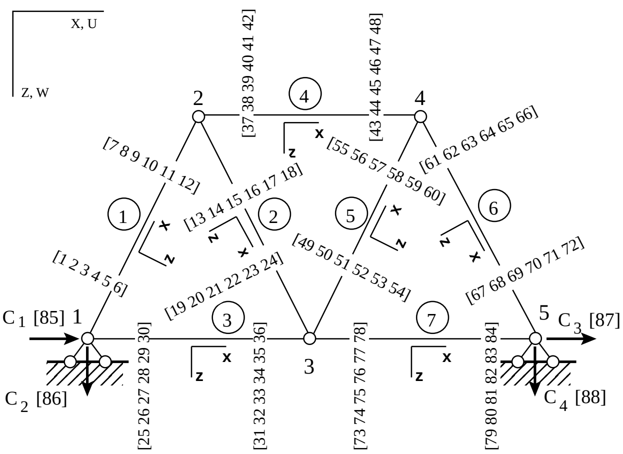


Fig. 11. Indices of state vector components for a seven-bar truss with distributed mass.

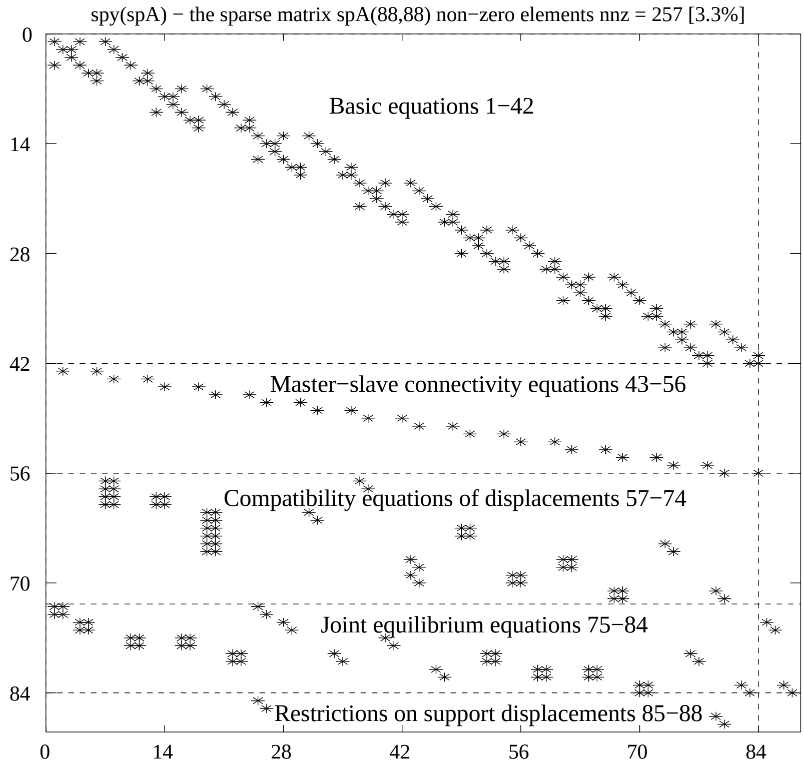


Fig. 12. Sparsity pattern of the matrix spA of a seven-bar truss.

Table 2. Natural frequencies of the seven-bar truss

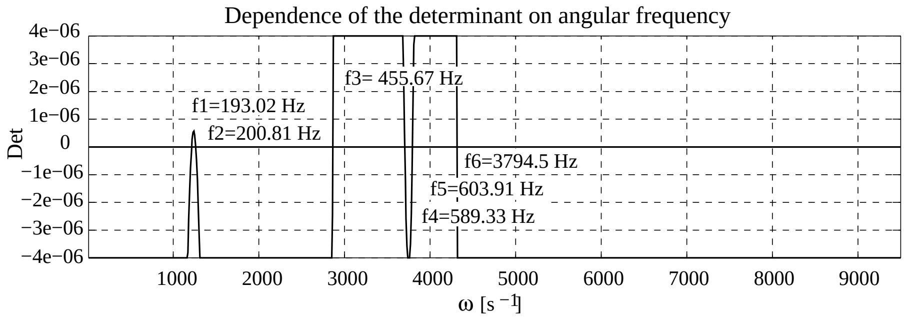
Natural frequency	EST method					
	DLM lumping frequency f [Hz]	Difference from DLM [%]	DM frequency f [Hz]	Difference from CM [%]	CM lumping frequency f [Hz]	DM frequency $\omega\,[\mathrm{s}^{-1}]$
1	176.72	7.91	191.89	− 0.59	193.02	1 205.659502
2	181.40	9.19	199.76	− 0.53	200.81	1 255.115771
3	344.44	21.29	437.62	− 4.12	455.67	2 749.664532
4	451.49	20.833	570.30	− 3.34	589.33	3 583.285360
5	466.14	19.631	580.00	− 4.12	603.91	3 644.248531
6	516.23	24.135	680.46	− 0.96	686.98	4 275.458665
7			1 168.50			7 342.131902
8			1 280.90			8 047.936312

In Table 3, the calculation results of the seven-bar truss without master–slave connections are presented. The first six frequencies determined by the EST method coincide exactly with those obtained by the adaptive GFEM in [11, p. 208].

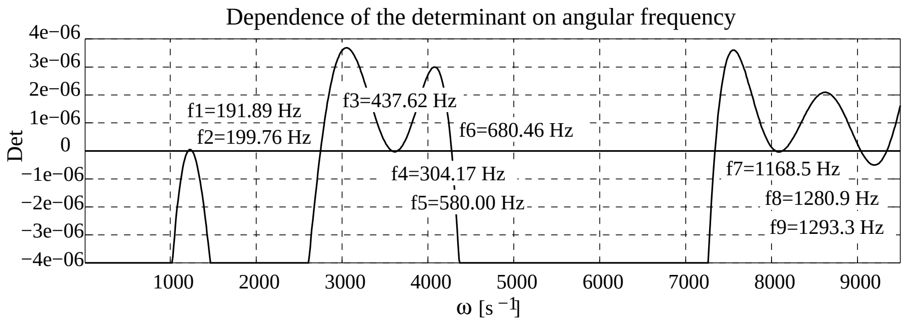
Figure 13 b shows the dependence of the determinant of the coefficient matrix of Eq. (44) on the angular frequency ω of the seven-bar truss. Figures 13 a and 13 c depict the dependence of the determinant on the angular frequency calculated by the EST method with CM and DLM, respectively.

Table 3. Calculation results of the seven-bar truss without master–slave connections

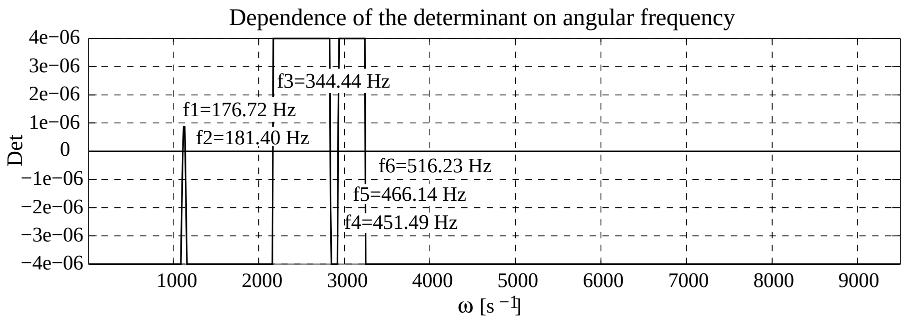
Pseudonatural frequency	EST method	
	Frequency f [Hz]	Frequency ω $\left[s^{-1}\right]$
1	262.25	1 647.784428
2	277.06	1 740.839797
3	495.18	3 111.322715
4	726.04	4 561.817307
5	767.64	4 823.248678
6	1 174.50	7 379.482322
7	1 193.50	7 499.144048



(a) Consistent mass (CM)



(b) Distributed mass (DM) in the longitudinal direction of the bar



(c) Directly lumped mass (DLM)

Fig. 13. Natural frequencies of the seven-bar truss.

Computing diary excerpt 2.1 (TrussAMShapes.m)

```
ModeShape 1
wf =  1205.7, baasi0=E, E=2.1E+11 # Pa
# baasi0 - scaling multiplier for displacements
===============================================
Displacements in global coordinates X-Z at the
beginning and end of the bar
  No        UA        WA        UL        WL
-----------------------------------------------

   1     0.000    -0.000     0.166     0.657
   2     0.166     0.657    -0.000     1.000
   3     0.000    -0.000    -0.000     1.000
   4     0.166     0.657    -0.166     0.657
   5    -0.000     1.000    -0.166     0.657
   6    -0.166     0.657    -0.000    -0.000
   7    -0.000     1.000    -0.000    -0.000
-----------------------------------------------
```

Computing diary excerpt 2.2 (TrussAMShapes.m)

```
ModeShape 2
wf =  1255.1, baasi0=E, E=2.1E+11 # Pa
# baasi0 - scaling multiplier for displacements
===============================================
Displacements in global coordinates X-Z at the
beginning and end of the bar
  No        UA        WA        UL        WL
-----------------------------------------------

   1     0.000    -0.000     1.000     0.074
   2     1.000     0.074     0.248     0.000
   3     0.000    -0.000     0.248     0.000
   4     1.000     0.074     1.000    -0.074
   5     0.248     0.000     1.000    -0.074
   6     1.000    -0.074     0.000     0.000
   7     0.248     0.000     0.000     0.000
-----------------------------------------------
```

2.5. Mode displacements of a seven-bar truss

For the nontrivial solution $\mathbf{\Phi}_i$ of the homogeneous system (46), we will choose a free variable in accordance with the natural frequency ω_i of a seven-bar truss.

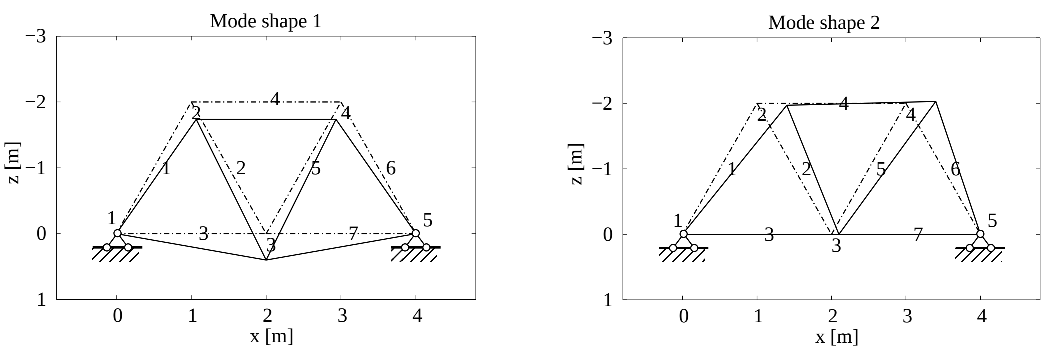
$$\mathbf{spA}\,(\omega_i)\cdot\mathbf{\Phi}_i = 0, \tag{46}$$

where the components of vector $\mathbf{\Phi}_i$ are $\Phi_{1,i}$, $\Phi_{2,i}$, ..., $\Phi_{N,i}$. Here N is the number of components of the state vectors and support reactions (in this example, N = 88).

The next step is to find a vector $\mathbf{\Phi}_i$ that matches ω_i. If we scale the free variable $\Phi_{j,i}$ by any real number, the scaled nontrivial solution $\mathbf{\Phi}_i$ will still be a solution to Eq. (46).

Next we extract the input ends (initial parameters of the state vectors) $\mathbf{Z_A}$ and $\mathbf{Z_L}$ (11) of the elements from vector $\mathbf{\Phi}_i$. We calculate the transversal displacement shapes 1–10 for the seven-bar truss modes with the master–slave connection (see Fig. 14 and computing diary excerpts 2.1, 2.2).

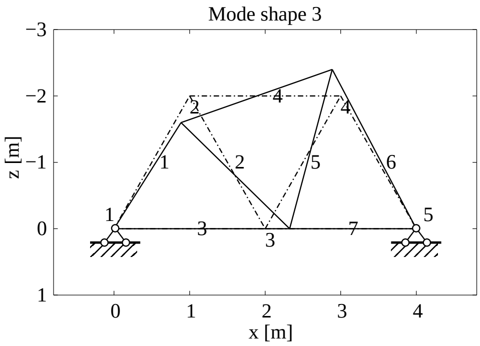
The rotary inertia of a rigid bar was not taken into account in calculating frequencies in [8, p. 241; 17, p. 653] and displacement shapes for seven-bar truss modes in [17, p. 654].



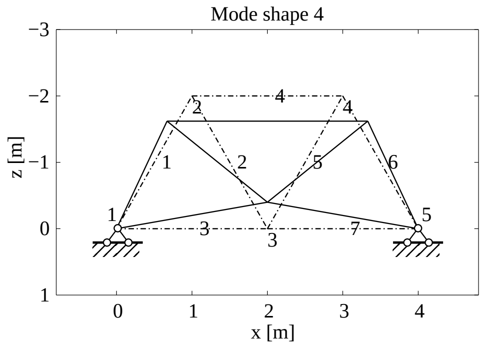
(a) Shape at $\omega_1 = 1.205659502 \times 10^3\,\mathrm{s}^{-1}$

(b) Shape at $\omega_2 = 1.255115771 \times 10^3\,\mathrm{s}^{-1}$

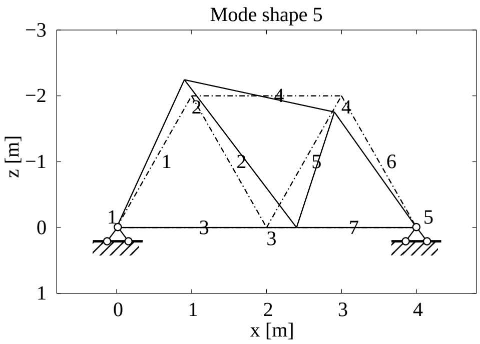
Fig. 14. Transversal displacement shapes of bars for seven bar truss modes. (*Continued on the next page.*)



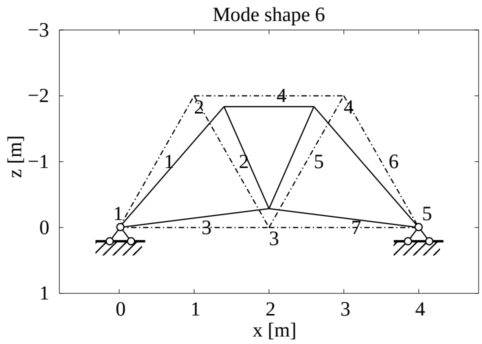
(c) Shape at $\omega_3 = 2.749664532 \times 10^3\,\mathrm{s}^{-1}$



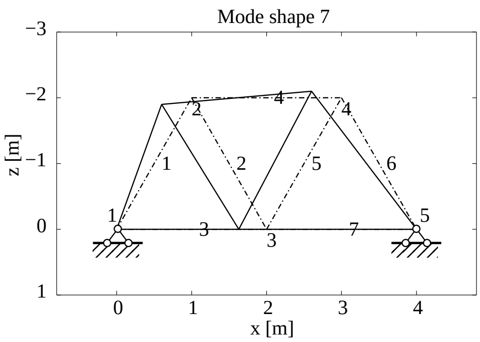
(d) Shape at $\omega_4 = 3.583285360 \times 10^3\,\mathrm{s}^{-1}$



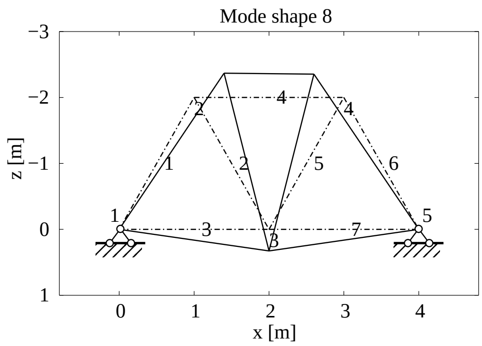
(e) Shape at $\omega_5 = 3.644248531 \times 10^3\,\mathrm{s}^{-1}$



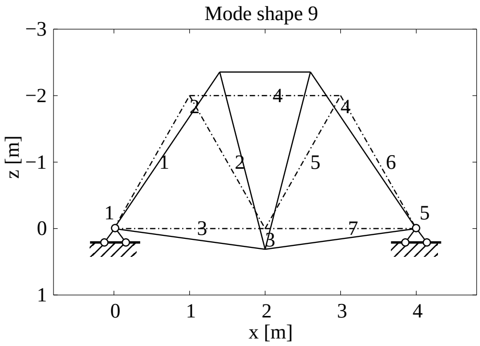
(f) Shape at $\omega_6 = 4.275458665 \times 10^3\,\mathrm{s}^{-1}$



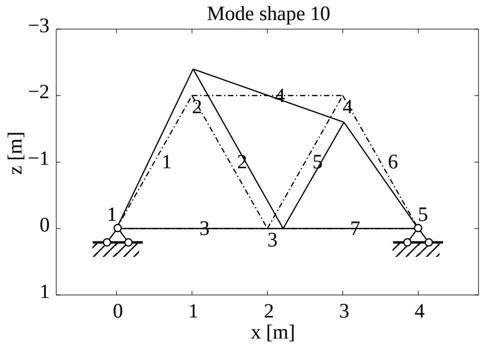
(g) Shape at $\omega_7 = 7.342131902 \times 10^3\,\mathrm{s}^{-1}$



(h) Shape at $\omega_8 = 8.047936312 \times 10^3\,\mathrm{s}^{-1}$



(i) Shape at $\omega_9 = 8.125812636 \times 10^3\,\mathrm{s}^{-1}$



(j) Shape at $\omega_{10} = 9.041134841 \times 10^3\,\mathrm{s}^{-1}$

Fig. 14. *Continued.*

3. CONCLUSIONS

Transfer equations for truss bar vibration have been developed. An exact solution for boundary value problems of truss vibration is presented. The method is exact as no approximations were made to derive the transfer equations and, in principle, all the natural frequencies can be found by finding more roots for the determinant of the coefficient matrix. In addition, the differential equations related to the system of transfer equations are all solved as exactly as needed by the bisection method. The equilibrium equations of longitudinal and transverse inertial forces of the bar serve as natural boundary conditions at joints. In the transverse direction, no bending can occur (the bar remains straight), the displacements vary linearly, and transversely the bar vibrates as a rigid body.

By using the EST method, the solutions gained by the directly lumped mass (DLM) or consistent mass (CM) matrices – approximations often used by the finite element analysis – give either lower (DLM) or higher (CM) natural frequencies than the distributed mass (DM) solution.

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Sõrestiku võnkumisülesande täpne lahendus

Andres Lahe, Andres Braunbrück ja Aleksander Klauson

On välja töötatud EST-meetodi rakendus sõrestike omavõnkesageduste täpseks määramiseks. Sõrestiku varda ülekandevõrrandid on koostatud vastavalt sõrestikskeemi määratlusele. Sõrestiku varraste ülekandevõrrandid ühendatakse süsteemiks oluliste ehk kinemaatiliste rajatingimuste ja loomulike ehk staatiliste rajatingimuste abil. Saadud hõreda ülekandevõrrandite süsteemi determinandi nullid määravad täpselt sõrestiku omavõnkesagedused. Meetodit võib nimetada täpseks, kuna sõrestiku definitsiooni piires pole varda ülekandevõrrandite süsteemi tuletamisel tehtud lihtsustusi ega ole lähendatud ühtki süsteemi diferentsiaalvõrrandit. Omavõnkesagedused on määratud numbriliselt, kasutades determinandi nullkoha määramiseks lõigu poolitamise meetodit. Lõplike elementide meetodiga arvutades koondatakse sõrestiku varraste mass sõlmedesse. EST-meetodiga võrreldes saadakse siin konsistentse massimaatriksiga arvutades suuremad omavõnkesagedused ja diagonaalile keskendatud massimaatriksiga arvutades väiksemad omavõnkesagedused.