



Popov form and the explicit equations of inverse systems

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Abstract. The paper addresses the invertibility problem for discrete-time nonlinear control systems, described by the input–output equations. The necessary and sufficient conditions for the existence of left and right inverse systems are given. The explicit equations of inverse systems are found by transforming the system equations into the strong Popov form with respect to inputs. The results are obtained under the assumption that the equations are transformable into the strong Popov form using linear equivalence transformations over the field of meromorphic functions.

Key words: discrete-time systems, nonlinear systems, input–output models, non-commutative polynomials, strong Popov form.

1. INTRODUCTION

In [7] the problem of right inversion is addressed for nonlinear control systems, described by the set of input–output (*i/o*) difference equations. The solution of the problem, that is necessary and sufficient conditions of right invertibility, is given based on the inversion algorithm (IA), extended for this class of systems. The IA is traditionally expressed in a form involving the implicit function theorem (IFT). However, in [7] the IA is presented in terms of differential one-forms, exactly like in [5] for nonlinear systems, described by the state equations. Therefore, the algorithm does not use the IFT. This form of the IA is certainly efficient for checking invertibility. To find the explicit equations of the right inverse, one has to integrate the set of one-forms, obtained at the last step of the IA, which may be a difficult task.

In this paper an alternative approach is suggested, based on the strong Popov form with respect to the control variables of the set of *i/o* equations. One can easily find the explicit equations of the inverse system when the set of original equations will be transformed into the strong Popov form with respect to the control variable. Note that our results are obtained under the assumption that the equations are transformable into the strong Popov form using *linear* equivalence transformations, see [9]. Our results address also the left inversion problem, not studied so far for this class of systems to the authors' knowledge. The new approach is computationally more efficient and transparent, though both approaches result, in principle, in the same

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equations of the inverse system¹. As shown via the example, our results agree with those developed for nonlinear systems, described in terms of the state equations. However, our results are more general since not all nonlinear i/o equations are realizable in the state space form. According to our knowledge, the approach is also new for linear systems.

2. PRELIMINARIES

2.1. I/o equations

Consider a discrete-time multi-input multi-output nonlinear system, described by the explicit set of higher-order difference equations, relating the inputs $u_k, k = 1, \dots, m$, the outputs $y_i, i = 1, \dots, p$, and a finite number of their time shifts:

$$y_i(t + n_i) = \phi_i \left(y_j(t), \dots, y_j(t + n_{ij}), u_l(t), \dots, u_l(t + s_{il}) \right), \quad i = 1, \dots, p, \quad (1)$$

where $j = 1, \dots, p, \iota = 1, \dots, m$, and ϕ_i are real meromorphic functions. The word ‘explicit’ means that the variable $y_i(t + n_i)$ does not appear on the right-hand side of the i th equation, i.e. $n_{ii} < n_i$. It is assumed that system (1) is strictly causal, i.e. $s_{il} < n_i$. The functions ϕ_i are defined on an open and dense subset of $\mathbb{R}^{(n+1)(p+m)}$, whereas $n = \max n_i$.

Definition 1. *The set of i/o equations (1) is said to be in the strong Popov form with respect to the output if*

- (a) $n_1 \leq n_2 \leq \dots \leq n_p$;
- (b) for all $\phi_i, i = 1, \dots, p$ the following conditions hold:
 - (i) $n_{ik} < n_i$ if $k \leq i$;
 - (ii) $n_{ik} \leq n_i$ if $k > i$;
 - (iii) $n_{ki} < n_i$ if $k \neq i$.

Compared with the definition of the strong Popov form for implicit equations in [2], we have made in Definition 1 technical simplification $j_i = i$ for the explicit equation (1). With this additional assumption condition (v) in the strong Popov form (see [2]) is always satisfied. This assumption allows us to avoid double indices and does not bring along any restrictions since this is always doable by renumbering the output coordinates², see also Remark 3.

Assumption 1. System (1) is assumed to be in the strong Popov form with respect to output.

We associate a multiplicative set S_Σ with system (1). If (1) involves any denominators, then these denominators have to be included in S_Σ together with their shifts and powers. The typically infinite set S_Σ can be generated by a *finite* generator set S_Σ^0 . The set S_Σ^0 generates S_Σ if each element of S_Σ can be obtained from a finite number of elements of S_Σ^0 by applying a finite number of multiplications and backward and forward shifts to these elements. If (1) does not include any denominators, then we set $S_\Sigma = \{1\}$ and only in this case S_Σ is a finite set (i.e. $S_\Sigma = S_\Sigma^0$).

Hereinafter we use the notation ζ for a variable $\zeta(t), \zeta^{[k]}$ for its k -step time shift $\zeta(t + k), k \in \mathbb{Z}$. In such notation (1) takes the form

$$\Sigma : y_i^{[n_i]} = \phi_i \left(y_j, \dots, y_j^{[n_{ij}]}, u_l, \dots, u_l^{[s_{il}]} \right), \quad i = 1, \dots, p, \quad (2)$$

where $\iota = 1, \dots, m$ and $j = 1, \dots, p$.

¹ It has to be mentioned that both the inversion algorithm and transformation of equations into the strong Popov form allow some freedom in their choices, not affecting the invertibility property but possibly the inverse system equations if $m \neq p$.

² Note that renumbering the output coordinates is not a row operation on globally linearized system equations.

We also consider the equations in the form

$$\Sigma_u : u_k^{[\sigma_k]} = \chi_k \left(u_1, \dots, u_l^{[\sigma_{kl}]}, y_j, \dots, y_j^{[v_{kj}]} \right), \quad k = 1, \dots, \mu, \quad (3)$$

where $l = 1, \dots, m$, $j = 1, \dots, p$, and $\mu = \min(m, p)$. Together with Σ_u the multiplicative set S_{Σ_u} is considered.

Definition 2. *I/o equations (3) are said to be in the strong Popov form with respect to the input if*

- (a) $\sigma_1 \leq \sigma_2 \leq \dots \leq \sigma_\mu$;
- (b) for all χ_k , $k = 1, \dots, \mu$ the following conditions hold:
 - (i) $\sigma_{k\ell} < \sigma_k$ if $\ell \leq k$;
 - (ii) $\sigma_{k\ell} \leq \sigma_k$ if $\ell > k$;
 - (iii) $\sigma_{\ell k} < \sigma_k$ if $\ell \neq k$.

In Definition 2, like in Definition 1, we have made a technical simplification for the explicit equations that in the k th equation of Σ_u the variable u_k appears with the highest shift.

With system Σ , described by equations (2), we associate a vector function $\Phi := [\tilde{\phi}_1, \dots, \tilde{\phi}_p]^T$, where $\tilde{\phi}_i := y_i^{[n_i]} - \phi_i(\cdot)$. The system Σ defines the inversive difference field \mathcal{Q}_Σ with the shift operator δ_Σ . In particular, the shift of $y_i^{[n_i-1]}$ is defined by the right-hand side of equation (2), see more in [2]³. Each element of \mathcal{Q}_Σ is the image of a meromorphic function under the map e_Σ . Basically the map e_Σ allows us to exclude (or include) the zeros, defined by equations (2), from (into) the elements of the field \mathcal{Q}_Σ , and in this way to find the simplest representatives of the functions in \mathcal{Q}_Σ . See [2] for a precise definition and Example 1 below.

2.2. Non-commutative polynomials

The field \mathcal{Q}_Σ and the shift operator δ_Σ induce the ring of non-commutative polynomials in a variable Z over \mathcal{Q}_Σ , denoted by $\mathcal{Q}_\Sigma[Z; \delta_\Sigma]$. The multiplication is defined by the linear extension of the following rules:

$$Z \cdot a := (\delta_\Sigma a)Z \quad \text{and} \quad a \cdot Z := aZ,$$

where $a \in \mathcal{Q}_\Sigma$ and $\delta_\Sigma a$ means δ_Σ evaluated at a (so for example $(aZ^\mu) \cdot (bZ^\nu) = a(\delta_\Sigma^\mu b)Z^{\mu+\nu}$).

Let $\mathcal{Q}_\Sigma[Z, \delta_\Sigma]^{p \times q}$ be the set of $p \times q$ -dimensional matrices, whose entries are polynomials in $\mathcal{Q}_\Sigma[Z, \delta_\Sigma]$. Let us denote the i th row of the matrix $W \in \mathcal{Q}_\Sigma[Z, \delta_\Sigma]^{p \times q}$ by $w_{i\bullet}$. For the non-zero row $w_{i\bullet}$ we define its degree $\deg w_{i\bullet} \equiv \sigma_i$ as the exponent of the highest power in Z present in $w_{i\bullet}$ for $i = 1, \dots, p$. If $w_{i\bullet} \equiv 0$, we define $\sigma_i = -\infty$. The vector of the row degrees is denoted by $\sigma := [\sigma_1, \dots, \sigma_p]$. The degree of the matrix W is defined as $\deg W := \max\{\sigma_1, \dots, \sigma_p\}$. Let $N = \deg W$, $e = [1, \dots, 1]$, and $M = [m_1, \dots, m_p] := N \cdot e - \sigma$. By Z^M we denote the diagonal $p \times p$ matrix with the diagonal elements Z^{m_1}, \dots, Z^{m_p} .

Definition 3. *The matrix $L(W)$ such that $Z^M W = L(W)Z^N +$ lower degree terms is called the leading row coefficient matrix of W .*

Definition 4. [9,11] *A polynomial matrix $W \in \mathcal{Q}_\Sigma[Z, \delta_\Sigma]^{p \times q}$ with non-zero rows is called row-reduced if its leading row coefficient matrix $L(W)$ has full row rank⁴ over the field \mathcal{Q}_Σ . If W contains zero rows, then W is called row-reduced if its submatrix consisting of non-zero rows is row-reduced.*

Definition 5. [11] *Matrix $W \in \mathcal{Q}_\Sigma[Z, \delta_\Sigma]^{p \times q}$ is in the Popov form if W is row-reduced with the rows sorted with respect to their degrees ($\sigma_1 \leq \dots \leq \sigma_p$) and for all non-zero rows $w_{i\bullet}$ there is a column index j_i (called the pivot index) such that*

³ In [2] the notations \mathcal{Q}_Σ^Φ and δ_Φ were used respectively for \mathcal{Q}_Σ and δ_Σ .

⁴ The matrix $W \in \mathcal{Q}_\Sigma[Z, \delta_\Sigma]^{p \times q}$ is said to have full row rank if $\text{rank} W = \min(p, q)$.

- (i) w_{ij} is monic;
- (ii) $\deg w_{ik} < \deg w_{ij}$ if $k < j$;
- (iii) $\deg w_{ik} \leq \deg w_{ij}$ if $k \geq j$;
- (iv) $\deg w_{kj} < \deg w_{ij}$ if $k \neq i$;
- (v) if $\deg w_{ij} = \deg w_{jk}$ and $i < k$, then $j_i < j_k$ (if the degrees of the rows are equal, then the pivot indices are increasing).

Proposition 1. [11] For any matrix $W \in \mathcal{Q}_\Sigma[Z, \delta_\Sigma]^{p \times q}$ there exists a unimodular matrix $U \in \mathcal{Q}_\Sigma[Z, \delta_\Sigma]^{p \times p}$ such that $U \cdot W$ is in the Popov form.

2.3. Linearized i/o equations

Our goal is to represent system (2) in terms of polynomials from $\mathcal{Q}_\Sigma[Z; \delta_\Sigma]$. For that purpose we apply the differential operation d to equations (2) to obtain

$$dy_i^{[n_i]} - \sum_{j=1}^p \sum_{\alpha=0}^n \frac{\partial \phi_i}{\partial y_j^{[\alpha]}} dy_j^{[\alpha]} - \sum_{t=1}^m \sum_{\beta=0}^n \frac{\partial \phi_i}{\partial u_t^{[\beta]}} du_t^{[\beta]} = 0 \tag{4}$$

for $i = 1, \dots, p$. The polynomial variable Z is interpreted as the shift operator. Defining $Z^\alpha dy_j := dy_j^{[\alpha]}$ and $Z^\beta du_t := du_t^{[\beta]}$ allows us to rewrite (4) as

$$P(Z)dy + Q(Z)du = 0, \tag{5}$$

where $P \in \mathcal{Q}_\Sigma[Z; \delta_\Sigma]^{p \times p}$ and $Q \in \mathcal{Q}_\Sigma[Z; \delta_\Sigma]^{p \times m}$ are polynomial matrices, whose elements $p_{ij}, q_{it} \in \mathcal{Q}_\Sigma[Z; \delta_\Sigma]$ are

$$p_{ij} = \delta_{ij} Z^{n_i} - \sum_{\alpha=0}^n \frac{\partial \phi_i}{\partial y_j^{[\alpha]}} Z^\alpha, \quad q_{it} = - \sum_{\beta=0}^n \frac{\partial \phi_i}{\partial u_t^{[\beta]}} Z^\beta, \tag{6}$$

δ_{ij} is the Kronecker symbol, and $dy = [dy_1, \dots, dy_p]^T$, $du = [du_1, \dots, du_m]^T$. Equation (5) describes the (globally) linearized system, associated with equations (2). For P and Q we define the equivalence classes $\bar{P} := e_\Sigma(P)$, where $e_\Sigma(P)_{ij} := e_\Sigma(p_{ij})$, $\bar{Q} := e_\Sigma(Q)$, where $e_\Sigma(Q)_{it} := e_\Sigma(q_{it})$. The following example explains the action of operator e_Σ .

Example 1. Consider the system taken from [9]:

$$y_1^{[2]} + \frac{y_2^{[2]} y_3^{[1]}}{y_1} - \frac{y_2^{[2]} u_3}{y_1} - u_1 = 0, \quad y_1^{[3]} u_1 + y_2^{[1]} - u_1^{[1]} u_1 - u_2 = 0, \quad y_3^{[1]} - u_3 = 0. \tag{7}$$

By (5) and (6) the matrix

$$P = \begin{pmatrix} Z^2 - \frac{y_2^{[2]}(y_3^{[1]} - u_3)}{y_1} & \frac{y_3^{[1]} - u_3}{y_1} Z^2 & \frac{y_2^{[2]}}{y_1} Z \\ u_1 Z^3 & Z & 0 \\ 0 & 0 & Z \end{pmatrix}.$$

Next we define the equivalence class \bar{P} . To find one of the simplest representatives of the class, we take into account system equations (7) in matrix P and obtain

$$\hat{P} := \begin{pmatrix} Z^2 & 0 & \frac{y_2^{[2]}}{y_1} Z \\ u_1 Z^3 & Z & 0 \\ 0 & 0 & Z \end{pmatrix}.$$

In examples we often make computations using representatives of elements from classes. By abuse of notation we then write $\bar{P} = \hat{P}$, where \bar{P} is the equivalence class and \hat{P} is the representative of this class.

Let us define the action of the ring $\mathcal{Q}_\Sigma[Z; \delta_\Sigma]$ on the field \mathcal{Q}_Σ by the linear extension of the formula $Z^s \uparrow a := \delta_\Sigma^s a$, where $a \in \mathcal{Q}_\Sigma$. Note that $Z \uparrow a = \delta_\Sigma a$, as the action of the polynomial on the element of the field, is different from $Za = (\delta_\Sigma a)Z$ as a product of polynomials Z and a in the ring of polynomials. Assume that the unimodular matrix U transforms the matrix Q into the Popov form. The matrix $U \in \mathcal{Q}_\Sigma[Z, \delta_\Sigma]^{p \times p}$, applied as an operator to system equations (2), i.e. $U \uparrow \Phi$, is called the *linear equivalence transformation*.

Assumption 2. It is assumed that system (1) (or equivalently, system (2)) can be transformed into the strong Popov form (3) with respect to input u using linear equivalence transformations.

3. RIGHT INVERSE SYSTEM

In this section it is assumed that $p \leq m$. Consider the set of i/o equations in the strong Popov form (2), satisfying Assumptions 1 and 2 together with the associated set S_Σ . Denote by \mathbf{u} a control sequence $\{u(t), t \geq 0\}$ and by \mathbf{y} the output sequences $\{y(t), t \geq 0\}$. We also consider the system

$$\Lambda : u_k^{[\sigma_k]} = G_k(u_1, \dots, u_i^{[\sigma_{k1}]}, y_j, \dots, y_j^{[v_{kj}]}), \quad k = 1, \dots, p, \tag{8}$$

where $i = 1, \dots, m, j = 1, \dots, p$ in the strong Popov form with respect to $u = [u_1, \dots, u_m]^T$. Note that the systems Λ and Σ_u have similar structure and both are related to the original system Σ . However, they are introduced for different purposes and do not have to coincide. In particular, we assume that $p \leq m$ for Λ . Together with Λ we consider a multiplicative set S_Λ . Let S be the smallest multiplicative set containing S_Σ and S_Λ .

- The pair (\mathbf{y}, \mathbf{u}) is acceptable with respect to the multiplicative set S (shortly S -acceptable) if for any $\psi \in S$ and any $t \geq 0, \psi(y(t), \dots, y(t + p_\psi), u(t), \dots, u(t + m_\psi)) \neq 0$.
- The sequence \mathbf{y} is S -acceptable if there is \mathbf{u} such that (\mathbf{y}, \mathbf{u}) is an S -acceptable pair.

Remark 1. If S is finitely generated, then the set of S -acceptable pairs (\mathbf{y}, \mathbf{u}) is generic in the following sense. For every $k \geq 0$ the set of finite sequences $(y(0), \dots, y(k), u(0), \dots, u(k))$, obtained from an acceptable pair (\mathbf{y}, \mathbf{u}) , is open and dense in some subset of $\mathbb{R}^{(k+1)(p+m)}$. This follows from the fact that non-acceptable pairs (\mathbf{y}, \mathbf{u}) satisfy a finite number of analytic equations. For a similar reason for an acceptable \mathbf{y} there is a generic set of sequences \mathbf{u} such that (\mathbf{y}, \mathbf{u}) is an acceptable pair.

Definition 6. System Λ is a right inverse of Σ if for any S -acceptable $\tilde{\mathbf{y}}$ there exists $\tilde{\mathbf{u}}$ such that $(\tilde{\mathbf{y}}, \tilde{\mathbf{u}})$ is acceptable and $(\tilde{\mathbf{y}}, \tilde{\mathbf{u}})$ solves Λ and after substituting $\mathbf{u} = \tilde{\mathbf{u}}$ to Σ and setting $y_i(k) = \tilde{y}_i(k), k = 0, \dots, n_i - 1$, we get the solution \mathbf{y} of Σ , satisfying $y_i(k) = \tilde{y}_i(k), k \geq n_i$. The right inverse of Σ is denoted by Σ_R^{-1} .

Proposition 2. For an S -acceptable sequence $\tilde{\mathbf{y}}$ there are infinitely many solutions $\tilde{\mathbf{u}}$ of Λ such that $(\tilde{\mathbf{y}}, \tilde{\mathbf{u}})$ is S -acceptable. They correspond to a generic set of initial values $u_k(l), k = 1, \dots, p; l = 0, \dots, \sigma_k - 1$ and parameters $u_\kappa(l), \kappa = p + 1, \dots, m, l \geq 0$.

Proof. Follows from Remark 1. □

Definition 7. System (2) is said to be right invertible if its right inverse system Σ_R^{-1} exists in the sense of Definition 6.

From the above, if the desired trajectory for the system Σ is fed into the right inverse system Σ_R^{-1} , then the outputs of the right inverse generate the inputs $\tilde{\mathbf{u}}$, resulting in $\tilde{\mathbf{y}}$ at the output of Σ ; see Fig. 1.

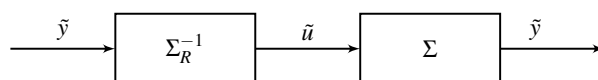


Fig. 1. Right inverse.

Definition 8. [3] The rank of the matrix $W \in \mathcal{Q}_\Sigma[Z, \delta_\Sigma]^{p \times m}$, denoted as $\rho(W)$, is defined to be equal to the maximum number of $\mathcal{Q}_\Sigma[Z, \delta_\Sigma]$ -linearly independent rows of W .

Lemma 1. ([4], Lemma 3.7) Multiplying the matrix $W \in \mathcal{Q}_\Sigma[Z, \delta_\Sigma]^{p \times m}$ by the unimodular matrix $U \in \mathcal{Q}_\Sigma[Z, \delta_\Sigma]^{p \times p}$ from the left does not change its rank, i.e. $\rho(W) = \rho(UW)$.

Theorem 1. Let $p \leq m$. Under Assumption 2 system (2) is right invertible iff $\rho(\bar{Q}) = p$.

Proof. Necessity. The proof is by contradiction. Assume that (2) is right invertible but, contrary to the claim, $\rho(\bar{Q}) < p$. If $\rho(\bar{Q}) < p$, then by Proposition 1 and Lemma 1, the matrix $\tilde{Q} = U\bar{Q}$ in the Popov form has at least one zero row. Thus globally linearized system equations (5) involve relation between differentials dy_1, \dots, dy_p , solely, not involving any of inputs u_1, \dots, u_m . Such a system is not right invertible by Definitions 6 and 7, since one cannot guarantee that $y_i(k) = \tilde{y}_i(k)$, $k \geq n_i$.

Sufficiency. Assume that $\rho(\bar{Q}) = p$ and show that then system (2) is right invertible. According to Definitions 6 and 7, this is so when one can provide the rules for computing the input sequence $\{\tilde{u}(t), t \geq 0\}$ such that $y(t) = \tilde{y}(t)$ for $t \geq 0$. Following [2,9], one can find the matrix $\tilde{Q} := U\bar{Q}$ in the Popov form with no zero rows. Under the assumptions of the theorem, by Lemma 1, also $\rho(\tilde{Q}) = p$. The application of the linear equivalence transformation U to equations (2) yields the system in the strong Popov form (3) with respect to the inputs. We show that (3) together with the multiplicative set S_{Σ_u} is really the right inverse of (2). Thus we set S to be the smallest multiplicative set containing S_Σ and S_{Σ_u} . For the sake of transparency the remaining part of the proof is presented for the multi-input single-output case, where $p = 1$ and $m = 2$. The simplification does not change the idea of the proof. Let Σ be given by

$$\tilde{\phi}(y^{[n]}, \xi, u_1^{[s_1]}, u_2^{[s_2]}) := y^{[n]} - \phi(\xi, u_1^{[s_1]}, u_2^{[s_2]}) = 0, \tag{9}$$

where $\xi = (y, y^{[1]}, \dots, y^{[n-1]}, u_1, \dots, u_1^{[s_1-1]}, u_2, \dots, u_2^{[s_2-1]})$. Assume that⁵ $s_1 \geq s_2$. Due to Assumption 2 one can transform (9) via linear equivalence transformation to

$$\tilde{\psi}(y^{[n]}, \xi, u_1^{[s_1]}, u_2^{[s_2]}) := u_1^{[s_1]} - \psi(\xi, u_2^{[s_2]}, y^{[n]}) = 0, \tag{10}$$

where $\tilde{\psi} = \alpha\tilde{\phi}$ for $e_\Sigma(\alpha) \in \mathcal{Q}_\Sigma$. We will show that (10) is the right inverse of Σ . Let $\tilde{\mathbf{u}}_1$ be the solution of

$$\tilde{\psi}(\tilde{y}^{[n]}, \xi, u_1^{[s_1]}, \tilde{u}_2^{[s_2]}) := u_1^{[s_1]} - \psi(\xi, \tilde{u}_2^{[s_2]}, \tilde{y}^{[n]}) = 0$$

for S -acceptable $\tilde{\mathbf{y}}$, some initial values $u_1(0), \dots, u_1(s_1 - 1)$, and some $\tilde{u}_2(k)$, $k \geq 0$, such that $(\tilde{\mathbf{y}}, \tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2)$ is S -acceptable (by Proposition 2). So $(\tilde{\mathbf{y}}, \tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2)$, where $\tilde{\xi} = (\tilde{y}, \dots, \tilde{y}^{[n-1]}, \tilde{u}_1, \dots, \tilde{u}_1^{[s_1-1]}, \tilde{u}_2, \dots, \tilde{u}_2^{[s_2-1]})$ satisfies

$$\tilde{u}_1^{[s_1]} - \psi(\tilde{\xi}, \tilde{u}_2^{[s_2]}, \tilde{y}^{[n]}) = 0. \tag{11}$$

Let \mathbf{y} be the solution of (9) for this $(\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2)$ and initial conditions $y(\ell) = \tilde{y}(\ell)$ for $\ell = 0, \dots, n - 1$. So

$$y^{[n]} - \phi(\tilde{\xi}, \tilde{u}_1^{[s_1]}, \tilde{u}_2^{[s_2]}) = 0. \tag{12}$$

On the other hand, one can transform (11) to

$$\tilde{y}^{[n]} = \phi(\tilde{\xi}, \tilde{u}_1^{[s_1]}, \tilde{u}_2^{[s_2]}) \tag{13}$$

via multiplying by α^{-1} . From (12) and (13) one gets $y^{[n]} = \tilde{y}^{[n]}$ and consequently, $y^{[k]} = \tilde{y}^{[k]}$ for $k \geq n$. \square

⁵ There is no loss of generality in assuming $s_1 \geq s_2$. If $s_1 < s_2$, then renumbering u_1, u_2 allows us to bring the system into the necessary form.

Example 2. Consider the system

$$\Sigma : y^{[2]} = uyy^{[1]} + u^{[1]}. \tag{14}$$

Since equation (14) does not contain denominators, the set $S_\Sigma = \{1\}$. The explicit equation of the right inverse system is

$$\Sigma_u : u^{[1]} = y^{[2]} - uyy^{[1]} \tag{15}$$

and the set $S_{\Sigma_u} = \{1\}$. Assume that the reference sequence $\{\tilde{y}(t), t \geq 0\}$ is given such that

$$\tilde{y}(0) = y(0), \tilde{y}(1) = y(1). \tag{16}$$

Now one can find the input sequence $\{\tilde{u}(t), t \geq 0\}$ as follows. First, $\tilde{u}(0)$ can be chosen arbitrarily. By (15),

$$\begin{aligned} \tilde{u}(1) &= \tilde{y}(2) - \tilde{u}(0)\tilde{y}(0)\tilde{y}(1), \\ \tilde{u}(2) &= \tilde{y}(3) - \tilde{u}(1)\tilde{y}(1)\tilde{y}(2) = \tilde{y}(3) - [\tilde{y}(2) - \tilde{u}(0)\tilde{y}(0)\tilde{y}(1)]\tilde{y}(1)\tilde{y}(2), \\ &\dots \end{aligned} \tag{17}$$

We show that substituting $\tilde{u}(t), t \geq 1$ into equations (14) yields, by (16), $y(t, y(0), y(1), \tilde{u}(0), \dots, \tilde{u}(t-1)) = \tilde{y}(t), t \geq 2$. First, note that $y(0) = \tilde{y}(0), y(1) = \tilde{y}(1)$. Applying $u(t) = \tilde{u}(t)$ in (14) yields for $t = 2$: $y(2) = \tilde{u}(0)y(0)y(1) + \tilde{u}(1)$. Due to (16) and (17), $y(2) = \tilde{u}(0)y(0)y(1) + \tilde{y}(2) - \tilde{u}(0)y(0)y(1) = \tilde{y}(2)$. Taking $t = 1$ and $u(t) = \tilde{u}(t)$ for $t = 1, 2$ in (14), we get

$$y(3) = \tilde{u}(1)y(1)y(2) + \tilde{u}(2). \tag{18}$$

Substituting $\tilde{u}(2)$ from (17) into (18) yields

$$y(3) = \tilde{u}(1)y(1)y(2) + \tilde{y}(3) - \tilde{u}(1)\tilde{y}(1)\tilde{y}(2). \tag{19}$$

Rewrite (19) as $y(3) = \tilde{u}(1)[y(1)y(2) - \tilde{y}(1)\tilde{y}(2)] + \tilde{y}(3)$. Note that $y(1) = \tilde{y}(1)$ due to initial conditions, and on the previous step we have shown that $y(2) = \tilde{y}(2)$. Thus the equalities $y(1)y(2) - \tilde{y}(1)\tilde{y}(2) = 0$ and $y(3) = \tilde{y}(3)$ hold. In a similar manner, one can show that $y(t) = \tilde{y}(t)$ for $t \geq 4$. Consequently, system (14) is right invertible by Definition 7.

Example 3. Consider the system with 3 inputs and 2 outputs:

$$\Sigma : \begin{aligned} \tilde{\phi}_1 &:= u_1^{[1]} + y_1^{[2]} + u_2^{[1]}y_2 = 0, \\ \tilde{\phi}_2 &:= u_2^{[1]} + u_3^{[1]}y_1 + y_2^{[3]} = 0 \end{aligned} \tag{20}$$

with the set $S_\Sigma = \{1\}$. After rewriting system (20) in explicit form (2), one obtains the indices

$$\begin{aligned} n_1 = 2, \quad n_{11} = -\infty, \quad n_{12} = 0, \quad s_{11} = 1, \quad s_{12} = 1, \quad s_{13} = -\infty, \\ n_2 = 3, \quad n_{21} = 0, \quad n_{22} = -\infty, \quad s_{21} = -\infty, \quad s_{22} = 1, \quad s_{23} = 1. \end{aligned}$$

By Definition 1 the system is in the strong Popov form (with respect to outputs), but not in the strong Popov form with respect to inputs, since condition (iii) of Definition 2 is not fulfilled. For system (20) the matrix

$$\bar{Q} = Q = \begin{bmatrix} Z & y_2Z & 0 \\ 0 & Z & y_1Z \end{bmatrix}$$

and so the rank $\rho(\bar{Q}) = 2 = p$. Due to Theorem 1 the right inverse system exists for (20). The application of Algorithm 1 from [2] to \bar{Q} yields the transformation matrix

$$U = \begin{bmatrix} 1 & -y_2 \\ 0 & 1 \end{bmatrix}.$$

Applying the operator U to $[\tilde{\phi}_1, \tilde{\phi}_2]^T$ yields

$$U(Z) \uparrow \begin{bmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{bmatrix} = \begin{bmatrix} \tilde{\phi}_1 - y_2 \tilde{\phi}_2 \\ \tilde{\phi}_2 \end{bmatrix} = \begin{bmatrix} u_1^{[1]} + y_1^{[2]} - u_3^{[1]} y_1 y_2 - y_2 y_2^{[3]} \\ u_2^{[1]} + u_3^{[1]} y_1 + y_2^{[3]} \end{bmatrix} := \begin{bmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{bmatrix}.$$

The indices of the transformed equations (after rewriting the system in the explicit form) are

$$\begin{aligned} \sigma_1 &= 1, & \sigma_{11} &= -\infty, & \sigma_{12} &= -\infty, & \sigma_{13} &= 1, \\ \sigma_2 &= 1, & \sigma_{21} &= -\infty, & \sigma_{22} &= -\infty, & \sigma_{23} &= 1; \end{aligned}$$

conditions (i)–(iii) of Definition 2 are fulfilled. Therefore $u_1^{[1]}, u_2^{[1]}$ can be expressed from $\tilde{\phi}_1 = 0, \tilde{\phi}_2 = 0$ as

$$u_1^{[1]} = -y_1^{[2]} + u_3^{[1]} y_1 y_2 + y_2 y_2^{[3]}, \quad u_2^{[1]} = -u_3^{[1]} y_1 - y_2^{[3]}, \tag{21}$$

where u_3 is considered as a free parameter. System (21) is in the strong Popov form with respect to input.

Let us show that equations (21) allow us to compute the sequence $\{\tilde{u}(t), t \geq 0\}$, required in Definition 6. Assume that the reference sequence $\{\tilde{y}_1(t), \tilde{y}_2(t), t \geq 0\}$ is given and

$$\tilde{y}_1(0) = y_1(0), \quad \tilde{y}_1(1) = y_1(1), \quad \tilde{y}_2(0) = y_2(0), \quad \tilde{y}_2(1) = y_2(1), \quad \tilde{y}_2(2) = y_2(2). \tag{22}$$

To compute the input sequence $\{\tilde{u}_1(t), \tilde{u}_2(t), t \geq 0\}$, we choose the sequence $\{\tilde{u}_3(t), t \geq 0\}$ arbitrarily and compute by (21) and (22):

$$\begin{aligned} \tilde{u}_1(1) &= -\tilde{y}_1(2) + \tilde{u}_3(1) y_1(0) y_2(0) + y_2(0) \tilde{y}_2(3), & \tilde{u}_2(1) &= -\tilde{u}_3(1) y_1(0) - \tilde{y}_2(3), \\ \tilde{u}_1(2) &= -\tilde{y}_1(3) + \tilde{u}_3(2) y_1(1) y_2(1) + y_2(1) \tilde{y}_2(4), & \tilde{u}_2(2) &= -\tilde{u}_3(2) y_1(1) - \tilde{y}_2(4), \\ \dots & & \dots & \end{aligned} \tag{23}$$

We show next that substituting $\tilde{u}_1(t), \tilde{u}_2(t)$, and $\tilde{u}_3(t), t \geq 0$ into equations (20) yields, using (22),

$$y_i(t, y_1(0), y_1(1), y_2(0), y_2(1), y_2(2), \tilde{u}(0), \dots, \tilde{u}(t-1)) = \tilde{y}_i(t), \quad t \geq 0, \quad i = 1, 2.$$

Replacing $u_1(t)$ and $u_2(t)$ respectively by $\tilde{u}_1(t)$ and $\tilde{u}_2(t)$ from (23) yields $y_1(2) = -\tilde{u}_1(1) - \tilde{u}_2(1) y_2(0)$, $y_2(3) = -\tilde{u}_2(1) - \tilde{u}_3(1) y_1(0)$. Due to (22),

$$\begin{aligned} y_1(2) &= -[-\tilde{y}_1(2) + \tilde{u}_3(1) y_1(0) y_2(0) + y_2(0) \tilde{y}_2(3)] - [-\tilde{u}_3(1) y_1(0) - \tilde{y}_2(3)] y_2(0) = \tilde{y}_1(2), \\ y_2(3) &= -[-\tilde{u}_3(1) y_1(0) - \tilde{y}_2(3)] - \tilde{u}_3(1) y_1(0) = \tilde{y}_2(3). \end{aligned}$$

In a similar manner one can show that $y_1(t) = \tilde{y}_1(t), t \geq 3$ and $y_2(t) = \tilde{y}_2(t), t \geq 4$. Consequently, system (20) is right invertible by Definition 7.

Remark 2. Although the strong Popov form itself is unique, the right inverse system is not necessarily unique. Namely, if $m > p$, the equations of inverse are parametrized by $m - p$ inputs that can be chosen freely. Expressing $u_1^{[1]}, u_3^{[1]}$ from (20) yields alternative equations of the right inverse system, parametrized by u_2 and its shifts

$$u_1^{[1]} = -y_1^{[2]} - u_2^{[1]} y_2, \quad u_3^{[1]} = -(u_2^{[1]} + y_2^{[3]}) / y_1,$$

satisfying Definition 6. However, the equations are not in the strong Popov form with respect to inputs, since condition (i) of Definition 2 is not satisfied for the second equation.

Remark 3. Sometimes the existence of an inverse system depends on the choice of variables. For instance, the system $y^{[3]} = (u_1^{[2]})^2 + u_2^{[2]}$ cannot be transformed into the explicit form with respect to u_1 (required by the Popov form), using the *linear* transformations, because for that the *nonlinear* transformation is necessary. However, one can find the inverse system by transforming the equations into the explicit form with respect to variable u_2 via *linear* transformation, obtaining $u_2^{[2]} = y^{[3]} - (u_1^{[2]})^2$. Observe that the latter system is not in the strong Popov form according to Definition 2, because it does not match with the system description (3) where from the i th equation the variable u_i is expressed. Relaxing the assumption $j_i = i$, made in this paper, reveals that the system $u_2^{[2]} = y^{[3]} - (u_1^{[2]})^2$ satisfies the conditions of the strong Popov form, as defined in [2].

Example 4. The goal of this Example is to demonstrate that indices s_{il} in Σ are not the same as the indices in the inverse system; they change. For instance, given the system in the strong Popov form with respect to outputs

$$\Sigma : y_1^{[2]} = u_1^{[1]} + u_2, \quad y_2^{[4]} = y_2 u_1^{[3]} + y_1 u_2^2 \quad (24)$$

together with the set $S_\Sigma = \{1\}$, its right inverse is

$$\Lambda : u_1^{[1]} = y_1^{[2]} - u_2, \quad u_2^{[2]} = y_1^{[4]} + \frac{u_2^2 y_1 - y_2^{[4]}}{y_2} \quad (25)$$

with $S_\Lambda^0 = \{y_2\}$. The transformation matrix is

$$U = \begin{pmatrix} -1 & 0 \\ -Z^2 & \frac{1}{y_2} \end{pmatrix}.$$

The comparison of maximal input shifts in the original and inverse systems reveals that shifts in the inverse are lower than or equal to those appearing in the original system. Indeed, for (24) the indices

$$\begin{aligned} s_{11} &= 1, & s_{12} &= 0, \\ s_{21} &= 3, & s_{22} &= 0, \end{aligned}$$

while for inverse system (25)

$$\begin{aligned} \sigma_1 &= 1, & \sigma_{11} &= -\infty, & \sigma_{12} &= 0, \\ \sigma_2 &= 2, & \sigma_{21} &= -\infty, & \sigma_{22} &= 0. \end{aligned}$$

To find the explicit equations of the right inverse for a system described by i/o equations there is no need to realize the equations in the state space form. However, our approach is consistent with the earlier results for state equations. The example below demonstrates that the diagram in Fig. 2 commutes.

Example 5. (Continuation of Example 3) Consider system (20) in the strong Popov form with respect to outputs. Following the approach in this paper, we transform (20) into the Popov form with respect to inputs u_1 and u_2 , obtaining (21).

An alternative way is to start by transforming i/o equations (20) into the state space form as

$$x_1^{[1]} = u_2 x_4 - u_1, \quad x_2^{[1]} = x_3, \quad x_3^{[1]} = u_3 x_5 - u_2, \quad x_4^{[1]} = -x_2, \quad x_5^{[1]} = -x_1, \quad y_1 = x_1, \quad y_2 = x_2, \quad (26)$$

where $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_2^{[1]}$, $x_4 = (u_1 + y_1^{[1]})/u_2$, $x_5 = (u_2 + y_2^{[2]})/u_3$, (see [10]). Applying the inversion algorithm from [8] allows us to find the equations of the right inverse system of (26) as

$$\begin{aligned} x_1^{[1]} &= y_1^{[1]}, \quad x_2^{[1]} = x_3, \quad x_3^{[1]} = y_2^{[2]}, \quad x_4^{[1]} = -x_2, \quad x_5^{[1]} = -x_1, \\ u_1 &= -y_1^{[1]} - y_2^{[2]} x_4 + u_3 x_4 x_5, \quad u_2 = -y_2^{[2]} + u_3 x_5. \end{aligned} \quad (27)$$

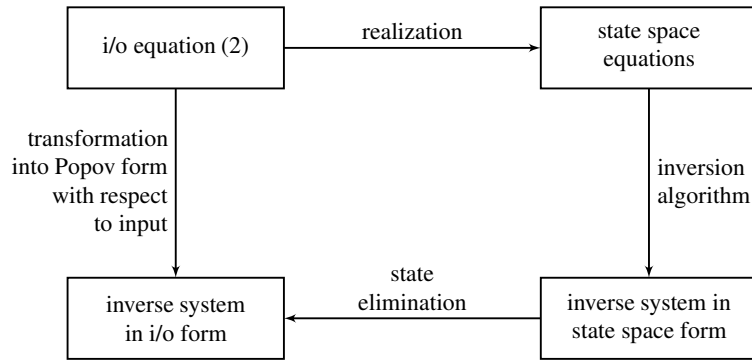


Fig. 2. Commutation scheme.

The order of inverse system (27) can be reduced if we take into account that $x_1 = y_1, x_2 = y_2$, and $x_3 = y_2^{[1]}$, meaning that the first three equations in (27) are just identities (see more in [8], p. 81):

$$\begin{aligned} \eta_1^{[1]} &= -y_2, \quad \eta_2^{[1]} = -y_1, \\ u_1 &= -y_1^{[1]} - y_2^{[2]} \eta_1 + u_3 \eta_1 \eta_2, \quad u_2 = -y_2^{[2]} + u_3 \eta_2, \end{aligned} \tag{28}$$

where $\eta_1 = x_4 = (u_1 + y_1^{[1]})/u_2$ and $\eta_2 = x_5 = (u_2 + y_2^{[2]})/u_3$. Eliminating state variables from (28) yields exactly i/o equations (21), obtained directly from (20).

Example 6. Consider Example 5.2 from [7]:

$$\Sigma: \quad y_1^{[1]} = u_1, \quad y_2^{[2]} = y_2^{[1]} u_1^{[1]} + u_2 \tag{29}$$

and the set $S_\Sigma = \{1\}$. Transforming the system into the strong Popov form with respect to inputs yields the following right inverse system:

$$\Lambda: \quad u_1 = y_1^{[1]}, \quad u_2 = y_2^{[2]} - y_1^{[2]} y_2^{[1]} \tag{30}$$

and the set $S_\Lambda = \{1\}$. The application of the IA from [7] to system (29) results also in (30).

4. LEFT INVERSE SYSTEM

In this section we assume that $p \geq m$. Let us consider the following system:

$$\Gamma: \quad u_k^{[\sigma_k]} = H_k(u_1, \dots, u_i^{[\sigma_{ki}]}, y_j, \dots, y_j^{[v_{kj}]}), \quad k = 1, \dots, m, \tag{31}$$

where $i = 1, \dots, m, j = 1, \dots, p$, together with the multiplicative set S_Γ . Let S be the smallest multiplicative set containing S_Σ and S_Γ .

Definition 9. System Γ is a left inverse of Σ if for any S -acceptable $\tilde{\mathbf{u}}$ there exists $\tilde{\mathbf{y}}$ such that $(\tilde{\mathbf{y}}, \tilde{\mathbf{u}})$ is S -acceptable and solves Σ and after substituting $\mathbf{y} = \tilde{\mathbf{y}}$ to Γ and setting $u_k(\ell) = \tilde{u}_k(\ell), \ell = 0, \dots, \sigma_k - 1$, we get solution \mathbf{u} of Γ , satisfying $u_k(l) = \tilde{u}_k(l), l \geq \sigma_k$. Then the left inverse of Σ is denoted by Σ_L^{-1} .

Definition 10. System Σ is left invertible if there exists a left inverse of Σ in the sense of Definition 9.

If system (2) is left invertible, then it is possible to reconstruct *uniquely* the input $\tilde{\mathbf{u}}$ from the knowledge of the observed output sequence $\tilde{\mathbf{y}}$.

Theorem 2. Let $p \geq m$. Under Assumption 2 system (2) is left invertible iff $\rho(\bar{Q}) = m$.

Proof. Sufficiency. The proof is, for transparency, presented for the single-input multi-output case where $p = 2, m = 1$, described by the equations

$$\begin{aligned} \tilde{\phi}_1(y_1^{[n_1]}, y_2^{[n_2]}, \xi, u^{[s]}) &= y_1^{[n_1]} - \phi_1(\xi, u^{[s]}) = 0, \\ \tilde{\phi}_2(y_1^{[n_1]}, y_2^{[n_2]}, \xi, u^{[s]}) &= y_2^{[n_2]} - \phi_2(\xi, u^{[s]}) = 0 \end{aligned} \tag{32}$$

together with the multiplicative set S_Σ , where $\xi = (y_1, \dots, y_1^{[n_1-1]}, y_2, \dots, y_2^{[n_2-1]}, u, \dots, u^{[s-1]})$. There exists, due to Assumption 2, a transformation operator U satisfying $e_\Sigma(U) \in \mathcal{Q}_\Sigma[Z, \delta_\Sigma]^{2 \times 2}$, which transforms equations (32) into the strong Popov form

$$U \uparrow \begin{bmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{bmatrix} = \begin{bmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{bmatrix},$$

where $\tilde{\psi}_1(y_1, \dots, y_1^{[v_1]}, y_2, \dots, y_2^{[v_2]}) = 0$ and the second equation

$$\Gamma : \tilde{\psi}_2(y_1^{[v_1]}, y_2^{[v_2]}, \xi', u^{[\sigma]}) = u^{[\sigma]} - \psi_2(\xi', y_1^{[v_1]}, y_2^{[v_2]}) = 0, \tag{33}$$

where $\xi' = (y_1, \dots, y_1^{[v_1-1]}, y_2, \dots, y_2^{[v_2-1]}, u, \dots, u^{[\sigma-1]})$, for some nonnegative integers $v_1, v_2, v_1, v_2, \sigma \leq s$. We demonstrate that (33) together with the multiplicative set S_Γ is really the left inverse of (32). In what follows we set S to be the smallest multiplicative set containing S_Σ and S_Γ .

Take S -acceptable \tilde{u} and solve Σ getting $(\tilde{y}_1, \tilde{y}_2)$. Then we obtain

$$\tilde{y}_1^{[n_1]} - \phi_1(\tilde{\xi}, \tilde{u}^{[s]}) = 0, \quad \tilde{y}_2^{[n_2]} - \phi_2(\tilde{\xi}, \tilde{u}^{[s]}) = 0.$$

From the linear equivalence transformation we get

$$\tilde{u}^{[\sigma]} - \psi_2(\tilde{\xi}', \tilde{y}_1^{[v_1]}, \tilde{y}_2^{[v_2]}) = 0. \tag{34}$$

Let us solve (33), setting $y_i = \tilde{y}_i, i = 1, 2$, and $u(k) = \tilde{u}(k)$ for $k = 0, \dots, \sigma - 1$. Then we get

$$u^{[\sigma]} - \psi_2(\tilde{\xi}', \tilde{y}_1^{[v_1]}, \tilde{y}_2^{[v_2]}) = 0. \tag{35}$$

From (34) and (35) it follows that $u^{[\sigma]} = \tilde{u}^{[\sigma]}$, as required and consequently, $u^{[k]} = \tilde{u}^{[k]}$ for $k \geq \sigma$.

Necessity. Given the system Σ and its left inverse Γ , their linearized descriptions are respectively (5) and

$$d\Gamma : R(Z)du + S(Z)dy = 0 \tag{36}$$

for some polynomial matrices $R \in \mathcal{Q}_\Sigma[Z; \delta_\Sigma]^{m \times m}$ and $S \in \mathcal{Q}_\Sigma[Z; \delta_\Sigma]^{m \times p}$. From (36) we obtain $du = -R(Z)^{-1}S(Z)dy$. Let us substitute dy by $-P^{-1}(Z)Q(Z)du$ from (5): $du = -R(Z)^{-1}S(Z)(-P^{-1}(Z)Q(Z)du)$, resulting in $I = R^{-1}SP^{-1}Q$. This cannot be true unless the rank $\rho(Q) = m$. Note that the elements of P^{-1} and R^{-1} are from the quotient field of the Ore ring $\mathcal{Q}_\Sigma[Z, \delta_\Sigma]$. The both inverse matrices P^{-1} and R^{-1} exist, because $\mathcal{Q}_\Sigma[Z; \delta_\Sigma]$ is the Ore ring, see [6]. \square

From the above, if the outputs of the original system Σ are fed into the left inverse system Σ_L^{-1} , the latter can reconstruct the inputs u of the original system on its output; see Fig. 3.

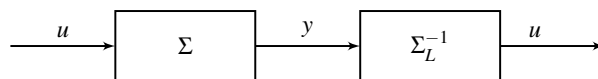


Fig. 3. Left inverse.

Example 7. Consider the set of equations in the strong Popov form with respect to y_1, y_2, y_3

$$\Sigma : \begin{aligned} y_1^{[2]} &= u_1 u_2^{[1]} - u_2^{[2]}, \\ y_2^{[3]} &= u_1^{[2]} - y_1, \\ y_3^{[3]} &= u_1^{[1]} - u_1^{[1]} u_2^{[2]} + u_2^{[3]} + y_1 y_2 \end{aligned} \quad (37)$$

with the set $S_\Sigma = \{1\}$. From (37) one obtains $p = 3, m = 2$. The matrix

$$\bar{Q} = Q = \begin{bmatrix} -u_2^{[1]} & Z^2 - u_1 Z \\ -Z^2 & 0 \\ (u_2^{[2]} - 1)Z & -Z^3 + u_1^{[1]} Z^2 \end{bmatrix}$$

is row-reduced, since its leading coefficient matrix has rank 2. The application of Algorithm 1 from [2] to \bar{Q} gives the transformation matrix

$$U = \begin{bmatrix} -Z^2 & 1 & -Z \\ -Z & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

and the set $S_\Gamma = \{1\}$. Applying the transformation U to system equations (37) yields

$$U(Z) \Gamma \begin{bmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \\ \tilde{\phi}_3 \end{bmatrix} = \begin{bmatrix} y_1 - y_1^{[4]} + y_1^{[1]} y_2^{[1]} + y_2^{[3]} - y_3^{[4]} \\ u_1^{[1]} - y_1^{[3]} + y_1 y_2 - y_3^{[3]} \\ u_2^{[2]} - u_1 u_2^{[1]} + y_1^{[2]} \end{bmatrix} =: \begin{bmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\psi}_3 \end{bmatrix}, \quad (38)$$

being in the strong Popov form with respect to u_1, u_2 . The explicit equations of the left inverse system are

$$\Gamma : u_1^{[1]} = y_1^{[3]} + y_3^{[3]} - y_1 y_2, \quad u_2^{[2]} = u_1 u_2^{[1]} - y_1^{[2]}. \quad (39)$$

Alternatively, the left inverse system can be found via state equations, as shown in Fig. 2. For that purpose the following steps are necessary. First, the realization of original equations (37) is constructed:

$$x_1^{[1]} = u_1 + x_2, \quad x_2^{[1]} = x_3, \quad x_3^{[1]} = u_1 - x_4, \quad x_4^{[1]} = u_2 x_5, \quad (40a)$$

$$x_5^{[1]} = u_1, \quad x_6^{[1]} = x_7 - u_2 x_5, \quad x_7^{[1]} = u_1 + x_8, \quad x_8^{[1]} = x_1(x_4 - u_2),$$

$$y_1 = x_4 - u_2, \quad y_2 = x_1, \quad y_3 = x_6 + u_2. \quad (40b)$$

Note that this is not always doable since not all i/o equations are realizable. Second, the left inverse system of state equations (40) is found. This can be done by expressing $u_1 = y_2^{[1]} - x_2, u_2 = x_4 - y_1$ from (40) and substituting u_1, u_2 into (40a):

$$\begin{aligned} x_1^{[1]} &= y_2^{[1]}, \quad x_2^{[1]} = x_3, \quad x_3^{[1]} = -y_1, \quad x_4^{[1]} = x_5(x_4 - y_1), \quad x_5^{[1]} = y_2^{[1]} - x_2, \\ x_6^{[1]} &= x_7 - x_5(x_4 - y_1), \quad x_7^{[1]} = x_8 - x_2 + y_2^{[1]}, \quad x_8^{[1]} = x_1 y_1, \\ u_1 &= y_2^{[1]} - x_2, \quad u_2 = x_4 - y_1. \end{aligned} \quad (41)$$

The third and last step is transforming the left inverse (41) into the form of i/o equations

$$u_1^{[2]} = y_1 + y_2^{[3]}, \quad u_2^{[2]} = u_1 u_2^{[1]} - y_1^{[2]}. \quad (42)$$

At first sight the first equations of (39) and (42) are different; moreover, the first equation of (39) depends on y_3 , whereas the first equation of (42) does not. However, note that (39) do not include the equation $\tilde{\psi}_1 = 0$, being part of (38). We rewrite $\tilde{\psi}_1 = 0$ as

$$y_3^{[4]} = y_1 - y_1^{[4]} + y_1^{[1]}y_2^{[1]} + y_2^{[3]}. \quad (43)$$

Shifting the first equation of (39) forward to obtain $u_1^{[2]} = y_1^{[4]} + y_3^{[4]} + y_1^{[1]}y_2^{[1]}$ and eliminating then $y_3^{[4]}$ using (43) yield $u_1^{[2]} = y_1 + y_2^{[3]}$, the first equation of (42). The second equations of (42) and (39) coincide.

5. CONCLUSIONS

It was shown that transforming the system into the strong Popov with respect to inputs enables one to find the explicit equations of right and left inverse systems for the set of i/o equations, under the assumptions $m \geq p$ and $p \geq m$, respectively.

Note that the *linear* equivalence transformations that construct the explicit equations of the inverse system are valid globally in the entire space besides a certain set S that consists of zeros of some functions. Therefore, the approach avoids using the IFT (yielding, in general, only local results) or integrating the set of 1-forms obtained by the IA (yielding the generic results that are valid almost everywhere). However, when one needs to apply *nonlinear* equivalence transformations, the inverse system is not necessarily defined globally. Moreover, such transformations are difficult to find, see [1], and it is unclear whether the approach, based on the strong Popov form, outperforms the earlier approach based on the IA.

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Popovi kuju ja pöördsüsteemi ilmutatud võrrandid

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On uuritud sisend-väljundvõrranditega antud mittelineaarsete diskreetaja süsteemide pööratavust. On leitud tarvilikud ja piisavad tingimused süsteemi parem- ning vasakpoolse pöördsüsteemi olemasoluks. Pöördsüsteemi ilmutatud võrrandite leidmiseks teisendatakse originaalsüsteem sisendite suhtes tugevale Popovi kujule. Artiklis on eeldatud, et võrrandeid saab tugevale Popovi kujule viia lineaarsete ekvivalentsiteisenduste abil, mis on defineeritud üle meromorfsete funktsioonide korpuse. Tugeva Popovi kuju leidmiseks kasutatakse mittekommutatiivsete polünoomide ringi teooriat, konstruktiivne meetod selleks on esitatud varasemas artiklis.