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MATHEMATICS

Morita contexts, ideals, and congruences for semirings with local units

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Abstract. We consider Morita contexts for semirings that have certain local units but not necessarily an identity element. We show that the existence of a Morita context with unitary bisemimodules and surjective maps implies that the two semirings involved have isomorphic quantales of ideals and lattices of congruences.

Key words: semiring, semimodule, Morita context, ideal, congruence.

1. INTRODUCTION

In the classical case [1;2, Chapter 6], *Morita equivalence* is an equivalence relation on the class of rings with identity, where two rings are considered equivalent if the categories of left (equivalently, right) modules over them are equivalent. This equivalence of categories turns out to be equivalent to the existence of a *Morita context* – a pair of bimodules over the two rings together with a pair of bimodule homomorphisms from their tensor products onto the original rings, satisfying certain conditions. While the definition of a Morita context may seem complex at first, it is often easier to prove statements about Morita equivalence using them rather than the categorical definition.

There have been several successful attempts to generalize Morita equivalence to settings other than rings with identity. Many results have be proven for rings with various kinds of local units [3,4]. In this case, unitary modules are considered instead of arbitary modules in the definition of Morita equivalence as well as in Morita contexts.

Generalizing in another direction, rings and modules have been replaced with semigroups and acts over them. For monoids [5,6], Morita equivalence turns out to be very close to isomorphism and thus not very interesting. However, using local unit conditions like those for rings, as well as unitary acts instead of arbitary ones, gives a meaningful theory for semigroups where several results analogous to those of rings hold [7].

A *semiring* is an algebraic structure where the additive structure in the definition of a ring has been changed from an Abelian group to a monoid. The analogues for modules of rings are called semimodules. It is a natural question whether a Morita theory could be developed for semirings, and whether it is closer to the theory for rings or semigroups. For semirings with identity, Morita equivalence was first studied by Katsov and Nam [8] and further by Sardar, Gupta, and Saha [9–11]. Morita equivalence for semirings with local units was first considered by Liu [12].

In this article, we approach Morita theory for semirings with local units from a different direction: that of Morita contexts. The relationship between Morita equivalence and the existence of a Morita context in this case has not been studied yet. We show, however, that the existence of a Morita context with conditions analogous to those used for rings and semigroups with local units implies that the two semirings have isomorphic lattices of ideals and congruences. These results are analogous to those obtained for semigroups with local units by Laan and Márki in [13] and for semirings with identity by Sardar and Gupta in [10].

2. DEFINITIONS

Definition 1. *A* **semiring** *[14] is an algebra* $(S,+,\cdot,0)$ *such that* $(S,+,0)$ *is a commutative monoid, multiplication is associative and distributes over addition from both sides, and* 0 *is a zero element with respect to multiplication.*

Note that we do not require the existence of a multiplicative identity element. Golan [14] uses the term *hemiring* for the above definition and reserves *semiring* for semirings with identity.

Definition 2. *A* **left semimodule** *over a semiring* S *is an algebra* ${}_SM=(M,+,0,(s\cdot)|_{s\in S})$ *such that* $(M,+,0)$ *is a commutative monoid and the following identities hold for all* $s,s'\in S$*,* $m,m'\in M$*:*
1. $s(m+m')=sm+sm'$*,*
2. $(s+s')m=sm+s'm$*,*
3. $(ss')m=s(s'm)$*,*
4. $s0_M=0_M$*,*
5. $0_Sm=0_M$*.*
Right semimodules *are defined analogously.*

Definition 3. *A* **bisemimodule** *over semirings* R *and* S *is an algebra* ${}_RM_S=(M,+,0_M,(r\cdot)|_{r\in R},(\cdot s)|_{s\in S})$ *such that* ${}_RM$ *is a left* R*-semimodule,* M_S *is a right* S*-semimodule, and* $(rm)s=r(ms)$ *for all* $r\in R$*,* $m\in M$*,* $s\in S$*.*

Definition 4. *Let* S *be a semiring and* ${}_SM$ *a left semimodule. For* $A\subseteq S$ *and* $U\subseteq M$*, we define*

$$AU=\left\{\sum_{i=1}^{n}s_im_i : n\in\mathbb{N}_0, s_i\in A, m_i\in U\right\}$$

and analogously for right semimodules.

Since a semiring is a semimodule over itself, Definition 4 also defines the product of two subsets of a semiring. This multiplication of subsets of a semiring is easily seen to be associative.

Definition 5. *For a semiring* S*, a left (right)* S*-semimodule* M *is* **unitary** *if* $SM=M$ *(*$MS=M$*). For semirings* S *and* T*, a bisemimodule* ${}_SM_T$ *is* **unitary** *if* ${}_SM$ *and* M_T *are unitary.*

The following local unit conditions are chosen to cover an as large as possible class of semirings in the results to be proven. Both are implied by the notion of local units in [4, Definition 1].

Definition 6. *A semiring* S *has* **weak local units** *if for every* $s\in S$ *there exist* $e,e'\in S$ *with* $es=s=se'$*.*

Definition 7. *A semiring* S *has* **common joint weak local units** *if for every* $s,s'\in S$ *there exist* $e,e'\in S$ *with* $s=ese'$ *and* $s'=es'e'$*.*

Definition 8. *An* **ideal** *of a semiring* S *is a set* $I\subseteq S$ *that is a submonoid of* $(S,+)$ *and for which* $SI\subseteq I$ *and* $IS\subseteq I$*.* **Finitely generated** *ideals are defined as in ring theory.*

Definition 9. *A* **quantale** *is a complete lattice endowed with an associative multiplication that is distributive from both left and right with respect to joins of any cardinality. An* **isomorphism** *of quantales is a bijection from one quantale to another that preserves joins and meets of any cardinality and multiplication.*

It is a well-known fact that the lattice $\mathrm{Id}(S)$ of ideals of a ring forms a quantale (see e.g. [15, p. 17]), where the multiplication of two ideals is given by Definition 4. It is easy to verify that the same fact holds for semirings.

Definition 10. *For S-semimodules M_S and $_SN$, their* **tensor product** *$M \otimes N$ is defined as the factor semigroup of the free commutative additive semigroup $F = F(M \times N)$ generated by the set $M \times N$, factorized by the congruence ρ generated by all ordered pairs of the form*

$$((m+m',n),(m,n)+(m',n)),$$
$$((m,n+n'),(m,n)+(m,n')),$$
$$((ms,n),(m,sn)),$$

where $m,m' \in M$, $n,n' \in N$, $s \in S$. The congruence class containing a generator (m,n) of F is denoted by $m \otimes n$.

Note that the elements $m \otimes n$ form a system of generators for the semigroup $M \otimes N$, i.e. every element of $M \otimes N$ is a finite sum of such elements. From the generating pairs of ρ we obtain the following basic identities:

$$(m+m') \otimes n = m \otimes n + m' \otimes n,$$
$$m \otimes (n+n') = m \otimes n + m \otimes n',$$
$$ms \otimes n = m \otimes sn.$$

The semigroup $M \otimes N$ is actually a monoid, the zero element being $0_M \otimes 0_N$:

$$m \otimes n + 0_M \otimes 0_N = m \otimes n + 0_M \otimes 0_S n = m \otimes n + 0_M 0_S \otimes n = (m+0_M) \otimes n = m \otimes n.$$

The tensor product of semimodules was first introduced in [16] and further studied in [17]. The following proposition can be proven as described in the paragraph preceding Theorem 3.1 of [17].

Proposition 1. *Let R, S, and T be semirings and $_SM_R$ and $_RN_T$ bisemimodules. Then the monoid $M_R \otimes {}_RN$ can be turned in a unique way into a bisemimodule $_SM_R \otimes {}_RN_T$, retaining its addition and zero element, such that for any $m \in M$, $n \in N$*

$$s(m \otimes n) = sm \otimes n, \ (m \otimes n)t = m \otimes nt.$$

Definition 11. *A* **Morita context** *is a sextuple $(S,T,{}_SP_T,{}_TQ_S,\theta,\phi)$ where*
1. S and T are semirings;
2. $_SP_T$ and $_TQ_S$ are bisemimodules as indicated by the subscripts;
3. $\theta\colon {}_S(P \otimes Q)_S \to {}_SS_S$ and $\phi\colon {}_T(Q \otimes P)_T \to {}_TT_T$ are bisemimodule homomorphisms;
4. for every $p,p' \in P$, $\theta(p \otimes q)p' = p\phi(q \otimes p')$;
5. for every $q,q' \in Q$, $\phi(q \otimes p)q' = q\theta(p \otimes q')$.

We say that a Morita context $(S,T,{}_SP_T,{}_TQ_S,\theta,\phi)$ is **unitary** *if $_SP_T$ and $_TQ_S$ are unitary bisemimodules.*

Example. We give an example (inspired by the proof of [18, Theorem 9]) of a unitary Morita context with surjective mappings where the semirings are non-isomorphic. Let F be a free semiring with two generators x and y. Let ρ be the congruence on F generated by the pair (y,y^2), and let $R := F/\rho$. Then $e := y/\rho$ is an idempotent. Let S be the subsemiring ReR of R; then $S = SeS \neq eSe$. Now one can verify that $(S,eSe,{}_SSe_{eSe},{}_{eSe}eS_S,\theta,\phi)$, where $\theta(se \otimes es') := ses'$ and $\phi(es \otimes s'e) := ess'e$, is a unitary Morita context with surjective mappings.

3. RESULTS

Our first result concerns ideals. The proof is analogous to that of Theorem 3 in [13] or Theorem 2.2 in [10].

Theorem 1. *If two semirings S and T have weak local units and there exists a unitary Morita context $(S,T,{}_SP_T,{}_TQ_S,\theta,\phi)$ with θ,ϕ surjective, then there is a quantale isomorphism* $\mathrm{Id}(S)\to\mathrm{Id}(T)$ *that takes finitely generated ideals to finitely generated ideals.*

Proof. Let $(S,T,{}_SP_T,{}_TQ_S,\theta,\phi)$ be a unitary Morita context with θ, ϕ surjective. Define

$$\Theta\colon \mathrm{Id}(T)\to\mathrm{Id}(S),\ \Theta(J):=\theta(PJ\otimes Q)=\left\{\theta\left(\sum_{i=1}^{n}p_it_i\otimes q_i\right):n\in\mathbb{N},p_i\in P,q_i\in Q,t_i\in J\right\},$$

$$\Phi\colon \mathrm{Id}(S)\to\mathrm{Id}(T),\ \Phi(I):=\phi(QI\otimes P)=\left\{\phi\left(\sum_{i=1}^{n}q_is_i\otimes p_i\right):n\in\mathbb{N}_0,p_i\in P,q_i\in Q,s_i\in I\right\}.$$

It is easily seen that the sets on the right side are indeed ideals. We show that Θ and Φ are mutually inverse bijections. Due to symmetry, it suffices to show that $\Theta(\Phi(I))=I$ for any $I\in\mathrm{Id}(S)$. First,

$$\Theta(\Phi(I))=\theta(P\phi(QI\otimes P)\otimes Q).$$

We now show that

$$\theta(P\phi(QI\otimes P)\otimes Q)=\theta(P\otimes Q)I\theta(P\otimes Q).$$

To see this, observe that, according to Definitions 4 and 10, the subset of S on the left side consists of all finite sums of elements of the form $\theta(p\phi(qs\otimes p')\otimes q')$, where $s\in I$, $p,p'\in P$, $q,q'\in Q$. This transforms into

$$\theta(p\phi(qs\otimes p')\otimes q')=\theta(\theta(p\otimes qs)p'\otimes q')=\theta(p\otimes q)s\theta(p'\otimes q'),$$

and elements of this form generate the set on the right side. Now $\theta(P\otimes Q)I\theta(P\otimes Q)=SIS\subseteq I$. Using weak local units, we can see that $I\subseteq SIS$, concluding the proof that Θ and Φ are mutually inverse.

It is easy to see that for $J'\subseteq J$, $\Theta(J')\subseteq\Theta(J)$ and the same for Φ; thus Θ and Φ are order-preserving bijections and therefore preserve all meets and joins.

To see that Φ preserves multiplication of ideals (the proof for Θ is analogous), we have to demonstrate for $I_1,I_2\in\mathrm{Id}(S)$ that $\Phi(I_1)\Phi(I_2)=\Phi(I_1I_2)$, or equivalently,

$$\phi(QI_1\otimes P)\phi(QI_2\otimes P)=\phi(QI_1I_2\otimes P).$$

The set $\phi(QI_1\otimes P)\phi(QI_2\otimes P)$ consists of all finite sums of elements of the form

$$\begin{aligned}\phi(q_1s_1\otimes p_1)\phi(q_2s_2\otimes p_2) &= \phi(q_1s_1\otimes p_1\phi(q_2s_2\otimes p_2))\\ &= \phi(q_1s_1\otimes\theta(p_1\otimes q_2s_2)p_2)=\phi(q_1s_1\theta(p_1\otimes q_2)s_2\otimes p_2),\end{aligned}\tag{1}$$

where $p_1,p_2\in P$, $q_1,q_2\in Q$, $s_1\in I_1$ and $s_2\in I_2$. Since $s_1\theta(p_1\otimes q_2)s_2\in I_1I_2$, we have shown $\Phi(I_1)\Phi(I_2)\subseteq\Phi(I_1I_2)$.

For the opposite inclusion, the set $\phi(QI_1I_2\otimes P)$ consists of all finite sums of elements of the form $\phi(qs_1s_2\otimes p)$, where $p\in P$, $q\in Q$, $s_1\in I_1$ and $s_2\in I_2$. Let $u\in S$ be chosen such that $us_2=s_2$, and let $u=\theta(p'\otimes q')$. Now applying (1) in reverse gives

$$\phi(qs_1s_2\otimes p)=\phi(qs_1\theta(p'\otimes q')s_2\otimes p)=\phi(qs_1\otimes p')\phi(q's_2\otimes p)\in\phi(QI_1\otimes P)\phi(QI_2\otimes P).$$

Now let $I=\sum_{i=1}^{m} Sa_iS$ be a finitely generated ideal. Using the existence of weak local units, let $a_i=u_ia_iv_i$ for some $u_i,v_i\in S$. Using surjectivity of θ, let $u_i=\theta\left(\sum_{j=1}^{n_i} p_{ij}\otimes q_{ij}\right)$, $v_i=\theta\left(\sum_{k=1}^{n_i'} p_{ij}'\otimes q_{ik}'\right)$. Now

$$\begin{aligned}\Phi(I)=&\phi(QI\otimes P)=\sum_{i=1}^{m}\{\phi(qa_i\otimes p):p\in P,q\in Q\}\\ =&\sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{k=1}^{n_i'}\left\{\phi\left(q\theta(p_{ij}\otimes q_{ij})a_i\theta(p_{ik}'\otimes q_{ik}')\otimes p\right):p\in P,q\in Q\right\}\\ =&\sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{k=1}^{n_i'}\left\{\phi(q\otimes p_{ij})\phi(q_{ij}a_i\otimes p_{ik}')\phi(q_{ik}'\otimes p):p\in P,q\in Q\right\}\subseteq\sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{k=1}^{n_i'}T\phi(q_{ij}a_i\otimes p_{ik}')T.\end{aligned}$$

The opposite inclusion also holds, since for $t,t'\in T$, $t\phi(q_{ij}a_i\otimes p_{ik}')t'=\phi(tq_{ij}a_i\otimes p_{ik}'t')\in\Phi(I)$. Therefore

$$\Phi(I)=\sum_{i=1}^{m}\sum_{j=1}^{n_i}\sum_{k=1}^{n_i'}T\phi(q_{ij}a_i\otimes p_{ik}')T$$

is finitely generated. □

Next, we consider congruences. The following result is the analogue of Theorem 6 in [13] and Theorem 2.15 in [10]. However, we give a slightly different proof, which does not need the use of transitive closure.

Theorem 2. *If two semirings S and T have common joint weak local units and there exists a unitary Morita context $(S,T,{}_SP_T,{}_TQ_S,\theta,\phi)$ with θ,ϕ surjective, then there exists a lattice isomorphism $\Theta:\mathrm{Con}(S)\to\mathrm{Con}(T)$. Furthermore, for each $\sigma\in\mathrm{Con}(S)$, S/σ and $T/\Theta(\sigma)$ are themselves contained in a unitary Morita context with surjective mappings.*

Proof. For $\sigma\in\mathrm{Con}(S)$, define

$$\Theta(\sigma)=\left\{(t,t')\in T^2:\forall p\in P,\ \forall q\in Q\colon\ \theta(pt\otimes q)\sim_\sigma\theta(pt'\otimes q)\right\}.$$

Clearly $\Theta(\sigma)$ is an equivalence relation. It is actually a congruence: for $(t_1,t_2),(t_3,t_4)\in\Theta(\sigma),p\in P,q\in Q$

$$\theta(p(t_1+t_3)\otimes q)=\theta((pt_1+pt_3)\otimes q)=\theta(pt_1\otimes q)+\theta(pt_3\otimes q)\sim_\sigma\theta(pt_2\otimes q)+\theta(pt_4\otimes q)=\theta(p(t_2+t_4)\otimes q)$$

and

$$\theta(pt_1t_3\otimes q)=\theta((pt_1)t_3\otimes q)\sim_\sigma\theta((pt_1)t_4\otimes q)=\theta(pt_1\otimes t_4q)\sim_\sigma\theta(pt_2\otimes t_4q)=\theta(pt_2t_4\otimes q).$$

The map $\Theta\colon\mathrm{Con}(S)\to\mathrm{Con}(T)$ is easily seen to be order-preserving. Let Φ be analogous to Θ in the opposite direction:

$$\Phi(\tau)=\left\{(s,s')\in S^2:\forall p\in P,\ \forall q\in Q\colon\ \phi(qs\otimes p)\sim_\tau\phi(qs'\otimes p)\right\},\ \tau\in\mathrm{Con}(T).$$

It remains to show that Φ is the inverse of Θ. Due to symmetry, it suffices to prove that $\Phi\Theta=1_{\mathrm{Con}(S)}$.

Let $\sigma\in\mathrm{Con}(S)$ and $s\sim_{\Phi(\Theta(\sigma))}s'$. From the definition of Φ, for all $p\in P$ and $q\in Q$

$$\phi(qs\otimes p)\sim_{\Theta(\sigma)}\phi(qs'\otimes p),$$

and from that and the definition of Θ, for all $p,p'\in P$ and $q,q'\in Q$

$$\theta(p'(\phi(qs\otimes p))\otimes q')\sim_\sigma\theta(p'(\phi(qs'\otimes p))\otimes q'). \tag{2}$$

The left side of (2) transforms to

$$\theta(p'(\phi(qs\otimes p))\otimes q') = \theta(\theta(p'\otimes qs)p\otimes q') = \theta(p'\otimes q)s\theta(p\otimes q').$$

Simplifying the right side of (2) in the same way, we get

$$\forall p,p'\in P,\ \forall q,q'\in Q\colon\ \theta(p'\otimes q)s\theta(p\otimes q') \sim_\sigma \theta(p'\otimes q)s'\theta(p\otimes q').$$

Since σ is compatible with addition and $P\otimes Q$ consists of finite sums of elements of the form $p\otimes q$, we get

$$\forall\alpha,\alpha'\in P\otimes Q\colon\ \theta(\alpha)s\theta(\alpha') \sim_\sigma \theta(\alpha)s'\theta(\alpha')$$

and from the surjectivity of θ

$$\forall s_1,s_2\in S\colon\ s_1ss_2 \sim_\sigma s_1s's_2.$$

Taking the common joint weak local units for s and s' as the values of s_1 and s_2, we get $s\sim_\sigma s'$. We have shown that $\Phi(\Theta(\sigma))\subseteq\sigma$.

In the opposite direction, $s\sim_\sigma s'$ implies

$$\forall p,p'\in P,\ \forall q,q'\in Q\colon\ \theta(p'\otimes q)s\theta(p\otimes q') \sim_\sigma \theta(p'\otimes q)s'\theta(p\otimes q')$$

or, applying the previously used transformation in reverse,

$$\forall p,p'\in P,\ \forall q,q'\in Q\colon\ \theta(p'(\phi(qs\otimes p))\otimes q') \sim_\sigma \theta(p'(\phi(qs'\otimes p))\otimes q'),$$

which is equivalent to $s\sim_{\Phi(\Theta(\sigma))} s'$. Thus $\Phi(\Theta(\sigma))=\sigma$, concluding the proof that the congruence lattices are isomorphic.

Let $\tau=\Theta(\sigma)$. We proceed to construct a Morita context for S/σ and T/τ.

Let μ be the bisemimodule congruence on ${}_SP_T$ generated by the set μ_0 of all pairs $(sp,s'p)$ and (pt,pt') where $(s,s')\in\sigma$, $(t,t')\in\tau$ and $p\in P$. Multiplications $S/\sigma\times P/\mu\to P/\mu$ and $P/\mu\times T/\tau\to P/\mu$,

$$(s/\sigma)(p/\mu) := (sp)/\mu,\ (p/\mu)(t/\tau) := (pt)/\mu,$$

are well defined. Now P/μ can be verified to be a unitary $(S/\sigma,T/\tau)$-module. From now on, we write P/μ to mean ${}_{S/\sigma}(P/\mu)_{T/\tau}$.

Analogously, we define $\nu\in\mathrm{Con}({}_TQ_S)$ generated by the set of pairs ν_0, and Q/ν becomes a unitary $(T/\tau,S/\sigma)$-bisemimodule.

Denote by (μ,ν) the equivalence relation $\{((p,q),(p',q')) : (p,p')\in\mu,(q,q')\in\nu\}$ on the set $P\times Q$. Define a map $\hat\theta_0\colon P\times Q\to S/\sigma$ by

$$\hat\theta_0(p,q) := \theta(p\otimes q)/\sigma.$$

We now verify that $(\mu,\nu)\subseteq\mathrm{Ker}(\hat\theta_0)$. Clearly, (μ,ν) is generated by the set

$$\{((p_1,q),(p_2,q)) : (p_1,p_2)\in\mu_0, q\in Q\}\cup\{((p,q_1),(p,q_2)) : (q_1,q_2)\in\nu_0, p\in P\},$$

and it suffices to show that this set is contained in $\mathrm{Ker}(\hat\theta_0)$. Consider the case of $(p_1,p_2)\in\mu_0, q\in Q$ (the case of $(q_1,q_2)\in\nu_0, p\in P$ is analogous). There are two possibilities.

(a) $(p_1,p_2)=(sp,s'p)$ for $s\sim_\sigma s'$, $p\in P$. Then

$$\theta(sp\otimes q) = s\theta(p\otimes q) \sim_\sigma s'\theta(p\otimes q) = \theta(s'p\otimes q)$$

and thus $\hat\theta_0(sp,q)=\hat\theta_0(s'p,q)$.

(b) $(p_1,p_2)=(pt,pt')$ for $t\sim_\tau t'$, $p\in P$. Then the definition of $\tau=\Theta(\sigma)$ implies that $\theta(pt\otimes q)\sim_\sigma \theta(pt'\otimes q)$, and thus $\hat\theta_0(pt,q)=\hat\theta_0(pt',q)$.

By the above, the map $\hat{\theta} : P/\mu \times Q/\nu \to S/\sigma$,

$$\hat{\theta}(p/\mu, q/\nu) := \hat{\theta}_0(p,q) = \theta(p \otimes q)/\sigma,$$

is well defined. This extends to a monoid homomorphism from the free monoid $F(P/\mu \times Q/\nu)$ to S/σ, which we also denote by $\hat{\theta}$. We can easily verify that the ordered pairs generating the congruence ρ given in Definition 10 are contained in $\mathrm{Ker}(\hat{\theta})$. Thus $\hat{\theta}$ factors through ρ, giving a monoid homomorphism $\tilde{\theta} : P/\mu \otimes Q/\nu \to S/\sigma$,

$$\tilde{\theta}(p/\mu \otimes q/\nu) = \Theta(p \otimes q)/\sigma.$$

The surjectivity of θ implies that $\tilde{\theta}$ is also surjective.

Now we verify that $\tilde{\theta}$ is an $(S/\sigma, S/\sigma)$-bisemimodule homomorphism. Due to additivity, it suffices to consider the tensor product's generators, and due to symmetry, to verify multiplication from the left:

$$\tilde{\theta}((s/\sigma)(p/\mu \otimes q/\nu)) = \tilde{\theta}((sp)/\mu \otimes q/\nu) = \theta(sp \otimes q)/\sigma = (s/\sigma)(\theta(p \otimes q)/\sigma) = (s/\sigma)\tilde{\theta}(p/\mu \otimes q/\nu).$$

By analogy, we get a surjective $(T/\tau, T/\tau)$-bisemimodule homomorphism $\tilde{\phi} : Q/\nu \otimes P/\mu \to T/\tau$,

$$\tilde{\phi}(q/\nu \otimes p/\mu) = \Phi(q \otimes p)/\tau.$$

It remains to verify the Morita equations. Due to symmetry, it is enough to verify just one of them. As above, it suffices to consider the tensor product's generators:

$$\begin{aligned}\tilde{\phi}(q/\nu \otimes p/\mu)(q'/\nu) &= (\phi(q \otimes p)/\tau)(q'/\nu) = (\phi(q \otimes p)q')/\nu \\ &= (q\theta(p \otimes q'))/\nu = (q/\nu)(\theta(p \otimes q')/\sigma) = (q/\nu)(\tilde{\theta}(p/\mu \otimes q'/\nu)).\end{aligned}$$

Thus $(S/\sigma, T/\tau, P/\mu, Q/\nu, \tilde{\theta}, \tilde{\phi})$ is a unitary Morita context with surjective mappings. □

4. CONCLUSIONS

It seems likely that the existence of a unitary Morita context with surjective mappings would imply Morita equivalence for semirings with local units, as is the case for semirings with identity, and for semigroups and rings with local units. Verifying this is left for future research. If it is true, our results imply that the quantale of ideals and the lattice of congruences are Morita invariants for semirings with suitable local unit conditions.

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Lokaalsete ühikutega poolringide Morita kontekstid ja ideaalid ning kongruentsid

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Klassikaliselt [1;2, ptk 6] on *Morita ekvivalentsus* defineeritud ühikelemendiga ringide klassil: kaht ringi peetakse ekvivalentseks, kui nende vasakpoolsete (ja samaväärselt parempoolsete) moodulite kategooriad on ekvivalentsed. See tingimus on samaväärne nn *Morita konteksti* olemasoluga. Morita kontekst koosneb kahest bimoodulist üle nende kahe ringi ja kahest homomorfismist nende bimoodulite tensorkorrutistest ringidesse, mis peavad rahuldama teatud tingimusi. Morita konteksti definitsioon võib tunduda keeruline, aga selle abil on mitmeid väiteid Morita ekvivalentsuse kohta tõestada lihtsam kui kategoorse definitsiooni abil.

Morita ekvivalentsuse mõistet on mitmel viisil edukalt üldistatud muudele struktuuridele kui ühikelemendiga ringid. Esiteks on vaadeldud mitmesuguste lokaalsete ühikutega ringe [3,4]. Sel juhul tuleb Morita ekvivalentsuse ja Morita konteksti definitsioonides asendada suvalised moodulid unitaarsete moodulitega.

Teine üldistussuund on olnud ringide ja moodulite asendamine poolrühmade ning polügoonidega. Monoidide korral [5,6] on Morita ekvivalentsus väga lähendane isomorfismile ja seetõttu ei paku eriti huvi. Kui aga nõuda (analoogiliselt ringidega) poolrühmadelt ühikelemendi asemel lokaalsete ühikute olemasolu ja vaadelda suvaliste polügoonide asemel unitaarseid, tekib sisukas teooria, kus kehtivad mitmed ringidega analoogilised tulemused [7].

Poolring on algebraline struktuur, kus Abeli rühma aditiivne struktuur ringi definitsioonis on asendatud monoidiga. Ringi moodulitele vastavad poolringi poolmoodulid. On loomulik küsimus, kas Morita teooriat on võimalik arendada ka poolringide korral ja kas see on lähedasem ringide või poolrühmade Morita teooriale. Ühikelemendiga poolringide jaoks uurisid Morita ekvivalentsust esimestena Katsov ja Nam [8] ning edasi Sardar, Gupta ja Saha [9–11]. Lokaalsete ühikutega poolringide jaoks uuris Morita ekvivalentsust esimesena Liu [12].

Käesolevas artiklis läheneme lokaalsete ühikutega poolringide Morita teooriale teisest suunast – Morita kontekstide poolt. Sel juhul ei ole veel tõestatud Morita ekvivalentsuse samaväärsus Morita konteksti olemasoluga. Näitame, et Morita konteksti leidumisest koos tingimustega, mis on analoogilised lokaalsete ühikutega ringide ja poolrühmade jaoks vajalike tingimustega, järeldub, et kahel poolringil on isomorfsed ideaalide ning kongruentside võred. Analoogilisi tulemusi on varem poolrühmade jaoks saanud Laan ja Márki [13] ning ühikelemendiga poolringide jaoks Sardar ja Gupta [10].