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CONTROL THEORY

The strong Popov form of nonlinear input–output equations

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Abstract. The equivalence transformations are applied to bring a system of nonlinear input–output (i/o) equations into a nonlinear equivalent of the Popov form, called the strong Popov form, under the assumption that the i/o equations already are in the strong row-reduced form.

Key words: discrete-time systems, input–output models, non-commutative polynomials, strong Popov form.

1. INTRODUCTION

The representation of nonlinear input–output (i/o) equations in the strong Popov form is a good starting point both for theoretical studies and implementation since such representation makes the computation of shift operators explicit. The strong Popov form simplifies the definition of the field of functions, associated with the nonlinear control system and also the software development.

This paper builds on two earlier papers [9] and [1] that address the transformation of the set of equations into the row-reduced and the strong row-reduced form, respectively. The transformations are the equivalence transformations that, by definition, do not change the zeros (solutions) of the system equations. Note that in [9] only linear transformations defined over the field of certain functions are applied. The field elements are meromorphic functions of system output and input variables and their shifts. Unfortunately, though linear transformations are enough to transform the system equations into the row-reduced form, they do not *always* result in the strong row-reduced form. The reason is that the concept of the row-reduced form in the nonlinear case is related rather to the linearized system equations than the original equations themselves. The paper [1] extends the results of [9], allowing the nonlinear equivalence transformations.

In this paper the same tools as in [9,1] are applied to transform the system equations into the (strong) Popov form. First, the linearized system description $P\mathrm{d}y - Q\mathrm{d}u = 0$ in terms of two polynomial matrices P and Q will be found. Under the assumption that the matrix $\bar{P}$, related closely to P, is in the row-reduced form

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(and respective i/o equations are in the strong row-reduced form), $\bar{P}$ will be transformed into the Popov form, multiplying from the left by a certain unimodular matrix. Then the obtained unimodular transformation matrix is applied to the original system equations. This approach works well in many situations. However, there exist numerous examples when this method leads to equations in the Popov form, which contain higher-order shifts of output variables than allowed by the definition of the strong Popov form. The reason is that the Popov form is actually a property of linearized equations and not the property of equations themselves, whereas the strong Popov form is the property of equations. In particular, the set of i/o equations is said to be in the Popov form if and only if its linearization is in the Popov form. However, in general, the Popov form of the linearized equations cannot be easily translated back to the system equations, i.e. to the strong Popov form. When the resulting equations are not in the strong Popov form, one has to look for nonlinear transformations, like in [1].

For linear time-varying systems the transformation of the system equations into the Popov form has been addressed in numerous papers (see for instance [4,6,10] and the references therein). Though general ideas in our approach are similar to those used in the theory of linear time-varying systems, the application of the results from linear time-varying theory requires special attention and is not, in most cases, directly realizable. A few issues need additional attention. First, whereas the coefficients of polynomials in the linear time-varying case belong to the field $\mathbb{R}(t)$, those in the nonlinear case are from the field of meromorphic functions in independent system variables. Moreover, as a result of computations it may happen that the dependent variables show up and these have to be replaced via the independent ones, otherwise the presented approach does not yield correct results. Second, in the construction of the respective field in the nonlinear case, a multiplicative set S has been introduced, depending on system variables. This means that certain expressions in system variables belonging to this set are not allowed to be equal to zero. Third, transforming an arbitrary nonlinear system into the strong Popov form may require nonlinear transformations, as mentioned earlier. In this case, it can happen that one may be able to transform the system equations into the strong Popov form only locally. That is, in different domains the equations may take different forms.

The paper is the extension of the conference paper [2]. Compared to [2], the following additions are made in this paper. (i) The algorithm for the Popov form has been improved. Now it also constructs a set S of inequations, ensuring that certain expressions are nonzero. (ii) The motivation behind the strong Popov form has been discussed in more details; comparison of the Popov and strong Popov forms has been added as well as the algorithm, transforming the set of i/o equations into the strong Popov form. (iii) Examples have been elaborated.

The paper is organized as follows. Section 2 introduces essential algebraic structures, related to the set of i/o equations. In Section 3 the Popov form and the strong Popov form of the set of i/o equations are discussed, whereas two examples are presented in Subsection 3.4. Section 4 draws the conclusions.

2. PRELIMINARIES

2.1. System of implicit i/o equations

We recall first the concepts and the language introduced in [9]. Consider the infinite sequences $Y = (\ldots, y(-1), y(0), y(1), y(2), \ldots,)$ and $U = (\ldots, u(-1), u(0), u(1), u(2), \ldots)$, where $y(k) \in \mathbb{R}^p$ and $u(k) \in \mathbb{R}^m$ for $k \in \mathbb{Z}$. We think of components of Y and U as independent variables. Let $\mathcal{A}$ be the set of all analytic functions with real values depending on finitely many elements of Y and U. Each function may depend on different elements of Y and U. $\mathcal{A}$ is a ring with addition and multiplication. Let $\delta : \mathcal{A} \to \mathcal{A}$ be defined as follows: $\delta y_i(k) = y_i(k+1)$, $\delta u_i(k) = u_i(k+1)$ and for $\varphi \in \mathcal{A}$, $(\delta\varphi)(Y,U) := \varphi(\delta Y, \delta U)$, where $\delta Y = \tilde{Y} := (\ldots, \tilde{y}(0), \tilde{y}(1), \tilde{y}(2), \ldots)$, $\tilde{y}(k) := y(k+1)$, and $\delta U = \tilde{U} := (\ldots, \tilde{u}(0), \tilde{u}(1), \tilde{u}(2), \ldots)$, $\tilde{u}(k) := u(k+1)$. Then $\mathcal{A}$ is a difference ring with a difference operator δ. Observe that δ is injective and onto, so it is an automorphism. Moreover, $\delta^{-1}\tilde{Y} = Y$ where $y(k) = \tilde{y}(k-1)$ and $\delta^{-1}\tilde{U} = U$ where $u(k) = \tilde{u}(k-1)$.

Let S be a multiplicative subset of the ring $\mathcal{A}$. This means that $1 \in S$, $0 \notin S$ and if a and b belong to S, so does ab. We shall assume that S is invariant with respect to both δ and δ^{-1}. Then $S^{-1}\mathcal{A}$ denotes the localization of the ring $\mathcal{A}$ with respect to S. It consists of meromorphic functions whose denominators belong to S. Observe that $S^{-1}\mathcal{A}$ is an inversive difference ring with the difference operator δ and, via the natural injection $\alpha \mapsto \frac{\alpha}{1}$, S may be interpreted as a subset of $S^{-1}\mathcal{A}$.

Consider a discrete-time multi-input multi-output nonlinear system, described by the set of implicit higher-order difference equations, relating the inputs u_k, $k = 1,\dots,m$, the outputs y_i, $i = 1,\dots,p$, and a finite number of their time shifts:

$$\varphi_i(y_j(t), y_j(t+1),\dots,y_j(t+n_{ij}), u_k(t), u_k(t+1),\dots,u_k(t+s_{ik})) = 0, \tag{1}$$

where $i, j = 1,\dots,p$, $k = 1,\dots,m$, and φ_i is a real meromorphic function belonging to $S^{-1}\mathcal{A}$, defined on an open and dense subset of $\mathbb{R}^{(n+1)(p+m)}$. If the ith equation does not depend on $y_j(t+\ell)$, $\ell = 0,1,\dots$, we set $n_{ij} := -\infty$. Let n_i be the highest output shift in the ith equation, i.e. $n_i := \max_j n_{ij}$ and let n be the highest output shift of the system, i.e. $n := \max_{i,j} n_{ij}$. Hereinafter we use the notation ξ for a variable $\xi(t)$, $\xi^{[k]}$ for its k-step time shift $\xi(t+k)$, $k \in \mathbb{Z}$.

For system (1) we define the matrix

$$M := \left[\frac{\partial \varphi_i}{\partial y_j^{[n_i]}} \right]_{ij}$$

and integers $m_i := n - n_i$ for $i = 1,\dots,p$. By $\operatorname{diag}\{\delta^{m_1},\dots,\delta^{m_p}\}$ we mean the diagonal operator matrix with the elements $\delta^{m_1},\dots,\delta^{m_p}$ on the main diagonal. The matrix $L := \operatorname{diag}\{\delta^{m_1},\dots,\delta^{m_p}\}M$ is called the *leading coefficient matrix* of system (1).

Definition 1. *The set of i/o equations (1) is said to be in the* strong row-reduced form *if its leading coefficient matrix has full rank.*

Assumption 1. In this paper it is assumed that system (1) is in the strong row-reduced form.

Note that an arbitrary implicit system of the form (1) can be transformed into the strong row-reduced form as shown in [9,1], though analytic transformation may not always exist.

Let us recall that an *ideal I* of a commutative ring A is a subset of A with the property that if $a, b \in I$, then $a + b \in I$ and if $c \in A$ and $a \in I$, then $ca \in I$. If A is a difference ring, then an ideal I of A is a *difference ideal* of A if it is closed with respect to the difference operator.

Let $\Phi = \{\varphi_1,\dots,\varphi_p\}$ be a finite subset of $S^{-1}\mathcal{A}$. Φ may be interpreted as a system of implicit i/o equations. Let $\langle\!\langle \Phi \rangle\!\rangle_S$ denote the smallest ideal of $S^{-1}\mathcal{A}$ that contains all the shifts $\delta^k(\varphi_i)$ for $i = 1,\dots,p$ and $k \in \mathbb{Z}$, i.e. the forward and backward shifts of φ_i. Observe that $\langle\!\langle \Phi \rangle\!\rangle_S$ is a difference ideal of $S^{-1}\mathcal{A}$ and $\delta(\langle\!\langle \Phi \rangle\!\rangle_S) = \langle\!\langle \Phi \rangle\!\rangle_S = \delta^{-1}(\langle\!\langle \Phi \rangle\!\rangle_S)$. Observe that Φ may be considered as a subset of $\tilde{S}^{-1}\mathcal{A}$ for some other multiplicative set $\tilde{S}$. For that reason we put S in the notation of the ideal $\langle\!\langle \Phi \rangle\!\rangle_S$. We make the following assumption:

Assumption 2. The ideal $\langle\!\langle \Phi \rangle\!\rangle_S$ is prime, i.e. if $\alpha, \beta \in S^{-1}\mathcal{A}$ and $\alpha\beta \in \langle\!\langle \Phi \rangle\!\rangle_S$, then $\alpha \in \langle\!\langle \Phi \rangle\!\rangle_S$ or $\beta \in \langle\!\langle \Phi \rangle\!\rangle_S$, and is proper, i.e. different from the entire ring.

The properness of the ideal $\langle\!\langle \Phi \rangle\!\rangle_S$ is equivalent to the condition $S \cap \langle\!\langle \Phi \rangle\!\rangle_S = \emptyset$. In particular, numerators of φ_i's do not belong to S. Let $S^{-1}\mathcal{A}/\langle\!\langle \Phi \rangle\!\rangle_S$ be the quotient ring. It consists of cosets $\bar{\varphi} = \varphi + \langle\!\langle \Phi \rangle\!\rangle_S$ for $\varphi \in S^{-1}\mathcal{A}$. We define "+" and "·" in this new ring by $\bar{\varphi} + \bar{\psi} := \overline{\varphi + \psi}$ and $\bar{\varphi} \cdot \bar{\psi} := \overline{\varphi \cdot \psi}$. These definitions do not depend on the choice of a representative in a coset. In particular $\bar{\varphi}_i = 0$, for $i = 1,\dots,p$. Since, by Assumption 2, $\langle\!\langle \Phi \rangle\!\rangle_S$ is a prime ideal, $S^{-1}\mathcal{A}/\langle\!\langle \Phi \rangle\!\rangle_S$ is an integral ring. Now we can redefine δ on $S^{-1}\mathcal{A}/\langle\!\langle \Phi \rangle\!\rangle_S$ (denoted now by δ_Φ to indicate its dependence on Φ) as follows: $\delta_\Phi \bar{\varphi} = \overline{\delta\varphi}$. This again is well defined, for if $\bar{\varphi} = \bar{\psi}$, then $\varphi + \langle\!\langle \Phi \rangle\!\rangle_S = \psi + \langle\!\langle \Phi \rangle\!\rangle_S$. Since $\delta(\langle\!\langle \Phi \rangle\!\rangle_S) \subset \langle\!\langle \Phi \rangle\!\rangle_S$ and $\delta(\langle\!\langle \Phi \rangle\!\rangle_S) + \langle\!\langle \Phi \rangle\!\rangle_S = \langle\!\langle \Phi \rangle\!\rangle_S$, $\delta\varphi + \delta(\langle\!\langle \Phi \rangle\!\rangle_S) = \delta\psi + \delta(\langle\!\langle \Phi \rangle\!\rangle_S)$. Moreover, the operator δ_Φ is bijective, so δ_Φ^{-1} is well defined on $S^{-1}\mathcal{A}/\langle\!\langle \Phi \rangle\!\rangle_S$. Let $\mathcal{Q}_S^\Phi$ denote the field of fractions of the ring $S^{-1}\mathcal{A}/\langle\!\langle \Phi \rangle\!\rangle_S$. As δ_Φ can be naturally extended to the field of fractions, $\mathcal{Q}_S^\Phi$ is now an inversive difference field with the difference operator δ_Φ.

Proposition 1. [9] *Assume that S and $\tilde{S}$ are multiplicative subsets of $\mathcal{A}$ and $S \subset \tilde{S}$, invariant with respect to δ and δ^{-1}. Let $\Phi \subset S^{-1}\mathcal{A}$ and the ideal $\langle\!\langle \Phi \rangle\!\rangle_S$ be prime and proper. Let $\tilde{S} \cap \langle\!\langle \Phi \rangle\!\rangle_{\tilde{S}} = \varnothing$. Then (a) $S^{-1}\mathcal{A} \subset \tilde{S}^{-1}\mathcal{A}$, (b) the ideal $\langle\!\langle \Phi \rangle\!\rangle_{\tilde{S}}$ of $\tilde{S}^{-1}\mathcal{A}$ is prime and proper, (c) there is a natural monomorphism of difference rings $\tau : S^{-1}\mathcal{A}/\langle\!\langle \Phi \rangle\!\rangle_S \to \tilde{S}^{-1}\mathcal{A}/\langle\!\langle \Phi \rangle\!\rangle_{\tilde{S}}$, and (d) τ may be extended to a monomorphism of difference fields $\mathcal{Q}_S^{\Phi} \to \mathcal{Q}_{\tilde{S}}^{\Phi}$.*

2.2. Non-commutative polynomials

The field $\mathcal{Q}_S^{\Phi}$ and the shift operator δ_{Φ} induce the ring of polynomials in a variable Z over $\mathcal{Q}_S^{\Phi}$, denoted by $\mathcal{Q}_S^{\Phi}[Z;\delta_{\Phi}]$. A polynomial $a \in \mathcal{Q}_S^{\Phi}[Z;\delta_{\Phi}]$ is written as $a = a_{\mu}Z^{\mu} + a_{\mu-1}Z^{\mu-1} + \ldots + a_1 Z + a_0$, where $a_i \in \mathcal{Q}_S^{\Phi}$ for $0 \leqslant i \leqslant \mu$. The addition of polynomials from $\mathcal{Q}_S^{\Phi}[Z;\delta_{\Phi}]$ is standard. The multiplication is defined by the linear extension of the following rules $Z \cdot a := (\delta_{\Phi}a)Z$ and $a \cdot Z := aZ$, where $a \in \mathcal{Q}_S^{\Phi}$ and $\delta_{\Phi}a$ means δ_{Φ} evaluated at a (so for example $(aZ^{\mu}) \cdot (bZ^{\nu}) = a(\delta_{\Phi}^{\mu}b)Z^{\mu+\nu}$). Observe that an element $a \in \mathcal{Q}_S^{\Phi} \subset \mathcal{Q}_S^{\Phi}[Z;\delta_{\Phi}]$ does not commute with Z in general, so the ring $\mathcal{Q}_S^{\Phi}[Z;\delta_{\Phi}]$ is non-commutative. It is called the *twisted polynomial ring* and it satisfies both the left and right Ore conditions, i.e. it is an Ore ring [7]. Moreover, for the polynomials a and b it holds that $\deg(a \cdot b) = \deg a + \deg b$. Let us define the action of the ring $\mathcal{Q}_S^{\Phi}[Z;\delta_{\Phi}]$ on the field $\mathcal{Q}_S^{\Phi}$ by the linear extension of the formula $Z^s \curvearrowright a := \delta_{\Phi}^s a$, where $a \in \mathcal{Q}_S^{\Phi}$.

Let $\mathcal{Q}_S^{\Phi}[Z,\delta_{\Phi}]^{p\times q}$ be the set of $p \times q$-dimensional matrices, whose entries are polynomials in $\mathcal{Q}_S^{\Phi}[Z,\delta_{\Phi}]$.

Definition 2. *For the matrices from $\mathcal{Q}_S^{\Phi}[Z,\delta_{\Phi}]^{p\times q}$ we define the following* elementary row operations: *(1) interchange of rows i and j, (2) multiplication of the row i by non-zero scalar from $\mathcal{Q}_S^{\Phi}$, (3) replacement of the row i by itself plus any polynomial multiplied by any other row j.*

Observe that all elementary row operations are invertible and any elementary row operation on matrix $W \in \mathcal{Q}_S^{\Phi}[Z,\delta_{\Phi}]^{p\times q}$ is equivalent to premultiplication (left multiplication) of W by an appropriate invertible matrix $E \in \mathcal{Q}_S^{\Phi}[Z,\delta_{\Phi}]^{p\times p}$. A matrix $U \in \mathcal{Q}_S^{\Phi}[Z,\delta_{\Phi}]^{p\times p}$ is called *unimodular* if it has an inverse matrix $U^{-1} \in \mathcal{Q}_S^{\Phi}[Z,\delta_{\Phi}]^{p\times p}$ such that $UU^{-1} = U^{-1}U = I_p$, where I_p is identity matrix.

2.3. Linearized i/o equations

Our goal is to represent system (1) in terms of polynomials from $S^{-1}\mathcal{A}[Z;\delta_{\Phi}]$. For that purpose we apply the differential operation d to (1) to obtain $\sum_{j=1}^{p}\sum_{\alpha=0}^{n}(\partial\varphi_i/\partial y_j^{[\alpha]})\mathrm{d}y_j^{[\alpha]} + \sum_{k=1}^{m}\sum_{\beta=0}^{n}(\partial\varphi_i/\partial u_k^{[\beta]})\mathrm{d}u_k^{[\beta]} = 0$ for $i = 1,\ldots,p$. Defining $Z^{\alpha}\mathrm{d}y_j := \mathrm{d}y_j^{[\alpha]}$ and $Z^{\beta}\mathrm{d}u_k := \mathrm{d}u_k^{[\beta]}$ like in [9] enables us to rewrite (1) as

$$P(Z)\mathrm{d}y + Q(Z)\mathrm{d}u = 0, \tag{2}$$

where $P \in S^{-1}\mathcal{A}[Z;\delta]^{p\times p}$ and $Q \in S^{-1}\mathcal{A}[Z;\delta]^{p\times m}$ are polynomial matrices, whose elements $p_{ij}, q_{ik} \in S^{-1}\mathcal{A}[Z;\delta]$ are $p_{ij} = \sum_{\alpha=0}^{n}(\partial\varphi_i/\partial y_j^{[\alpha]})Z^{\alpha}$, $q_{ik} = \sum_{\beta=0}^{n}(\partial\varphi_i/\partial u_k^{[\beta]})Z^{\beta}$ and $\mathrm{d}y = [\mathrm{d}y_1,\ldots,\mathrm{d}y_p]^{\mathrm{T}}$, $\mathrm{d}u = [\mathrm{d}u_1,\ldots,\mathrm{d}u_p]^{\mathrm{T}}$. Equation (2) describes the *(globally) linearized system*, associated with equations (1).

Let e_S^{Φ} denote the map $S^{-1}\mathcal{A} \to \mathcal{Q}_S^{\Phi} : \varphi \mapsto (\varphi + \langle\!\langle \Phi \rangle\!\rangle_S)/1$ (later we usually skip the denominator). If $p = \sum_i p_i Z^i$ with $p_i \in S^{-1}\mathcal{A}$, then we define $e_S^{\Phi}(p) := \sum_i e_S^{\Phi}(p_i)Z^i$. This is a polynomial in $\mathcal{Q}_S^{\Phi}[Z;\delta_{\Phi}]$. For a matrix P with elements in $S^{-1}\mathcal{A}[Z;\delta]$, we define the matrix $\bar{P} := e_S^{\Phi}(P)$ where $e_S^{\Phi}(P)_{ij} := e_S^{\Phi}(p_{ij})$.

3. POPOV FORM

3.1. Linearized equations in the Popov form

Let us denote the ith row of the matrix $W \in \mathcal{Q}_S^{\Phi}[Z,\delta_{\Phi}]^{p\times q}$ by $w_{i\bullet}$. For the non-zero row $w_{i\bullet}$ we define its degree $\deg w_{i\bullet} \equiv \sigma_i$ as the exponent of the highest power in Z present in $w_{i\bullet}$ for $i = 1,\ldots,p$. If $w_{i\bullet} \equiv 0$, we define $\sigma_i = -\infty$. The vector of the row degrees is denoted by $\sigma := [\sigma_1,\dots,\sigma_p]$. The column degrees $\tau_1,\dots,\tau_q$ are defined as the row degrees of the matrix W^{T}. The degree of the matrix W is defined as $\deg W := \max\{\sigma_1,\dots,\sigma_p\}$. Let $N = \deg W$, $e = [1,\dots,1]$, and $M = [m_1,\dots,m_p] := N\cdot e - \sigma$. By Z^M we denote the diagonal $p\times p$ matrix with the diagonal elements $Z^{m_1},\dots,Z^{m_p}$.

Definition 3. *The matrix $L_{\mathrm{row}}(W)$ such that $Z^M W = L_{\mathrm{row}}(W)Z^N +$ lower degree terms is called the* leading row coefficient matrix *of W.*

The matrix $W \in \mathcal{Q}_S^{\Phi}[Z,\delta_\Phi]^{p\times q}$ is said to have *full rank* if $\operatorname{rank} W = \min(p,q)$.

Definition 4. [3,9,10] *A polynomial matrix $W \in \mathcal{Q}_S^{\Phi}[Z,\delta_\Phi]^{p\times q}$ with non-zero rows is called* row-reduced *if its leading row coefficient matrix $L_{\mathrm{row}}(W)$ has full row rank over the field $\mathcal{Q}_S^{\Phi}$. If W contains zero rows, then W is called row-reduced if its submatrix consisting of non-zero rows is row-reduced.*

Definition 5. [3] *A polynomial matrix $W \in \mathcal{Q}_S^{\Phi}[Z,\delta_\Phi]^{p\times q}$ is said to be in the* weak Popov form *if the leading coefficient of the submatrix formed from the non-zero rows of W is in upper triangular form up to row permutations.*

Definition 6. [8,10] *Matrix $W \in \mathcal{Q}_S^{\Phi}[Z,\delta_\Phi]^{p\times q}$ is in the* Popov form *if W is row-reduced with the rows sorted with respect to their degrees ($\sigma_1 \leqslant \cdots \leqslant \sigma_p$) and for all non-zero rows $w_{i\bullet}$ there is a column index j_i (called the* pivot index*) such that*

(I) w_{ij_i} *is monic*

(II) $\deg w_{ik} < \deg w_{ij_i}$, *if* $k < j_i$

(III) $\deg w_{ik} \leqslant \deg w_{ij_i}$, *if* $k \geqslant j_i$

(IV) $\deg w_{kj_i} < \deg w_{ij_i}$, *if* $k \neq i$

(V) if $\deg w_{ij_i} = \deg w_{kj_k}$ *and* $i < k$, *then* $j_i < j_k$ *(if the degrees of the rows are equal, then the pivot indices are increasing).*

Proposition 2. [10] *For any matrix $W \in \mathcal{Q}_S^{\Phi}[Z,\delta_\Phi]^{p\times q}$ there exists a unimodular matrix $U \in \mathcal{Q}_S^{\Phi}[Z,\delta_\Phi]^{p\times p}$ such that $U\cdot W$ is in the Popov form.*

Remark 1. Every matrix in the Popov form is also in the weak Popov form while converse is not necessarily true. If we relax Definition 6 so that the condition $\sigma_1 \leqslant \cdots \leqslant \sigma_p$ is dropped and (I), (IV), and (V) are replaced by the requirement that pivot indices are different ($j_i \neq j_k$ whenever $i \neq k$), then we obtain the definition for the weak Popov form.

Definition 7. *The set of i/o difference equations (1) is said to be in the* Popov form *if there is a multiplicative subset $\widetilde{S}$ of $\mathcal{A}$, $\widetilde{S} \supseteq S$, such that the matrix $\bar{P} = e_{\widetilde{S}}^{\Phi}(P)$ is in the Popov form over $\mathcal{Q}_{\widetilde{S}}^{\Phi}[Z;\delta_\Phi]$.*

3.2. Algorithm: transforming the matrix into the Popov form

The following algorithm transforms the matrix W into the Popov form under the assumption that the $p\times q$ matrix W is row-reduced and its row degrees are non-decreasing: $0 \leqslant \sigma_1 \leqslant \sigma_2 \leqslant \cdots \leqslant \sigma_p$. The algorithm is a straightforward extension of the respective procedure for the linear time-varying case [10], except the steps related to the construction of the set S_0.

Algorithm 1
Input: Matrix W
Output: Matrix $\widetilde{W}$ in the Popov form and unimodular matrix U, such that $\widetilde{W} = UW$
Step 1. $\widetilde{W} := W$, $U := I_p$, $S_0 := \{1\}$
Step 2. $i := 1$
Step 3. Fix the column index j_i such that $\deg \widetilde{w}_{ij_i} = \sigma_i$ and j_i is minimal of all possible columns
Step 4. $k := 1$

Step 5. **If** $k \neq i$ **And** $\deg \widetilde{w}_{kj_i} \geqslant \sigma_i$ **Then**
(a) Find γ and r such that $\widetilde{w}_{kj_i} = \gamma \widetilde{w}_{ij_i} + r$ and $\deg r < \sigma_i$
(b) $E := I_p$
(c) $E_{k,i} := -\gamma$
(d) $\widetilde{W} := E \cdot \widetilde{W}$
(e) $U := E \cdot U$
(f) Add denominators of coefficients of γ to S_0 (if they are not in S_0 already).
End If
Step 6. $k := k+1$
Step 7. **If** $k \leqslant p$ **Then Goto** Step 6
Step 8. $i := i+1$
Step 9. **If** $i < p$ **Then Goto** Step 4
Step 10. **If** condition (IV) of Definition 6 is not fulfilled, **Then Goto** Step 2
Step 11. Make the pivot elements of $\widetilde{W}$ monic:
(a) Let $\alpha_1, \ldots, \alpha_p$ be the leading coefficients of the pivot elements $\widetilde{w}_{1j_1}, \ldots, \widetilde{w}_{pj_p}$
(b) Let A be the diagonal matrix with $\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_p}$ on the main diagonal
(c) $\widetilde{W} := A \cdot \widetilde{W}$
(d) $U := A \cdot U$
(e) Add functions $\alpha_1, \ldots, \alpha_p$ to S_0 (if they are not in S_0 already).
Step 12. **Return** $\widetilde{W}$, U

When we start, some functions φ_i in (1) may have denominators that, together with their forward and backward shifts, should be included in the set S. If the functions are analytic, one may set $S := \{1\}$, meaning that $S^{-1}\mathcal{A} = \mathcal{A}$. Of course, additional denominators that show up in the reduction algorithm should also be included in S together with their shifts and powers; this means denominators both in the equivalence transformations and in transformed matrix $\widetilde{W}$. That is, we extend our initial S by adding an infinite number of elements.

In extension one has to be careful not to include in S the denominators that may cancel the functions φ_i. This is guaranteed by the use of the field $\mathcal{Q}_S^{\Phi}$ (in the algorithm) whose elements are fractions with non-zero denominators. Since we work with cosets with respect to $\langle\!\langle \Phi \rangle\!\rangle_S$, the denominators are represented by functions that do not belong to $\langle\!\langle \Phi \rangle\!\rangle_S$. Since we multiply the functions φ_i by the functions, being representatives of the cosets, the functions φ_i cannot be cancelled (because they cannot appear in denominators). This explains why the property $S \bigcap \langle\!\langle \Phi \rangle\!\rangle_S = \varnothing$ must always be satisfied.

Remark 2. The infinite set S can be compactly described by its generator set S_0. The set S_0 generates S if each element of S can be obtained from a finite number of elements of S_0 by applying a finite number of multiplications and backward and forward shifts to these elements. For the sake of simplicity, we present in examples below rather the generator set S_0 than S itself.

If the assumptions of Algorithm 1 are not fulfilled, the pivot indices j_i can change during the execution of the algorithm. To avoid possible misunderstanding, we show that if the assumptions are satisfied, the pivot indices keep their values.

Lemma 1. *Assume that the matrix W is row-reduced and its row degrees satisfy $0 \leqslant \sigma_1 \leqslant \cdots \leqslant \sigma_p$. Consider the transformation on Step 5(a) of Algorithm 1, which can be written as*

$$\tilde{w}_{k\bullet} := w_{k\bullet} - \gamma w_{i\bullet}, \tag{3}$$

where $w_{k\bullet}$ is the old kth row and $\tilde{w}_{k\bullet}$ is the transformed row. Transformation (3) does not change the pivot indices that are already fixed on Step 3.

Proof. One has to show that transformation (3) does not change the pivot indices $j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_p$. Without loss of generality we assume in this proof that all fixed pivot indices are on the main diagonal, i.e. $j_\ell = \ell$ for $\ell = 1, \dots, p$.

Case 1, $k < i$. We show that after applying transformation (3) to matrix W the pivot index of the row $\tilde{w}_{k\bullet}$ is still k. *First*, we show that the transformed elements $\tilde{w}_{k\nu} := w_{k\nu} - \gamma w_{i\nu}$, $\nu = 1, \dots, k-1$ satisfy the condition $\deg \tilde{w}_{k\nu} < \sigma_k$. Since $\sigma_1 \leqslant \cdots \leqslant \sigma_p$, the transformation (3) occurs only if $\sigma_k = \sigma_i$ and thus $\deg w_{ki} = \deg w_{ii}$ and $\deg \gamma = 0$. Thus $\deg \tilde{w}_{k\nu} = \max\{\deg w_{k\nu}, \deg w_{i\nu}\} < \sigma_k$. *Second*, we consider the element $\tilde{w}_{kk} := w_{kk} - \gamma w_{ik}$ and show that $\deg \tilde{w}_{kk} = \sigma_k$. The degree of $\tilde{w}_{kk}$ would be less than σ_k only if the leading terms of w_{kk} and γw_{ik} would be equal, but this cannot happen, since $\deg w_{ik} < \deg \sigma_i = \deg \sigma_k = \deg w_{kk}$ and $\deg \gamma = 0$.

Case 2, $i < k$. Note that when passing the main loop (Steps 2–9) for the first time, the pivot indices are not yet fixed for the rows $w_{i+1,\bullet}, \dots, w_{p\bullet}$, thus the proof makes sense for the second or further execution of the main loop. We show that after applying transformation (3) to matrix W the pivot index of the row $\tilde{w}_{k\bullet}$ is still k. *First*, we show that the transformed elements $\tilde{w}_{k\ell} := w_{k\ell} - \gamma w_{i\ell}$, $\ell = 1, \dots, k-1$, satisfy the condition $\deg \tilde{w}_{k\ell} < \sigma_k$. The sum of two polynomials cannot have the degree higher than the addends. Thus,

$$\deg \tilde{w}_{k\ell} \leqslant \max\{\deg w_{k\ell},\ \deg(\gamma w_{i\ell})\}. \tag{4}$$

Since pivot of $w_{k\bullet}$ is k,

$$\deg w_{k\ell} < \sigma_k,\ \ \ell = 1, \dots, k-1. \tag{5}$$

The polynomial γ is determined in Step 5(a) as the right quotient of w_{ki} and w_{ii} such that $w_{ki} = \gamma w_{ii} + \tilde{w}_{ki}$, and $\deg \tilde{w}_{ki} < \deg w_{ii}$. Then $\deg \gamma = \deg w_{ki} - \deg w_{ii}$ and thus

$$\deg(\gamma w_{i\ell}) \leqslant \deg w_{ki} - \deg w_{ii} + w_{i\ell}. \tag{6}$$

Knowing that $\deg w_{ki} < \sigma_k$, $\deg w_{ii} = \sigma_i$, and $\deg w_{i\ell} \leqslant \sigma_i$ for $\ell = 1, \dots, p$ allows us to deduce $\deg(\gamma w_{i\ell}) < \sigma_k$. Due to (5) and (6) the inequality (4) yields $\deg \tilde{w}_{k\ell} < \sigma_k$ for $\ell = 1, \dots, k-1$. *Second*, we show that $\deg \tilde{w}_{kk} = \sigma_k$. By (3), $\tilde{w}_{kk} = w_{kk} - \gamma w_{ik}$, and since the pivot index of $w_{k\bullet}$ is k, one knows that $\deg w_{kk} = \sigma_k$. It is possible to get $\deg \tilde{w}_{kk} < \sigma_k$ if the leading terms of w_{kk} and γw_{ik} are equal, but this would mean that the rows of $L_{\mathrm{row}}(W)$ are dependent and thus W is not row-reduced, which contradicts the assumption. □

Theorem 1. *Assume that the matrix W is row-reduced and its row degrees satisfy the inequalities $0 \leqslant \sigma_1 \leqslant \sigma_2 \leqslant \cdots \leqslant \sigma_p$. Application of Algorithm 1 transforms W into the Popov form.*

Proof. Passing Steps 1–9 of the algorithm for the first time transforms matrix $\widetilde{W}$ into the form, where pivot indices j_i are fixed for each row and conditions (II) and (III) of Definition 6 are fullfilled. Due to Lemma 1, if the pivot index is once fixed, it cannot change later. Executing Steps 1–9 transforms the leading row coefficient matrix $L_{\mathrm{row}}(\widetilde{W})$ into the upper triangular form (up to permutation of the rows) so that the elements $L_{\mathrm{row}}(\widetilde{W})_{i,j_i}$, corresponding to the pivot indices, are non-zero, and those located left from $L_{\mathrm{row}}(\widetilde{W})_{i,j_i}$ are zero. It means that $\widetilde{W}$ is in the weak Popov form.

Step 10 checks whether condition (IV) is fulfilled. In the case of a negative answer Steps 2–9 are repeated until condition (IV) is satisfied. Condition (V) of Definition 6 is guaranteed, since on Step 3 we choose j_i as the smallest possible. Finally, Step 11 makes the pivot elements of $\widetilde{W}$ monic. □

Remark 3. The weak Popov form can be obtained as an intermediate step of computing the Popov form. For the weak Popov form, we remove Step 10 from Algorithm 1. For the Popov form the outer loop (Step 2–Step 9) of the algorithm is passed at most $p-1$ times, but for the weak Popov form these steps are passed just once.

3.3. Strong Popov form

The definition of the Popov form as given in Definition 7 is actually a property of linearized i/o equations (2) (where the matrix P is replaced by $\bar{P}$) and not the property of the original system equations (1). In particular, the set of i/o equations is said to be in the Popov form if the polynomial matrix $\bar{P}$ is in the Popov form. To transform the linearized i/o equations (2) into the Popov form, linear i/o equivalence transformation, as constructed in Algorithm 1, is used. Such transformation is defined in terms of unimodular polynomial matrix over the difference field of meromorphic functions $\mathcal{Q}^{\Phi}_{\widetilde{S}}$ in independent system variables and the polynomial indeterminate may be interpreted as a forward shift operator. Finally, the unimodular matrix defines the operator that will be applied to system equations (1). The application of the unimodular matrix to the system equations corresponds to the linear transformations (over $\mathcal{Q}^{\Phi}_{S}$) of the equations, however, sometimes nonlinear transformations are necessary to bring the system into the explicit form, see Example 3. This approach works well in many situations. However, sometimes this method leads to equations in the Popov form, which cannot be transformed into the explicit form. For this reason we introduce below the strong Popov form.

Definition 8. *The set of i/o equations (1) is said to be in the* strong Popov form *if*
(a) $0 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_p$;
(b) for each φ_i, $n_i \geqslant 0$, *there exists a variable* y_{j_i} *such that the following conditions hold*: *(i)* $\partial\varphi_i/\partial y_{j_i}^{[n_i]} = 1$; *(ii)* $n_{ik} < n_i$, *if* $k < j_i$; *(iii)* $n_{ik} \leqslant n_i$, *if* $k \geqslant j_i$; *(iv)* $n_{kj_i} < n_i$, *if* $k \neq i$; *(v) if* $n_i = n_k$ *and* $i < k$, *then* $j_i < j_k$.

Comparing Definitions 6 (Popov form) and 8 (strong Popov form) reveals that conditions (I)–(V) in Definition 6 regarding the degrees of polynomials match conditions (i)–(v) in Definition 8 regarding the structural indices n_{ij} of system (1). Though the definitions are analogous, it is important to note that the Popov form of system (1) is, by Definition 7, determined by matrix $\bar{P}$, whose entries do not necessarily satisfy the condition $\deg \bar{p}_{ij} = n_{ij}$, but rather $\deg \bar{p}_{ij} = \deg(e^{\Phi}_{\widetilde{S}}(p_{ij})) \leqslant \deg p_{ij} = n_{ij}$. Moreover, note that in the Popov form of matrix $\bar{P}$ zero rows are allowed while in the strong Popov form we require that all indices $n_i > -\infty$. Thus the strong Popov form of system (1) implies the Popov form, but the converse is not true, in general. An example of the system, being in the Popov but not in the strong Popov form, can be found in Example 3 below.

If system (1) is in the strong Popov form, it can be represented in the explicit form

$$y_{j_i}^{[n_i]} = \phi_i\Big(y_j, \ldots, y_j^{[\nu_{ij}]}, u_k, \ldots, u_k^{[s_{ik}]}\Big), \; i = 1, \ldots, p, \tag{7}$$

where $j = 1, \ldots, p$, $k = 1, \ldots, m$, and

$$\nu_{ij} := \begin{cases} n_{ij}, \text{ if } j \neq j_i \\ n_{ij} - 1, \text{ if } j = j_i. \end{cases}$$

Example 1. In the special case $p = m = 2$, assume, for the sake of simplicity, that $j_1 = 1$ and $j_2 = 2$. By Definition 8 this system is in the strong Popov form if the following conditions hold:
(a) since $j_i = i$, one has $n_1 := n_{11}$ and $n_2 := n_{22}$, which have to satisfy $n_1 \leqslant n_2$;
(b) (i) $\partial\varphi_1/\partial y_1^{[n_1]} = 1$ and $\partial\varphi_2/\partial y_2^{[n_2]} = 1$; (ii) $n_{21} < n_2$; (iii) $n_{12} \leqslant n_1$; (iv) $n_{12} < n_2$ and $n_{21} < n_1$; (v) in case $j_i = i$, the condition is always satisfied.

The following algorithm transforms the implicit i/o equations (1) into the strong Popov form using linear transformations, whenever possible.

Algorithm 2
Input: Implicit equations (1) in strong row-reduced form
Output: System $\tilde{\varphi}_i = 0$ in the strong Popov form
Step 1. Find the matrix $\bar{P} = e^{\Phi}_{\widetilde{S}}(P)$
Step 2. Transform the row-reduced matrix $\bar{P}$ into the Popov form by Algorithm 1. It means multiplying $\bar{P}$ by the unimodular matrix $U \in \mathcal{Q}^{\Phi}_{S}[Z, \delta_\Phi]^{p\times p}$. Denote $\widetilde{P} := U\bar{P}$
Step 3. Apply U as an operator to system (1): $[\tilde{\varphi}_i, \ldots, \tilde{\varphi}_p]^{\mathrm{T}} := U \curvearrowright [\varphi_1, \ldots, \varphi_p]^{\mathrm{T}}$
Step 4. **If** System $\tilde{\varphi}_i = 0$ is in the strong Popov form, i.e. Definition 8 is satisfied

Then Return $\tilde{\varphi}_i$
Else Print: "The system cannot be transformed into the strong Popov form via linear transformations."

Proposition 3. *An arbitrary nonlinear system (1) can be transformed (at least locally) in the strong Popov form (with possible zero rows), using either linear or nonlinear transformations.*

3.4. Examples

Example 2. Consider the system

$$\varphi_1 \equiv u_2 + y_2^{[1]} + y_3^{[1]} = 0,\ \varphi_2 \equiv y_2 + u_1 y_2^{[2]} + y_3 = 0,\ \varphi_3 \equiv u_3^{[1]} + y_1^{[3]} + y_1 y_3^{[3]} = 0. \tag{8}$$

Since no denominators appear in equations (8), we may set $S = S_0 := \{1\}$. The polynomial matrix

$$P = \bar{P} = \begin{bmatrix} 0 & Z & Z \\ 0 & u_1 Z^2 + 1 & 1 \\ Z^3 + y_3^{[3]} & 0 & y_1 Z^3 \end{bmatrix}. \tag{9}$$

The leading row coefficient matrix

$$L_{\mathrm{row}}(\bar{P}) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & u_1^{[1]} & 0 \\ 1 & 0 & y_1 \end{bmatrix} \tag{10}$$

is of full rank, thus (8) is row-reduced, however, the conditions of Definition 6 are not fulfilled and therefore, the system is not in the Popov form. The row degree vector of the matrix $\bar{P}$ is $\sigma = [1, 2, 3]$, thus row degrees are in non-decreasing order. Following Algorithm 1, we take the matrix $\widetilde{P} := \bar{P}$, $U = I_3$, and $i = 1$. Since $\sigma_1 = 1$, the pivot element of the 1st row is $\widetilde{p}_{12}$, i.e. $j_1 = 2$. Next, the aim of Steps 4–6 is to make the degrees of all the elements below $\widetilde{p}_{12}$ strictly less than $\sigma_1 = 1$. For the 2nd row ($k = 2$) we obtain $\widetilde{p}_{22} = \gamma \widetilde{p}_{12} + r$, consequently $\gamma = u_1 Z$, $r = 1$, and

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -u_1 Z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

Since γ has no denominators, the generator set $S_0 := \{1\}$ remains unchanged. Multiplying $\widetilde{P} = \bar{P}$ by E from the left corresponds to multiplying the 1st row by $-\gamma$ and adding it to the 2nd row, resulting in

$$\widetilde{P} := \begin{bmatrix} 0 & Z & Z \\ 0 & 1 & -u_1 Z^2 + 1 \\ Z^3 + y_3^{[3]} & 0 & y_1 Z^3 \end{bmatrix}. \tag{12}$$

We also let $U := E \cdot U = E$. For $k = 3$ the degree condition $\deg \widetilde{p}_{32} = -\infty \geqslant \sigma_1 = 1$ is fulfilled, therefore the **Then**-part in Step 5 is skipped.

Taking $i = 2$ yields that $j_2 = 3$ – the pivot element of the second row is $\widetilde{p}_{23}$. For $k = 1$ the degree condition is fulfilled, but for $k = 3$ we obtain $\gamma = -\frac{y_1}{u_1^{[1]}} Z$ and $r = \frac{y_1}{u_1^{[1]}} Z$, yielding

$$E := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{y_1}{u_1^{[1]}} Z & 1 \end{bmatrix}, \quad \widetilde{P} := \begin{bmatrix} 0 & Z & Z \\ 0 & 1 & -u_1 Z^2 + 1 \\ Z^3 + y_3^{[3]} & \frac{y_1}{u_1^{[1]}} Z & \frac{y_1}{u_1^{[1]}} Z \end{bmatrix}.$$

Due to the division by $u_1^{[1]}$ we set $S_0 = \{1, u_1\}$. The matrix

$$U := E \cdot U = \begin{bmatrix} 1 & 0 & 0 \\ -u_1 Z & 1 & 0 \\ -y_1 Z^2 & \frac{y_1}{u_1^{[1]}} Z & 1 \end{bmatrix}.$$

For $i = 3$ the pivot element is $\widetilde{p}_{31}$. Since $\deg \widetilde{p}_{11} = \deg \widetilde{p}_{21} = -\infty < \deg \widetilde{p}_{31} = 3$, the degree conditions are satisfied. We have reached Step 10, where we ascertain that condition (IV) of Definition 6 is not fulfilled and thus return to Step 2. For $i = 1$ and $j_1 = 2$ Steps 5 and 6 give the following results: if $k = 2$, the degree condition is fulfilled, if $k = 3$, then $\gamma = \frac{y_1}{u_1^{[1]}} Z$, $r = 0$ and thus

$$E := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{y_1}{u_1^{[1]}} & 0 & 1 \end{bmatrix}, \quad \widetilde{P} := \begin{bmatrix} 0 & Z & Z \\ 0 & 1 & -u_1 Z^2 + 1 \\ Z^3 + y_3^{[3]} & 0 & 0 \end{bmatrix}.$$

The transformed matrix $\widetilde{P}$ fulfils conditions (I)–(V) of Definition 6. Note that the row and column degrees of $\widetilde{P}$ are equal now, if regarding the position of the pivot elements. That is $\sigma_1 = \tau_2 = 1$, $\sigma_2 = \tau_3 = 2$, $\sigma_3 = \tau_1 = 3$. The unimodular matrix is

$$U := E \cdot U = \begin{bmatrix} 1 & 0 & 0 \\ -u_1 Z & 1 & 0 \\ -y_1 Z^2 - \frac{y_1}{u_1^{[1]}} & \frac{y_1}{u_1^{[1]}} Z & 1 \end{bmatrix}. \tag{13}$$

The application of U to the vector function $\varphi = [\varphi_1, \varphi_2, \varphi_3]^{\mathrm{T}}$ yields the system in the strong Popov form[1]:

$$U(Z) \Rsh \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{bmatrix} = \begin{bmatrix} u_2 + y_2^{[1]} + y_3^{[1]} \\ u_2^{[1]} + y_3^{[2]} - \frac{y_2 + y_3}{u_1} \\ u_3^{[1]} - \frac{u_2 y_1}{u_1^{[1]}} - u_2^{[2]} y_1 + y_1^{[3]} \end{bmatrix}. \tag{14}$$

To transform (8) into the strong Popov form (14), it is necessary that all the variables from the set S, generated by $S_0 = \{1, u_1\}$, are non-zero, see Remark 2. From (14) one can easily express the highest time-shifts of the output variables:

$$y_2^{[1]} = -u_2 - y_3^{[1]}, \quad y_3^{[2]} = -u_2^{[1]} + \frac{y_2 + y_3}{u_1}, \quad y_1^{[3]} = -u_3^{[1]} + u_2^{[2]} y_1 + \frac{u_2 y_1}{u_1^{[1]}}.$$

Example 3. Consider the model of the 27 tray binary distillation column, operating in high-purity regime [11], where u_1 is modular reflux, u_2 is steam flow rate (moles/minute), y_1 and y_2 are temperatures[2]:

$$\begin{aligned} \varphi_1 &:= y_2^{[1]} - 0.0018 + 0.22 u_1 + 1.7 u_2^2 - 0.92 y_2 - 30.4 u_2 y_2^2 = 0, \\ \varphi_2 &:= y_1^{[3]} - 0.0012 + 0.18 u_1^{[2]} - 1.1 u_2^{[2]} y_1 - 0.98 y_1^{[2]} + 1.8 u_1^{[2]} y_2^{[2]} = 0. \end{aligned} \tag{15}$$

[1] For $\Rsh$ see Subsection 2.2.

[2] To guarantee the condition $\deg \bar{p}_{1\bullet} \leqslant \deg \bar{p}_{2\bullet}$, we have reversed the order of equations, compared with the original equations in [11]. Note that though the condition $\deg \bar{p}_{1\bullet} \leqslant \deg \bar{p}_{2\bullet}$ is not necessary for the weak Popov form, it is required later for nonlinear transformation.

The generator set is $S_0 := \{1\}$. The matrix

$$\bar{P} = P = \begin{bmatrix} 0 & Z - 60.8u_2y_2 - 0.92 \\ Z^3 - 0.98Z^2 - 1.1u_2^{[2]} & 1.8u_1^{[2]}Z^2 \end{bmatrix} \tag{16}$$

and its leading row coefficient matrix

$$L_{\text{row}}(\bar{P}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus, (16) is row-reduced and in the weak Popov form, but it is not in the Popov form, since condition (IV) of Definition 6 is not fulfilled. Indeed, for $i = 1$, $j_1 = 2$, $k = 2$ we obtain $\deg \bar{p}_{22} = 2 \nless \deg \bar{p}_{12} = 1$, whenever $u_1^{[2]} \neq 0$. However, since P is in the weak Popov form and the pivot indices $j_1 = 2$, $j_2 = 1$ are fixed, system (15) can be rewritten in the form below:

$$y_2^{[1]} = 0.0018 - 0.22u_1 - 1.7u_2^2 + 0.92y_2 + 30.4u_2y_2^2, \tag{17a}$$
$$y_1^{[3]} = 0.0012 - 0.18u_1^{[2]} + 1.1u_2^{[2]}y_1 + 0.98y_1^{[2]} - 1.8u_1^{[2]}y_2^{[2]}. \tag{17b}$$

The first equation is in the proper form while the second is not, since it involves $y_2^{[2]}$. Substituting $y_2^{[2]}$ in (17b) by the right-hand side of forward-shifted (17a) gives $y_1^{[3]} = 0.0012 - 0.18u_1^{[2]} + 1.1u_2^{[2]}y_1 + 0.98y_1^{[2]} - 1.8u_1^{[2]} + \left[0.0018 - 0.22u_1^{[1]} - 1.7\left(u_2^{[1]}\right)^2 + 0.92y_2^{[1]} + 30.4u_2^{[1]}\left(y_2^{[1]}\right)^2\right]$. The latter equation still depends on $y_2^{[1]}$, which has to be again substituted by the right-hand side of (17a). That way one obtains the explicit equations

$$y_2^{[1]} = 0.0018 - 0.22u_1 - 1.7u_2^2 + 0.92y_2 + 30.4u_2y_2^2, \tag{18a}$$
$$\begin{aligned} y_1^{[3]} =& 0.0012 - 0.18u_1^{[2]} + 1.1u_2^{[2]}y_1 + 0.98y_1^{[2]} \\ & - 1.8u_1^{[2]} + \left[0.0018 - 0.22u_1^{[1]} - 1.7\left(u_2^{[1]}\right)^2 + 0.92A + 30.4u_2^{[1]}A^2\right], \end{aligned} \tag{18b}$$

where $A = 0.0018 - 0.22u_1 - 1.7u_2^2 + 0.92y_2 + 30.4u_2y_2^2$.

The application of Algorithm 1 leads to a transformation: for $i = 1$, $j_1 = 2$ and $k = 2$ we obtain from $\bar{p}_{22} = \gamma\bar{p}_{12} + r$ the right quotient $\gamma = 1.8u_1^{[2]}Z - 1.8u_1^{[2]}\left(0.92 + 60.8u_2^{[1]}y_2^{[1]}\right)$, the right remainder $r = 1.8u_1^{[2]}(0.92 + 60.8u_2y_2)\left(0.92 + 60.8u_2^{[1]}y_2^{[1]}\right)$, and the unimodular matrix

$$U = \begin{bmatrix} 1 & 0 \\ -1.8u_1^{[2]}Z + 1.8u_1^{[2]}\left(0.92 + 60.8u_2^{[1]}y_2^{[1]}\right) & 1 \end{bmatrix}.$$

Assume that $u_1^{[\ell]} \neq 0, \ell \in \mathbb{Z}$. Multiplying U and $\bar{P}$ yields

$$\widetilde{P} := U\bar{P} = \begin{bmatrix} 0 & Z - 60.8u_2y_2 - 0.92 \\ Z^3 - 0.98Z^2 - 1.1u_2^{[2]} & 1.8u_1^{[2]}(0.92 + 60.8u_2y_2)\left(0.92 + 60.8u_2^{[1]}y_2^{[1]}\right) \end{bmatrix}.$$

The matrix $\widetilde{P}$ is in the Popov form. However, if one tries to find the transformed equations using the linear transformation as

$$\begin{bmatrix} \tilde{\varphi}_1 \\ \tilde{\varphi}_2 \end{bmatrix} := U(Z) \vec{\ } \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{19}$$

then one encounters a failure. Namely,

$$\begin{aligned}\tilde{\varphi}_2 =& y_1^{[3]} - 0.0012 + 0.18u_1^{[2]} - 1.1u_2^{[2]}y_1 - 0.98y_1^{[2]}\\ &+ 1.8u_1^{[2]}\left[0.0018 - 0.22u_1^{[1]} - 1.7\left(u_2^{[1]}\right)^2 + 0.92A + 60.8Au_2^{[1]}y_2^{[1]} - 30.4u_2^{[1]}\left(y_2^{[1]}\right)^2\right]\end{aligned}$$

still depends on $y_2^{[1]}$; moreover, it also contains $\left(y_2^{[1]}\right)^2$, which cannot be removed by any *linear* transformation (over the polynomial ring). Thus, (19) is in the Popov form (since $\widetilde{P}$ is in the Popov form), but not in the *strong* Popov form, since $1 = n_{22} \nless n_1 = 1$ and condition (iv) of Definition 8 is not satisfied. This example illustrates the difficulties of adapting the theory of linear (time-varying) systems for nonlinear systems.

To transform equations (15) into the strong Popov form, one has to use nonlinear transformations in analogy with [1]. However, the simple structure of equations (15) enables one to find the nonlinear transformation almost by direct inspection. For system (15) the indices are: $n_{11} = 0$, $n_{11} = 1$, $n_{21} = 3$, $n_{22} = 2$. For the strong Popov form it is necessary to make n_{22} equal to zero. Our aim is to construct the nonlinear function $\chi(\varphi_1, \varphi_1^{[1]})$ such that $\hat{\varphi}_2 := \varphi_2 - \chi(\varphi_1, \varphi_1^{[1]})$ does not depend on $y_2^{[2]}, y_2^{[1]}$, i.e. for $\hat{\varphi}_2$ one gets the index $n_{22} = 0$. On the first step, to remove dependency on $y_2^{[2]}$, note that the coefficient of $y_2^{[2]}$ is $c_1 := 1.8u_1^{[2]}$. Compute

$$\begin{aligned}\tilde{\varphi}_2 :=& \varphi_2 - c_1\varphi_1^{[1]} = y_1^{[3]} - 0.0012 + 0.18u_1^{[2]} - 1.1u_2^{[2]}y_1 - 0.98y_1^{[2]}\\ &+ 1.8u_1^{[2]}\left[0.0018 - 0.22u_1^{[1]} - 1.7\left(u_2^{[1]}\right)^2 + 0.92y_2^{[1]} + 30.4u_2^{[1]}\left(y_2^{[1]}\right)^2\right],\end{aligned}$$

which obviously does not depend on $y_2^{[2]}$, i.e. for $\tilde{\varphi}_2$ the index $n_{22} = 1$. On the second step we remove the term $(y_2^{[1]})^2$. Note that the coefficient of $(y_2^{[1]})^2$ is $c_2 := 1.8 \cdot 30.4u_1^{[2]}u_2^{[1]} = 54.72u_1^{[2]}u_2^{[1]}$. Compute $\tilde{\tilde{\varphi}}_2 := \tilde{\varphi}_2 - c_2\varphi_1^2$. The explicit expression of $\tilde{\tilde{\varphi}}_2$ is omitted due to its complexity, but we indicate that $\tilde{\tilde{\varphi}}_2$ does not involve $(y_2^{[1]})^2$ any more. We also indicate that $\tilde{\tilde{\varphi}}_2$ depends on $y_2^{[1]}$ linearly while the coefficient of $y_2^{[1]}$ is $c_3 := 1.8u_1^{[2]}\left(0.92 + 60.8u_2^{[1]}A\right)$. On the third step $y_2^{[1]}$ is removed by linear transformation $\hat{\varphi}_2 := \tilde{\tilde{\varphi}}_2 - c_3\varphi_1$ and thus for $\hat{\varphi}_2$ the index $n_{22} = 0$. To conclude, the nonlinear transformation, bringing equations (15) into the strong Popov form, is

$$\begin{aligned}\hat{\varphi}_1 := \varphi_1, \quad \hat{\varphi}_2 :=& F(\varphi_1, \varphi_1^{[1]}, \varphi_2) = \varphi_2 - c_1\varphi_1^{[1]} - c_2\varphi_1^2 - c_3\varphi_1\\ =& \varphi_2 - 1.8u_1^{[2]}\varphi_1^{[1]} - 54.72u_1^{[2]}u_2^{[1]}\varphi_1^2 - 1.8u_1^{[2]}\left(0.92 + 60.8u_2^{[1]}A\right)\varphi_1.\end{aligned}$$

The transformed $\hat{\varphi}_2$ is

$$\begin{aligned}\hat{\varphi}_2 =& y_1^{[3]} - 0.0012 + 0.18u_1^{[2]} - 1.1u_2^{[2]}y_1 - 0.98y_1^{[2]}\\ &+ 1.8u_1^{[2]}\left[0.0018 - 0.22u_1^{[1]} - 1.7\left(u_2^{[1]}\right)^2 + 0.92A + 30.4u_2^{[1]}A^2\right].\end{aligned}$$

The result coincides with (18b), obtained from the weak Popov form, when we express $y_1^{[3]}$ as function of the other variables.

4. CONCLUSIONS

In this paper two algorithms were developed for transforming the set of nonlinear higher-order i/o difference equations into the Popov and the strong Popov form, respectively, using the equivalence transformations that, by definition, do not change the solutions of the system equations. The transformed system description includes, in general, in addition to system equations, also a number of inequations that guarantee certain expressions to be non-zero. That is, unlike the typical approaches assuming the generic setting (see for instance [5]), in our case the open and dense subset where the results are valid is specified by a certain set of functions S, being the byproduct of the developed algorithms. To resume, our linear transformations are valid globally in the entire space with removed zeros of functions from the set S. The approach avoids using the implicit function theorem, yielding only local results as in [12] for continuous-time nonlinear systems. However, when one needs to apply nonlinear transformations, the strong Popov form is not necessarily defined globally.

As for future research directions, the strong Popov form allows, in principle, finding explicit equations of the inverse system when the set of original equations will be transformed into the Popov form with respect to the control variables.

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Mittelineaarsete sisend-väljundvõrrandite tugev Popovi kuju

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On välja töötatud kaks algoritmi mittelineaarse diskreetajaga süsteemi sisend-väljundvõrrandite teisendamiseks (lineaarsete või mittelineaarsete ekvivalentsiteisendustega) vastavalt kas Popovi või tugevale Popovi kujule. Ekvivalentsiteisendused ei muuda definitsiooni põhjal süsteemi lahendeid. Teisendused eeldavad algse süsteemi esitust tugeval reapõhiselt taandatud kujul, mis varasemate tulemuste põhjal alati eksisteerib. Popovi (ja tugev Popovi) kuju sisaldab peale võrrandite veel avaldisi, mis ei tohi võrduda nulliga. Kui lineaarteisendusest piisab, on algoritmi tulemuseks vastav lineaarteisendus (üle meromorfsete funktsioonide korpuse). Mittelineaarse ekvivalentsiteisenduse leidmise algoritm on ka konstruktiivne, v.a samm, mis nõuab osatuletistega diferentsiaalvõrrandite süsteemi lahendamist. Viimane aspekt iseloomustab paljude mittelineaarsete juhtimisülesannete lahendusi. Lisaks ei pruugi tugev Popovi kuju globaalselt eksisteerida. Teoreetilisi tulemusi on illustreeritud mitme näitega.