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CONTROL
THEORY

Global linearization approach to nonlinear control systems: a brief tutorial overview

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Abstract. The paper gives an overview of an algebraic approach based on differential 1-forms, developed for the study of nonlinear control systems. The purpose of the paper is to describe the approach, comment on the necessary assumptions made, and demonstrate the effectiveness and limitations of the approach. Two very important aspects of the approach are as follows: (1) one works with differentials and not with functions, meaning that computations are, up to integration similar to the linear case and (2) the approach is used to study generic properties of control systems that hold for almost every point of a suitable domain. The first point means that solutions to various problems are found in terms of 1-forms and the integrability properties allow transformation of the solution back to the level of functions. The study of generic properties simplifies the presentation of the solutions, since there is no need to specify the working point and its neighbourhood. Finally, the paper includes an extensive list of publications, where the approach of 1-forms is studied or applied to solve different control problems.

Key words: nonlinear control systems, state equations, input-output equations, time-delay systems, systems on time scales, approach of one-forms, non-commutative polynomial rings, functions' algebra.

1. INTRODUCTION

Today nonlinearity plays an essential role in control systems theory. Both the geometric and algebraic methods have progressed substantially, providing crucial conceptual tools addressing system modelling, analysis, and control design. Among them the algebraic approach of differential 1-forms is transparent and intuitively more understandable than the most popular differential geometric approach that is based on complex geometric objects. The approach is based on the idea of working with differentials of nonlinear system equations rather than with equations themselves. Then the vector spaces of 1-forms over suitable differential/difference fields of nonlinear functions may be constructed. Thus, further analysis is very similar to that of the linear case except that the coefficients of the vectors are now meromorphic functions in independent system variables and not real numbers as in the linear theory. The benefit of such a framework is that the theoretical results are conceptually very

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similar to those in linear control theory though the computations are different. Since the ultimate goal is to study nonlinear systems, and not the globally linearized equations, the conditions similar to linear theory are accompanied with certain integrability conditions. In some rare cases it can be theoretically proved that those integrability conditions are always satisfied. This happens, for instance, in system reduction and in accessibility studies [33]. The framework developed suggests a wide range of mathematical tools and a systematic way to handle different control problems from the unified viewpoint. Moreover, upon the 1-forms approach polynomial methods have been developed. The tools are developed to address continuous- and discrete-time systems as well as systems, described on time scales. The latter allows handling also non-uniformly sampled control systems. Some results are also developed for nonlinear systems with delays.

The algebraic approach has been successfully applied to address a great number of structural control problems in nonlinear control. The methods based on differential 1-forms are strongly model-based, relying, in fact, in the majority of cases on direct cancellation. This approach is especially well-suited for checking generic solvability conditions of various problems. However, in order to find the control laws, at the last step one has to integrate certain sets of differential 1-forms to get back to the level of equations. For this reason the solvability conditions typically include certain integrability assumptions that in the linear case are always satisfied. Moreover, the integration of (in principle integrable) differential forms (or sets of such forms) is known to be a difficult task and in many cases no closed-form solution exists in terms of elementary functions.

The paper aims to give a tutorial overview with a focus on the word 'overview'. The adjective 'tutorial' means that different technical assumptions made in the approach and the necessity of their introduction are explained. It also shows, based on some simple examples, what happens if these assumptions do not hold. Sometimes at the expense of more complex theory one can relax some assumptions [41]; in the other cases it is not possible. Comparisons with alternative approaches in nonlinear control are briefly discussed with respective advantages/disadvantages. A number of future research directions are identified. The paper focuses on discrete-time systems for a number of reasons. First, they are less studied than continuous-time systems[1]. Second, the discrete-time case needs some additional assumptions, which we want to explain in detail and the construction of the inversive difference field is more complex.

Note that most results are obtained for time-invariant nonlinear systems, only few results have been obtained for time-varying nonlinear systems [75,106], or nonlinear time-delay systems (see Subsection 8.4). The most important solutions, given in terms of 1-forms, have been implemented in a *Mathematica*-based symbolic software NLControl [49] (see [20] for a brief overview of the software package).

The paper is organized as follows. In Section 2 the basic ideas of the approach are described for discrete-time systems and the submersivity assumption is explained. Sections 3 and 4 focus, respectively, on the ways one moves from functions to 1-forms and vice versa. Section 5 explains different aspects of the generic approach. In Section 6, based on the difference field and forward-shift operator, the polynomial tools are constructed for the study of discrete-time systems. Unification of the study of discrete- and continuous-time cases is described in Section 7. The rest of the paper is devoted to an overview of the problems solved by the approach (Section 8) and comparing different approaches/cases (Section 9). Finally, the conclusions are drawn and open problems and difficulties are discussed.

2. CONTROL SYSTEM

In this section the classes of nonlinear control systems addressed within the algebraic approach are defined. Both the state-space and the input-output (i/o) representations of the system are given. Note that throughout the paper we use the abridged notations. First, in order to simplify the exposition, we leave out the time argument t, so $\xi := \xi(t)$. Moreover, we use symbols $+$, $-$, and $^{[k]}$ instead of the shifted time arguments, so $\xi^+ := \xi(t+1)$, $\xi^- := \xi(t-1)$, and $\xi^{[k]} := \xi(t+k)$.

The approach requires that the system equations should be described by analytic functions. That is, smoothness is not enough. The choice of analytic functions allows us to define the ring $\mathscr{A}$ that is an integral

[1] For the continuous-time case there exists a good book on this approach [33].

domain, being the cornerstone of the approach. Note that, in general, the ring $\mathscr{A}$, associated to the control system, is the quotient ring (ring of cosets) of analytic functions, defined by the difference ideal, generated by system equations. Recall that the integral domain $\mathscr{A}$ can be naturally enlarged, i.e., embedded in a larger algebraic object, called its field of fractions $\mathscr{K}$ such that every element of $\mathscr{K}$ can be expressed as a fraction (quotient) $a^{-1}b$ of two elements of $\mathscr{A}$, whenever $a \neq 0$. The elements of $\mathscr{K}$ are meromorphic functions. The use of meromorphic functions is essential for carrying out division in computations. The use of the other functions, for instance smooth, will result in a non-integral domain with loss of many useful properties. Moreover, one defines also the skew polynomial ring of shift (or difference, or differential in the continuous-time case) operators over the field $\mathscr{K}$. For this ring to be an integral domain, the ring $\mathscr{A}$ has to be an integral domain. The term 'skew' means that the shift (or difference, or derivative) operator does not commute with every element in the coefficient field.

Additionally, the use of analytic or meromorphic functions allows us to study the generic properties of the systems (see more in Section 5). This means that the properties hold on some open and dense subsets of suitable domains if they hold at some point of the domain. That is, generic properties hold in almost all situations.

2.1. State-space description

Consider a multi-input multi-output (MIMO) nonlinear discrete-time dynamical system, described by the state equations

$$\begin{aligned} x(t+1) &= f(x(t),u(t)),\\ y(t) &= h(x(t)). \end{aligned} \tag{1}$$

In (1), $x(t) \in \mathscr{X} \subset \mathbb{R}^n$ is a state vector, $u(t) \in \mathscr{U} \subset \mathbb{R}^m$ is an input vector, $y(t) \in \mathscr{Y} \subset \mathbb{R}^p$ is an output vector, and $\mathscr{X}$, $\mathscr{U}$, $\mathscr{Y}$ are open subsets. Moreover, $f : \mathscr{X} \times \mathscr{U} \to \mathscr{X}$ and $h : \mathscr{X} \to \mathscr{Y}$ are assumed to be meromorphic functions. Assume that for $(x(t),u(t)) \in \mathscr{X} \times \mathscr{U}$, also $f(x(t),u(t)) \in \mathscr{X}$. Additionally, we assume that

$$\operatorname{rank}_{\mathscr{K}} \frac{\partial f(\cdot)}{\partial u} = m \tag{2}$$

and

$$\operatorname{rank}_{\mathscr{K}} \frac{\partial h(\cdot)}{\partial x} = p. \tag{3}$$

These assumptions are not restrictive; they only mean that all inputs and outputs, respectively, are independent.

2.2. Input-output description

Throughout the paper it is assumed that the indices $i, j = 1, \ldots, p$ and $\kappa = 1, \ldots, m$. Consider a MIMO nonlinear system, described by the set of difference equations

$$y_i(t+n_i) = \phi_i\left(y_j(t), y_j(t+1), \ldots, y_j(t+n_{ij}), u_\kappa(t), u_\kappa(t+1), \ldots, u_\kappa(t+s_{i\kappa})\right). \tag{4}$$

In (4), ϕ_i are supposed to be meromorphic functions. Let $n := n_1 + \cdots + n_p$ and $s_\kappa := \max_{1 \le i \le p}\{s_{i\kappa}\}$ and assume that the indices in (4) are supposed to satisfy the relations

$$\begin{aligned} &n_1 \le \cdots \le n_p, && n_{ij} < n_j, && s_{i\kappa} < n_i,\\ &n_{ij} < n_i,\ j \le i, && n_{ij} \le n_i,\ j > i. \end{aligned} \tag{5}$$

The conditions (5) are mostly introduced for technical reasons. Under such conditions the forward shift operator, defined by equations and applied to system variables, can be explicitly computed and independent system variables determined. However, in the case when the equations are defined by the set of implicit i/o difference

equations, this is not so simple. Since our long-term goal was to develop the *Mathematica*-based symbolic software, the computability aspect is important for us. Finally, let us mention that since the systems of the form (4) are often obtained as a result of the identification process, it is natural to select a model structure class from the following criterion – suitability for control and ease of model development.

The restrictions (5) mean that Eqs (4) are assumed to be in the so-called Popov form [10]. This guarantees that the indices n_i are unique up to permutations. Note that under mild conditions, one can always transform an arbitrary set of i/o equations, at least locally, into the Popov form.

2.3. Submersivity assumption

The submersivity assumption means that in (1),

$$\operatorname{rank}_{\mathscr{K}}\left[\frac{\partial f(\cdot)}{\partial(x,u)}\right]=n \tag{6}$$

or in (4),

$$\operatorname{rank}_{\mathscr{K}}\left[\frac{\partial \phi(\cdot)}{\partial(y,u)}\right]=p, \tag{7}$$

where $y=(y_1,\dots,y_p)^{\mathsf{T}}$, $u=(u_1,\dots,u_m)^{\mathsf{T}}$, and $\phi=(\phi_1,\dots,\phi_p)^{\mathsf{T}}$. Analogous conditions for systems defined on time scales having a more complex form are given in [79].

The submersivity property plays a crucial role in the study of discrete-time nonlinear control systems, described either in terms of the shift or difference operator. This concept was introduced into the nonlinear control theory in [41] and since then this assumption has been made in the majority of papers on discrete-time nonlinear control systems. Especially, this assumption is vital in the algebraic approach based on differential forms. Under the submersivity assumption the backward shift, playing an important role in the above approach, is a well-defined operator and the inversive closure of the difference field, defined by the control system can be constructed. The submersivity property guarantees that independent variables remain independent under the action of shift operator σ. In [46] it was proven that system (1) with rational f, is submersive if and only if the ideal, defined by the control system, is prime, proper, and reflexive. Finally, note that the submersivity assumption is not restrictive, since it is a necessary condition for the system to be generically accessible (controllable) [41].

In principle, one may relax the submersivity condition as shown in [41], but this can be done at the expense of much technical complexity. Sometimes a more restrictive condition of drift invertibility (alternatively called the reversibility of the map $f(\cdot,u)$) is assumed [27], i.e., $\operatorname{rank}_{\mathscr{K}}\left[\frac{\partial f}{\partial x}\right]=n$. However, as claimed (though not proven) in [28], under the submersivity assumption the system can be transformed via static state feedback into the form with invertible drift. Finally, note that in the linear case one works with the 1-forms, defined over the inversive difference field of real numbers. Thus, there is no need to extend the difference field $\mathscr{K}$ such that the extended field would be inversive, for which the submersivity assumption is necessary. In other words, the backward-shift of a constant (real number) is always the same constant and in the linear case there is no need to define the backward-shifts of system variables.

2.4. Difference field

In this subsection the construction of the inversive difference field, defined by system equations (4), is shown in detail. The case, related to state equations is commented briefly at the end of the subsection.

Let $\mathscr{K}$ denote the field of meromorphic functions in a finite number of *independent* system variables from the infinite set

$$\mathscr{C}=\left\{y_i,y_i^{[1]},\dots,y_i^{[n_i-1]},u_\kappa^{[l]},l\geq 0\right\}.$$

Observe that $y_i^{[k_i]}$, $k_i \geq n_i$, $i = 1,\ldots,p$ are dependent variables since they can be computed from Eqs (4). Note that one has to distinguish between signals $y(t)$ or $u(t)$ as time functions in (4) and signal variables y or u in $\mathscr{C}$ which are just variables that can take real values and are not considered to be functions of time. The field $\mathscr{K}$ is the field of meromorphic functions of variables y,u and their shifts. The forward-shift operator $\sigma : \mathscr{K} \to \mathscr{K}$ is defined as follows:

$$\sigma(F)\left(y_i, y_i^{[1]}, \ldots, y_i^{[n_i-1]}, u_\kappa, u_\kappa^{[1]}, \ldots, u_\kappa^{[l]}\right) := F\left(y_i^{[1]}, y_i^{[2]}, \ldots, y_i^{[n_i]}, u_\kappa^{[1]}, u_\kappa^{[2]}, \ldots, u_\kappa^{[l+1]}\right), \tag{8}$$

meaning that σ is applied to each argument of the function F, replacing the arguments by their forward shifts except for $y_i^{[n_i]}$, which as the *dependent* variables have to be replaced by $\phi_i(\cdot)$ from system equations (4). Of course, the forward shift of a constant function c is the same function: $\sigma c = c$.

Under the submersivity assumption (7) the operator σ is an injective endomorphism and the pair $(\mathscr{K},\sigma)$ a difference field (see[2] [32]). The inverse of σ, denoted by σ^{-1}, is called the backward shift, and defined analogously to (8). In general, the field $\mathscr{K}$ is not inversive, meaning that some $\zeta \in \mathscr{K}$ may not have pre-image in $\mathscr{K}$, i.e., $\sigma^{-1}\zeta \notin \mathscr{K}$. However, under condition (7), there exists, up to an isomorphism, a unique difference overfield $\mathscr{K}^*$, called the *inversive closure* of $\mathscr{K}$ such that $\mathscr{K} \subset \mathscr{K}^*$ and the extension of σ to $\mathscr{K}^*$ is an automorphism (see [32]). A detailed construction of $\mathscr{K}^*$ is given, for instance, in [4] and [46].

The construction of $\mathscr{K}^*$ requires the rule to compute the k-step backward shifts σ^{-k}, $k \geq 1$ of independent system variables. Note that the independent variables of the field $\mathscr{K}$ are given by the elements of the set $\mathscr{C}$, whereas the inversive closure $\mathscr{K}^*$ contains, in addition, the variables $\sigma^{-\iota}y_i$ and $\sigma^{-\iota}u_\kappa, \iota \geq 1$, where $\sigma^{-\iota}$ means the ι-time application of the backward-shift operator σ^{-1}. However, not all of those new variables are independent. For simplicity reasons we explain the situation for the single-input single-output case, when $p = m = 1$. The important point in the construction of $\mathscr{K}^*$ is that, in order to go backwards in time, one has two possibilities: solving Eq. (4), either with respect to $y(t)$ or with respect to $u(t)$. From (4), one can readily calculate the backward shifts of variable y, if the condition $\partial\phi/\partial y \not\equiv 0$ holds and $\sigma^{-k}u$ are given. This can be done by solving Eq. (4) with respect to y and applying the backward shift to the result the required number of times. This shows that $\sigma^{-k}y$ for $k \geq 1$ must not be considered as independent variables of the field extension $\mathscr{K}^*$ in the sense that they can be expressed as functions of the other variables such as $\sigma^{-k}u$ and those from $\mathscr{C}$. Note that an alternative possibility is specifying $\sigma^{-k}y$ as the independent variables of the field extension $\mathscr{K}^*$. Then, under the assumption that $\partial\phi/\partial u \not\equiv 0$ Eq. (4) can be solved for u and shifted back to compute $\sigma^{-k}u$ as dependent variables of $\mathscr{K}^*$. To conclude, we have two possibilities of solving Eq. (4), namely with respect to y or u, meaning that either $\sigma^{-k}u$ or $\sigma^{-k}y$ for $k \geq 1$, respectively, have to be chosen as the independent variables for the construction of the inversive closure $\mathscr{K}^*$. Of course, in the MIMO case, there are more options. Although the choice of independent variables is not unique, each possible choice brings up a field extension of $\mathscr{K}$ which is isomorphic to $\mathscr{K}^*$. There are no 'good' or 'bad' choices in the sense that all possible choices that define the field extension $\mathscr{K}^*$ are suitable. The only difference is that for some choices the computations are simpler than for others. Therefore we assume that the inversive closure of the difference field $\mathscr{K}$ is given and we will use the same symbol to denote the difference field and its inversive closure.

In case of state equations (1) the set $\mathscr{C}$ reads as

$$\mathscr{C} = \left\{x_i, i = 1,\ldots,n; u_\kappa^{[l]}, l \geq 0\right\},$$

which is extended by variables $w^{[-k]}$, $k \geq 1$, to define the field extension $\mathscr{K}^*$. Here $w^{[-k]}$ is the kth-order backward shift of a variable w, defined as $w = \chi(x,u)$. We have assumed that the map $f : \mathscr{X} \times \mathscr{U} \to \mathscr{X}$ can be extended to $\bar{f} = (f^\mathsf{T}, \chi^\mathsf{T})^\mathsf{T} : \mathscr{X} \times \mathscr{U} \to \mathscr{X} \times \mathbb{R}^m$ so that $\bar{f}$ has a generic (global) analytic inverse, defined on its image $\bar{f}(\mathscr{X} \times \mathscr{U})$. If system (1) is drift invertible, one can always take $w = u$, but in a general case there does not exist a simple rule for the best choice of χ. One typically prefers the choices of χ, which makes the inverse

[2] In general theory [32] the shift operator $\sigma : \mathscr{K} \to \mathscr{K}$ is only an endomorphism of $\mathscr{K}$, not necessarily injective. In such a case $a^\Delta = a^\sigma - a = 0$ does not imply $a^\sigma = a$ and the equality $\sigma\Delta = \Delta\sigma$ does not necessarily hold.

of the extended map $\bar{f}$ as simple as possible. The assumption that $\bar{f}$ has a global analytic inverse implies that $\operatorname{rank}_{\mathscr{K}}\left[\partial\bar{f}/\partial(x,u)\right]=n+m$. It also implies generic submersivity (6).

The following examples illustrate the necessity of the submersivity assumption and the construction of the inversive closure $\mathscr{K}^*$.

Example 1 (non-submersive system)**.** The system

$$\begin{aligned} x_1(t+1) &= x_1(t),\\ x_2(t+1) &= -x_1(t) \end{aligned} \tag{9}$$

illustrates the type of pathologies that can appear for non-submersive systems. Observe that $\sigma[x_1(t)+x_2(t)]=0$, but $x_1(t)+x_2(t)\neq 0$ necessarily. The shift operator, defined by system (9), is not injective since it does not have a trivial kernel: $a^{\sigma}=0 \not\Rightarrow a=0$. Next, define the function $F=1/(x_1+x_2)$. Straightforward computation shows that $\sigma(F)$ is not defined.

Example 2. Consider the system

$$\begin{aligned} x_1(t+1) &= u_1(t),\\ x_2(t+1) &= x_3(t)u_1(t),\\ x_3(t+1) &= u_2(t). \end{aligned} \tag{10}$$

For system (10), one can readily check that $x_3(0)=x_2(1)/x_1(1)$. Thus, a pre-image in $\mathscr{K}$ (through σ) of $x_3(0)$ is $x_2(0)/x_1(0)$. Whereas $x_3(0)$ has a pre-image in $\mathscr{K}$, $x_1(0)$ and $x_2(0)$ have none. For this example, $\mathscr{K}^*$ is the field of meromorphic functions in the variables $\{x(0),u(t),w(-l),\ t\geq 0, l\geq 1\}$, where $w(0)=[x_1(0),x_2(0)]$.

3. FROM EQUATIONS TO 1-FORMS

The approach is built up by introducing the notion of differential form in an abstract and formal way. In the rest of the study we are interested in the algebraic properties of differential forms. Consider next the infinite set of symbols $\mathrm{d}\mathscr{C}=\{\mathrm{d}\xi,\xi\in\mathscr{C}\}$ and define $\mathscr{E}$ as the vector space over the field $\mathscr{K}$ spanned by the elements of $\mathrm{d}\mathscr{C}$: $\mathscr{E}=\operatorname{span}_{\mathscr{K}}\mathrm{d}\mathscr{C}$. Observe that the span is defined over the field of functions, not over $\mathbb{R}$ like in the standard definition of codistribution. Elements of $\mathscr{E}$ are called differential 1-forms. For a function F depending on a finite number of real variables, $\mathrm{d}F$ denotes the standard differential of F. In particular, for $F\in\mathscr{K}$ the operator $\mathrm{d}:\mathscr{K}\to\mathscr{E}$ is defined (in case of system description (4)) as follows:

$$\mathrm{d}F=\sum_{i=1}^{p}\sum_{k=0}^{n_i-1}\frac{\partial F}{\partial y_i^{[k]}}\mathrm{d}y_i^{[k]}+\sum_{\kappa=1}^{m}\sum_{l\geq 0}\frac{\partial F}{\partial u_{\kappa}^{[l]}}\mathrm{d}u_{\kappa}^{[l]}+\sum_{s=1}^{m}\sum_{l\geq 1}\frac{\partial F}{\partial w_s^{[-l]}}\mathrm{d}w_s^{[-l]}$$

or in the case of system description (1) as

$$\mathrm{d}F=\sum_{i=1}^{n}\frac{\partial F}{\partial x_i}\mathrm{d}x_i+\sum_{\kappa=1}^{m}\sum_{l\geq 0}\frac{\partial F}{\partial u_{\kappa}^{[l]}}\mathrm{d}u_{\kappa}^{[l]}+\sum_{s=1}^{m}\sum_{l\geq 1}\frac{\partial F}{\partial w_s^{[-l]}}\mathrm{d}w_s^{[-l]}.$$

Taking the total differential of Eqs (1) results in the globally linearized system description, given in terms of differential 1-forms as

$$\begin{aligned} \mathrm{d}x_{\nu}^{+} &= \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\mathrm{d}x_i+\sum_{\kappa=1}^{m}\sum_{l\geq 0}\frac{\partial f}{\partial u_{\kappa}^{[l]}}\mathrm{d}u_{\kappa}^{[l]},\\ \mathrm{d}y_{\mu} &= \sum_{i=1}^{n}\frac{\partial h}{\partial x_i}\mathrm{d}x_i, \end{aligned} \tag{11}$$

where $\nu=1,\ldots,n$, $\mu=1,\ldots,p$.

If $\omega = \sum_j \alpha_j \mathrm{d}\xi_j$ is a 1-form with $\alpha_j \in \mathscr{K}$ and $\xi_j \in \mathscr{C}$, then the operator $\sigma : \mathscr{K} \to \mathscr{K}$ induces the operator $\sigma : \mathscr{E} \to \mathscr{E}$ by

$$\sigma(\omega) = \sum_j \sigma(\alpha_j)\mathrm{d}[\sigma(\xi_j)].$$

It has been shown that

$$\sigma(\mathrm{d}\alpha) = \mathrm{d}[\sigma(\alpha)].$$

Note that the operator $\sigma : \mathscr{E} \to \mathscr{E}$ is invertible and the inverse operator $\sigma^{-1} : \mathscr{E} \to \mathscr{E}$ is defined by

$$\sigma^{-1}(\omega) = \sum_i \sigma^{-1}(\alpha_j)\mathrm{d}[\sigma^{-1}(\xi_j)].$$

A vector space, defined in $\mathscr{E}$, is a coordinate-free (algebraic) object, independent of the chosen basis, though the coordinates of the vector from this vector space can vary from one basis to another. In the algebraic framework we understand a globally linearized nonlinear system rather as a vector space $\mathscr{E}$ and not as a set of differentials of nonlinear equations. The latter is only representative of the system and varies from one basis to another. Since many solvability conditions rely on the integrability property of a vector space, these conditions are also coordinate-free.

Observe that $\mathrm{span}_{\mathscr{K}}\{\omega_1, \ldots, \omega_\kappa\}$ is not a codistribution in the standard sense. When we multiply a 1-form by a meromorphic function, it may become *non-defined* at certain points of the space. Thus, there is no simple way of evaluating $\mathrm{span}_{\mathscr{K}}\{\omega_1, \ldots, \omega_\kappa\}$ at a point. The linear space over $\mathscr{K}$ has a fixed dimension. The codistributions evaluated at a point are linear spaces over $\mathbb{R}$ whose dimensions may have variable dimension, depending on the point and be smaller than the dimension of $\mathrm{span}_{\mathscr{K}}\{\omega_1, \ldots, \omega_\kappa\}$. However, the dimensions will be generically equal and one may assume, after a restriction to some open and dense subset, that $\mathrm{span}_{\mathscr{K}}\{\omega_1, \ldots, \omega_\kappa\}$ is a constant dimensional codistribution. For this reason, the subspaces of 1-forms are not dual objects to distribution, in the strict sense. However, in [95] the vector space of vector fields over $\mathscr{K}$ is constructed which can be understood as a dual object to the subspace of 1-forms. This vector space is defined by all possible linear combinations of $\partial/\partial x_i$ and $\partial/\partial u_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, over the field $\mathscr{K}$. Then, the concepts of forward and backward shifts of vector fields are defined and explicit formulas are given for their computation. In this way one can construct a geometric approach for discrete-time systems, which is dual to the approach of 1-forms.

3.1. Ordinary differentials versus Kähler differentials

There exist two main approaches to compute differentials of system variables. The first is the concept of Kähler differential introduced by Johnson [52] and used by Fliess' school [38]. This is a very natural choice from a purely algebraic point of view. Kähler differentials permit us to translate problems from algebra into linear algebra. The second approach is to use ordinary differential as known from the college calculus, as done in [33]. A natural question to be asked is whether these two concepts, Kähler differentials and ordinary differentials, coincide or not. The paper [39] shows that if the attention is restricted to polynomial, rational or algebraic functions, both concepts coincide. The ordinary differentials are relevant (Kähler differentials are not suitable) in cases when the functions in system descriptions under study or functions in problem solutions include transcendental functions. In principle, Kähler differentials can also be defined for such systems, but at the expense of introducing new variables and the rules for computing their differentials. For instance, even though one can introduce a new variable $\exp x$, algebraically independent of x over $\mathscr{R}$, its Kähler differential does not satisfy the relation

$$\mathrm{d}\exp x = \exp x \, \mathrm{d}x \tag{12}$$

unless we postulate this to hold using the quotient spaces modulo (12) since $\mathrm{d}x$ and $\mathrm{d}\exp x$ are linearly independent over $\mathscr{R}(x, \exp x)$.

4. FROM 1-FORMS TO FUNCTIONS

Below, the symbol $\mathrm{d}\omega$ denotes the exterior derivative of the 1-form ω and $\wedge$ means the exterior or wedge product.

One says that $\omega \in \mathscr{E}$ is an *exact* 1-form if $\omega = \mathrm{d}\zeta$ for some $\zeta \in \mathscr{K}$. A 1-form ω for which $\mathrm{d}\omega = 0$ is said to be *closed*. Note that exact 1-forms are closed, whereas closed 1-forms are only locally exact. A 1-form is called integrable if there exists an integrating factor $\lambda \in \mathscr{K}$ such that $\lambda\omega$ is an exact 1-form. The integrability of a 1-form can be checked by the Frobenius theorem below.

Theorem 1 ([30]). *Let $\omega_1, \dots, \omega_\nu$ be 1-forms in $\mathscr{E}$, linearly independent in a neighbourhood of a certain point. The system of 1-forms $\omega_1, \dots, \omega_\nu$ is completely integrable in this neighbourhood if and only if for all $\ell = 1, \dots, \nu$*

$$\mathrm{d}\omega_\ell \wedge \omega_1 \wedge \cdots \wedge \omega_\nu = 0.$$

The Frobenius theorem on which we rely and which holds for C^∞ functions is a local statement, valid in a neighbourhood of each point. It guarantees the integrability of a constant dimensional codistribution, generated by 1-forms. The functions whose differentials generate the codistribution are defined only locally. More precisely, they are defined at some neighbourhood of almost every point of the domain. That is, in different regions of the domain the integration may result in different functions. This is a kind of 'local generic' feature. That is, one cannot claim the existence of a global solution that is defined almost everywhere since there does not exist the generic version of the Frobenius theorem.

The following example demonstrates the importance of constructing and utilizing the extended field $\mathscr{K}^*$ in a correct way.

Example 3. Consider the system

$$y(t+2) = y(t)u(t+1)$$

and

$$\omega = y^{[-1]}\mathrm{d}y^{[1]} + \mathrm{d}u. \tag{13}$$

Note that in this system $y^{[-1]}$ has to be considered as a dependent variable and should be replaced by $y^{[1]}/u$. In doing so, the 1-form ω takes the form

$$\omega = \frac{y^{[1]}}{u}\mathrm{d}y^{[1]} + \mathrm{d}u$$

and is obviously integrable, whereas if we would erroneously treat $y^{[-1]}$ as an independent element of $\mathscr{K}^*$, the 1-form (13) would not be integrable.

In most cases one needs complete integrability, which means that the vector space of 1-forms can be generated locally by exact 1-forms. However, in some cases, partial integrability is enough, when only some generating 1-forms can be made exact. Another type of incomplete integrability is *constrained* integrability when the space of 1-forms is integrable only for a fixed value of some variable. Such aspect is important in the studies of singularities.

5. GENERIC APPROACH

The algebraic approach of 1-forms makes generic assumptions (for instance, submersivity assumption in Subsection 2.3) and suggests generic solutions to problems, i.e., the solutions valid for almost every point of the suitable domains of definition. The main reason to do so is to focus on the key aspects of the solution and omit the possible singularities. The study of generic properties allows expressing the solutions in a more compact way, since there is no need to specify the working point and its neighbourhood like in classical approaches [50,99]. Since ranks and dimensions of vector spaces are defined over the field of functions, singularities do not show up explicitly, and need additional attention in later analysis. In practice, once the system equations are fixed, these matters can be addressed quite easily. The generic approach ignores the distinction between local and global.

Typically, the 1-forms and functions are defined not on $\mathbb{R}^n$ but on open and dense[3] regions in $\mathbb{R}^n$ and the stated relations among them hold only on open subregions.

There exist two reasonable but different definitions of genericity in the literature and one does not imply the other [66]. One says that P is generic in Y if a property $P(y)$ holds for almost all y in Y to mean that the set $\{y \in Y : P(y) \text{ does not hold}\}$ has zero (Lebesque) measure. The other definition calls a property generic if $\{y \in Y : P(y) \text{ holds}\}$ contains an open and dense set. We rely on the second definition. That is, we are interested in the properties that hold on open and dense subsets of suitable domains of definitions, provided they hold at some point of such domains.

Let us mention that the concept of generic property does not make sense, in general, for systems defined by smooth functions, whereas it makes sense in case of analytic and meromorphic functions. The reason is that there exist smooth functions being neither generically equal to zero nor different from zero (see Example 4 below). Non-zero analytic functions (defined on $\mathbb{R}^n$) are different from zero at the points of an open and dense subset of $\mathbb{R}^n$, or said alternatively, are generically different from zero. Therefore, it makes sense to define the generic rank of the matrix whose entries are analytic functions.

Since we look at dimensions (or ranks) over a field of functions, not over $\mathbb{R}$, there is no argument either about the points where to evaluate dimensions or about constant dimensionality of codistributions. A generic rank is a maximal rank on an open and dense set. The rank may drop on some subset. Reducing the set, one can always achieve a constant rank over $\mathbb{R}$.

In the example below we will show why we do not consider smooth non-analytic functions.

Example 4. Consider the system

$$\begin{aligned} x_1(t+1) &= x_1(t) + F(x_2(t)), \\ x_2(t+1) &= u(t), \end{aligned} \tag{14}$$

where $F(x_2(t)) = \exp^{-\frac{1}{x_2(t)}}$, when $x_2(t) > 0$ and $F(x_2(t)) = 0$ elsewhere. Note that $F(\cdot)$ is smooth, but not analytic. For system (14) the ranks of vector spaces and properties are different, when $x_2(t) > 0$ or not. Thus, system (14) is accessible when $x_2(t) > 0$ and is not accessible otherwise, since then $x_1(t+1) - x_1(t) = 0$. Therefore, we cannot talk about generic accessibility property.

5.1. From generic to global

The 'generic focus' is fine, but it means that something holds on some *unspecified* open and dense subset of the entire space. It would be nice if the domain, where the results are valid, could be specified.

One option to do so is to introduce a multiplicative subset $\mathscr{S}$ of the difference ring $\mathscr{A}$ (ring of analytic functions in system variables) [71]. This means that 1 is in $\mathscr{S}$, but 0 is not and if a and b belong to $\mathscr{S}$, so does ab. Then $\mathscr{S}^{-1}\mathscr{A}$ denotes the localization of $\mathscr{A}$ with respect to $\mathscr{S}$ that consists of meromorphic functions whose denominators belong to $\mathscr{S}$ (are not equal to zero). When we start, some functions in system equations may have denominators that, together with their forward and backward shifts, should be included in the set $\mathscr{S}$. If the functions are analytic, then one may set $\mathscr{S} := \{1\}$, meaning that $\mathscr{S}^{-1}\mathscr{A} = \mathscr{A}$. Of course, additional denominators that show up in the algorithms or computations should also be included in $\mathscr{S}$ together with their shifts and powers. That is, the initial set $\mathscr{S}$ has to be extended. The infinite set $\mathscr{S}$ can be described by its finite generator set $\mathscr{S}_0$.

By choosing proper $\mathscr{S}$, the calculations can be made global. The domain is the entire space with removed zeros of functions from $\mathscr{S}$. This set is also open and dense, but specified by $\mathscr{S}$. However, there is a small problem. Such space is in principle infinite dimensional and without speaking its topology, the meaning of open and dense is not clear. One can overcome this difficulty by restricting oneself to finitely many variables, which makes the space finite-dimensional. Of course, it would probably be tedious to describe precisely the variables to which one must restrict oneself.

[3] Note that a meromorphic function is analytic on an open and dense set. Only open is not enough because (sometimes) one has to add the functions with different domains, and may end up with an empty domain of the sum.

6. POLYNOMIAL TOOLS

The polynomial approach [24,87] has proven to be very powerful for the analysis and synthesis of linear time-invariant systems. In this approach the polynomial indeterminate is interpreted as the derivative or shift operator. Later, this approach was generalized to linear time-varying systems, both continuous- and discrete-time (see, e.g., [25,51]). In this extension, the polynomial coefficients are no more constants but functions of time.

The difference field $(\mathscr{K},\sigma)$ induces a non-commutative skew polynomial ring, denoted by $\mathscr{K}[z;\sigma]$. An element of $\mathscr{K}[z;\sigma]$ is a polynomial of the form

$$p=\sum_{i=0}^{\gamma} p_i z^i,$$

where $p_i\in\mathscr{K}$, $i=0,\dots,\gamma$, and z denotes the polynomial indeterminate. The multiplication in $\mathscr{K}[z;\sigma]$ is defined by the rule

$$z\varphi=\sigma(\varphi)z$$

for a function $\varphi\in\mathscr{K}$, whereas addition is defined in a usual way. Recall that the ring $\mathscr{K}[z;\sigma]$

- is an integral domain (i.e., it does not contain zero divisors; see Section 2);
- satisfies the left Ore condition, i.e., for all non-zero $p,q\in\mathscr{K}[z;\sigma]$ there exist non-zero $p',q'\in\mathscr{K}[z;\sigma]$ such that $p'q=q'p$, since σ is an automorphism [37].

Define

$$z^k\mathrm{d}y_j:=\mathrm{d}y_j^{[k]},\quad z^l\mathrm{d}u_\kappa:=\mathrm{d}u_\kappa^{[l]} \tag{15}$$

for $k,l\geq 0$ to represent the nonlinear system (4) in terms of two polynomial matrices. Differentiate (4) to obtain the infinitesimal system description

$$\mathrm{d}y_i^{[n_i]}-\sum_{j=1}^{p}\sum_{\alpha=0}^{n_{ij}}\frac{\partial\phi_i}{\partial y_j^{[\alpha]}}\mathrm{d}y_j^{[\alpha]}-\sum_{\kappa=1}^{m}\sum_{\beta=0}^{s_{i\kappa}}\frac{\partial\phi_i}{\partial u_\kappa^{[\beta]}}\mathrm{d}u_\kappa^{[\beta]}=0 \tag{16}$$

and use relations (15) to rewrite (16) as

$$P(z)\mathrm{d}y+Q(z)\mathrm{d}u=0, \tag{17}$$

where $P(z)$ and $Q(z)$ are $p\times p$ and $p\times m$-dimensional matrices, respectively, whose elements $p_{ij}(z),q_{i\kappa}(z)$ are from $\mathscr{K}[z;\sigma]$ and

$$\begin{aligned} p_{ij}(z)&=\delta_{ij}z^{n_i}-\sum_{\alpha=0}^{n_{ij}}p_{ij,\alpha}z^\alpha, & p_{ij,\alpha}&=\frac{\partial\phi_i}{\partial y_j^{[\alpha]}}\in\mathscr{K},\\ q_{i\kappa}(z)&=-\sum_{\beta=0}^{s_{i\kappa}}q_{i\kappa,\beta}z^\beta, & q_{i\kappa,\beta}&=\frac{\partial\phi_i}{\partial u_\kappa^{[\beta]}}\in\mathscr{K} \end{aligned} \tag{18}$$

with δ_{ij} being Kronecker delta. Equation (17) describes the globally linearized system, corresponding to Eqs (4).

Recall that $\deg(pq)=\deg p+\deg q$ holds when the polynomial ring $\mathscr{K}[z;\sigma]$ is an integral domain. Observe that $\mathscr{K}[z;\sigma]$ is not a field because z is not a unit in $\mathscr{K}[z,\sigma]$, i.e., there does not exist a polynomial p such that $zp(z)=1$. In other words, there is no inverse for z in multiplication.

7. UNIFICATION

Many results concerning continuous-time control systems carry over quite easily to the corresponding results for discrete-time systems, while others seem to be completely different in nature from their continuous-time counterparts. The mathematical formalisms that unify and extend the study of continuous- and discrete-time cases help to reveal and explain such discrepancies. Two such formalisms are *time scales calculus* and *pseudo-linear*

algebra. Another practical advantage of the unification of the solutions is in the reduction of the implementation load since one may now write a single *Mathematica* or Maple function that covers all the special cases.

The term 'time scale' refers to the way dynamic systems behave over time. Most engineering applications assume time to be either continuous or uniformly discrete, both of which are merged in time scales formalism into general framework, and follow from the latter as special cases. Moreover, time scales formalism accommodates easily the non-uniformly (irregularly) sampled systems. The main tools of algebraic formalism of differential 1-forms have been extended for systems, defined on homogeneous time scales in [11] and on non-homogeneous but regular time scales in [12]. Compared with the homogeneous case the main difficulties in the non-homogeneous case are non-commutativity of delta-derivative and shift operators and the fact that the additional time variable t appears in the definition of the difference ring. The first difficulty only adds technical/computational complications. The second difficulty yields that the new variables of the inversive closure, depending on t, have to be chosen to be smooth at each dense point t of the time scale. However, it is important to stress that there are no dense points for non-uniformly sampled systems.

Note that homogeneous time scales include continuous-time and uniformly sampled (discrete-time) systems while a regular time scale includes also non-uniformly sampled systems. The main concept of time scales calculus is the so-called delta derivative, which is a generalization of both standard time-derivative and the difference operator. Though time scales calculus accommodates more possibilities, regarding the control theory, the most important special cases are continuous- and uniformly sampled discrete-time systems that are the examples of systems, defined on homogeneous time scales. However, the shift operator is not delta-derivative. For this reason the discrete-time systems addressed within the time scales formalism describe the system equations in terms of the difference operator and not in terms of the more conventional shift operator. A description of a dynamical system based on the difference operator is often referred to as delta-domain description [92]. When signals are sampled at a high sampling rate, the delta-domain models are less sensitive to round-off errors and do not yield ill-conditioned models like those based on the shift operators [48,92]. Moreover, the delta-domain models are better linked to the continuous-time plant.

For nonlinear systems, defined on homogeneous time scales, one constructs the ring of skew polynomials $\mathscr{K}[z;\sigma,\Delta]$, where σ is a shift operator (automorphism) like in Section 6 and Δ is the delta-derivative. Note that $\mathscr{K}[z;\sigma,\Delta]$ is not the left Ore ring unless σ is an automorphism. Polynomial multiplication is defined by the rule

$$z\varphi = \sigma(\varphi)z + \Delta(\varphi).$$

There exists another formalism, called pseudo-linear algebra, that can handle derivative, difference, and shift operators. However, this approach is unable to address the non-uniformly sampled systems. Pseudo-linear algebra [1,26], alternatively called Ore algebra, provides the tools to study the common properties of linear differential, difference, shift, and other types of operators such as q-shift and q-difference operators, expressed in terms of skew polynomials. Note that in order to make the tools of pseudo-linear algebra applicable to nonlinear systems, one has to find first the globally linearized system description in terms of differential 1-forms by applying the differential operator to system equations. The approach has been used to solve the realization problem [19] and for the reduction of nonlinear control systems [73].

8. PROBLEMS SOLVED

The approach of differential 1-forms has been used to solve many different problems. Some of them are explained in detail below, other solutions are just cited in Table 1.

8.1. Realization

Realization, a fundamental problem in nonlinear control theory, consists of transforming the set of i/o equations (4) into the state-space form (1). The task is motivated by the fact that most of nonlinear control theory is

Table 1. Solved problems: literature summary

Problems	Continuous		Discrete		Time scales	
	i/o	state	i/o	state	i/o	state
Reduction	–	–	[9]	–	[72]	–
Accessibility	[84]	–	–	[46]	[14]	–
Observability/observable space	–	–	–	[64,65,70]	–	[63]
Identifiability	–	–	[100]	–	–	–
Flatness/Exact linearization	–	–	–	[4,5,28,97,106]	–	[13,31]
Special forms	–	[61]	[6,10,18,71]	[27,62,69,94]	–	[31]
Decoupling	–	–	–	[2,3,67,68]	–	–

developed for systems described by state equations, while identification/modelling methods result typically in i/o equations, especially for modern application areas. Constructive solutions for the realization problem are found. Minimal realization (i.e., realization with the smallest number of state variables) requires, in general, two steps: system reduction (finding the irreducible i/o representation) and construction of the state coordinates. Exact formulas in terms of non-commutative polynomials were found for each subtask. The first subtask was studied separately both for discrete- and continuous-time systems (see [21] and [82], respectively). The paper [81] demonstrates how a non-realizable set of i/o difference equations can be made realizable by adding a postcompensator. Note that this is not possible in the continuous-time case [96]. The earlier results on realization in terms of 1-forms without using polynomial formalism were given in [86] and those using the transfer function approach in [45]. The results were then generalized for systems defined on homogeneous time scales [23,29], and finally the solution was extended to time-varying nonlinear systems [75]. Moreover, the results helped us to advance the realization theory for linear time-varying systems [83] and linear parameter-varying systems [22], which was quite an untypical situation. Usually the theory for more simple systems is developed first and then extended to more complex systems. Here the opposite happened. Finally, the results of our theory helped to work out nice subclasses of bilinear [17,77,78], quadratic [17,74] and general i/o equations [80], each of which guaranteed to have a state-space description. The relations between different realization methods have been addressed in [76,80].

8.2. Systems on time scales

Nonlinear control systems on time scales are a novel research topic. When we started investigating it, the relevant mathematical tools were practically missing. Therefore, we first generalized the algebraic formalism of differential 1-forms for systems defined on (i) homogeneous time scales [7,8,11] and later to (ii) non-homogeneous but regular time scales [12]. The application of this formalism has allowed us to study various control problems for systems, defined on homogeneous time scales such as reduction [72], realization [23,29], static state feedback linearization [13], checking accessibility [14], and observability [63] properties and transforming the state equations into certain observer forms [31] that help to develop observers with linear error dynamics.

8.3. Output feedback

One of the fundamental notions in control is feedback. Most commonly state feedback is used, however, this requires that the states should be known/measurable, which is often not the case. An alternative is to use output/measurement feedback. An additional advantage of an output feedback is its applicability to systems described by i/o equations, which is the only possibility for non-realizable i/o equations. In the case of output feedback only some functions of states are used; therefore the solvability conditions as well as solutions themselves are substantially more complicated to find than in the case of state feedback. The

output feedback has been used to solve the problems like i/o linearization, disturbance, and i/o decoupling. Necessary and sufficient conditions have been found for i/o linearizability by dynamic output feedback [56]. Then this solution is used to derive sufficient conditions to solve the disturbance decoupling problem by dynamic output/measurement feedback [54] and necessary and sufficient conditions to solve the i/o decoupling problem by dynamic output/measurement feedback [55]. In the i/o decoupling problem one searches a feedback such that every output of the closed-loop system depends exactly on a single input of the closed-loop system.

These results are mostly easily expandable to continuous-time systems. Only the solution to the i/o linearization problem in [56] needs modifications, since taking time-derivative of a function is more complex than forward-shifting it. Some preliminary results in continuous-time can be found, for example, in [33].

8.4. Time-delay systems

The method based on differential 1-forms has also been generalized for time-delay systems [91,108], whose evolution depends not only on the present state, but also on the past values of state variables. A typical assumption made, when working with time-delay system, is that different delays are commensurable, i.e., multiples of some fixed minimal delay. This assumption simplifies the study considerably . When talking about time-delay systems, the majority of papers address continuous-time systems with delays rather than discrete-time systems. The reason is that under a simple additional assumption (besides commensurable delays), a discrete-time system with delays can be transformed into a higher dimensional discrete-time system without delays and thus, one can apply the standard theory to such systems. Therefore, the majority of papers work with time-delay systems of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t),x(t-1),\dots,x(t-D),u(t),u(t-1),\dots,u(t-D)), \\ y(t) &= h(x(t),\dots,x(t-D)), \end{aligned} \tag{19}$$

where $D>0$, $x(t)\in X\subset\mathbb{R}^n$ is the state, $u(t)\in U\subset\mathbb{R}^m$ is the control input, and $y(t)\in Y\subset\mathbb{R}^p$ is the output of the system.

In the case of time-delay systems, one constructs a difference/differential field $\mathscr{K}$, which depends now on two operators – the time-derivative $\mathrm{d}/\mathrm{d}t$ and the delay δ, which is similar to the backward-shift operator in the discrete-time case. Unlike in the discrete-time case, here, one does not need the submersivity assumption, since one can always construct a well-defined inverse of δ. A major difference from the delay-free case is the fact that, instead of the field $\mathscr{K}$, one uses a polynomial ring $\mathscr{K}[\vartheta;\delta]$, which is constructed similarly to the ring $\mathscr{K}[z;\sigma]$ in Section 6. The reason for such a choice is the independence of x and $\delta(x)$ in $\mathscr{K}$ where the effects of the delays are thus not taken into consideration. In $\mathscr{K}[\vartheta;\delta]$ the variables x and $\delta(x)$ are dependent, because $\mathrm{d}[\delta(x)]=\vartheta\mathrm{d}x$. In fact, time-delay systems can be studied as systems over a ring [34].

The most important obstacle in generalizing the approach of differential forms for time-delay systems has been the problem of the integrability of 1-forms. Namely, time-delay systems are infinite dimensional and therefore, the standard Frobenius theorem is not applicable. The problem has been studied over 15 years (see [91,93]), and finally solved in [53]. Another important difficulty in extending the method is the causality issue. One typically requires that objects should not depend on the future values of the system variables. The source of causality problems is the fact that not every element in $\mathscr{K}[\vartheta;\delta]$ is invertible. This also makes the search for state/input transformations more complicated.

Compared to the delay-free case, the global linearization approach has been applied much less to the study of structural control problems of nonlinear time-delay systems. So far, the method has been used to study the accessibility property [89,108], system inversion [93], the disturbance decoupling problem [60,90,93], the realization problem [40], and the observability property [93,108] of time-delay systems.

References to some other problems and solutions, based on 1-forms approach, are given in Table 1. The table does not contain the references cited in the book [33], where many of the problems are studied for continuous-time systems.

9. COMPARISONS

In this section comparisons are made with other algebraic methods and also, discrete- and continuous-time cases are briefly compared. Note that, while the popular differential geometric approach assumes that the system is described by the state equations, the algebraic method of differential 1-forms can handle also the systems, defined by their i/o equations.

9.1. Generalized transfer functions

Since $\mathscr{K}[z;\sigma]$ is an integral domain, it is possible to construct its field of left fractions $\mathscr{K}(z;\sigma)$. Any element of $\mathscr{K}(z;\sigma)$ can be represented as a left fraction $p^{-1}q$ of two polynomials $p,q \in \mathscr{K}[z;\sigma]$ with $p \neq 0$. By definition, the elements of $\mathscr{K}(z;\sigma)$ are equivalence classes of pairs (q,p) when $p \neq 0$ with respect to the equivalence relation given by $(q,p) \sim (q',p')$ iff $p'q = q'p$. Thus, when we write $p^{-1}q$, we usually mean the whole equivalence class but use in computations its simplest representative. The computations in the field, in particular addition and multiplication, do not depend on the representatives. This is similar to working with ordinary fractions. When we write $1/2 = 2/4$, we mean that $1/2$ and $2/4$ are equivalent, i.e., belong to the same equivalence class.

The concept of transfer function $T(z)$ has been extended for nonlinear systems based on the polynomial system description (17), and is defined as $T(z) = -P^{-1}(z)Q(z)$. That is, according to this understanding, the transfer function is a 'multiple' object – the whole equivalence class. The concept is sometimes called 'generalized transfer function' (GTF). The elements of $T(z)$ belong to $\mathscr{K}(z;\sigma)$. The key tool behind the GTF is the non-commutative Ore polynomial ring [103]. The concept of GTF was originally introduced in [109] and independently reintroduced, studied, and used in [42] and [43]. The generalized transfer functions have been used in the controller design [15,16,44] and in the solution of the realization problem [45].

One aspect needs to be mentioned explicitly. In the linear case a control system can be associated to each proper rational function, i.e., to each element of $\mathscr{K}(z;\sigma)$. However, things are different in the nonlinear case. The reason is that the 1-form that corresponds to the element of $\mathscr{K}(z;\sigma)$, is not necessarily integrable. If the differential form is exact or can be made exact by multiplying it with an integrating factor, i.e., with an element of $\mathscr{K}$, then there exists a control system that corresponds to this rational function. To conclude, not every fraction of skew polynomials necessarily represents a control system. This aspect plays a crucial role in designing compensators.

Ever since the introduction of the GTF it has been obvious that one of its strongest limitations is computational complexity, and the development of the software tools that would fully eliminate the mathematical burden involved is almost an impossible task. Even the computation of the GTF itself from the state equations is a difficult task [102]. It is comparatively easy to eliminate dx from the globally linearized state equations (11) and as a result, to find the transfer matrix. The latter requires an effective method for inverting Ore polynomial matrices. The extension of the Gauss–Jordan elimination method for the Ore polynomial case in [101] suffers from coefficient growth which makes it suitable only for simple systems. Unfortunately, the obtained transfer matrix depends yet on the state coordinates (unlike in the linear case) that have to be replaced as functions of inputs, outputs, and their differentials (or shifts). Though always at least locally doable, in many cases the result cannot be expressed in terms of elementary functions. Moreover, on a global level, state elimination does not have a unique solution and may be represented by a great number of possible solutions that, besides functions, depend also on inequations [35]. The construction of the transfer matrix for i/o equations is a much easier task.

However, the troubles do not stop here. In linear transfer matrix-based control design for MIMO systems, a special form of the transfer matrix, the Smith–McMillan form plays a key role. The reason is that the important system structural parameters are not available by direct inspection (by looking at the individual elements of the transfer matrix). The Smith–McMillan form provides a way to determine these structural parameters. The first step here is to transform the polynomial matrix, associated to the transfer matrix, into the Smith form. This step was extended for the GTF and implemented in the *Mahematica*-based package NLControl [18] but the second step, though theoretically able to be done, would yield the extremely complex expressions of no practical value already for systems with two inputs and two outputs.

Some problems have been solved for single-input single-output systems using the transfer function approach [15,16,44], but in the solution one faces other difficulties. Namely, in the interconnected systems, the initially independent variables become dependent, and this has to be taken into account in the computation of the GTF. That is, one cannot handle the transfer functions on their own as self-contained objects, but has to keep in backstage the knowledge of the equations that define the independent coefficients of the GTF, which is extremely inconvenient. To a lesser degree it is also important in the algebraic approach of the 1-forms.

Finally, the weak point in the transfer function technique (already present in the case of linear systems) is the fact that the method is based on the assumption of zero initial conditions. For non-zero initial conditions the nice transfer function algebra – a field structure – would no longer work [24]. In [24] the authors explain why the polynomial system theory is preferable. This point is also supported by [107].

9.2. Functions' algebra

Functions' algebra is an algebraic method (see [58,85,110]) developed in analogy with the pair algebra of partitions [47], which was used to describe the behaviour of sequential automaton. The method has shown some success, solving the problems like disturbance decoupling [57–59], fault diagnosis [104,105], and feedback linearization [85] for continuous- and discrete-time systems as well as for hybrid and discrete event systems. When compared with other methods, many solutions look similar for discrete- and continuous-time systems, only the computations are different for different system classes. Unlike the method based on differential 1-forms, functions' algebra operates directly with vector functions depending on the state and input variables. The set S of all such vector functions is divided into equivalence classes on the basis of a preorder $\leq$, which defines an equivalence relation on S. Namely, two vector functions are said to be equivalent if one can write the first vector function in terms of the second and vice-versa. In functions' algebra one works with these equivalence classes, just like in the 1-forms formalism one works with vector spaces without specifying a particular basis of a given vector space. The set S_e of all the equivalence classes forms together with the preorder $\leq$ (which becomes partial order on S_e) a lattice. The main elements in functions' algebra are operations $\times$ and $\oplus$, operators $\mathbf{m}$ and $\mathbf{M}$. In a lattice for every two elements α and β there exists a minimal element γ, which is bigger than α and β (with respect to the partial order $\leq$) and a maximal element δ, which is smaller than α and β. The operations $\oplus$ and $\times$ are defined as $\alpha \oplus \beta = \gamma$ and $\alpha \times \beta = \delta$. Finally, the lattice $(S_e, \leq)$ is connected to the system dynamics through a binary relation Δ, which defines the operators $\mathbf{m}$ and $\mathbf{M}$. The vector function $\mathbf{m}(\alpha)$ represents the maximum amount of available information on the next state of the system, when knowing α. The vector function $\mathbf{M}(\beta)$ represents the minimum amount of information necessary to know β on the next state of the system.

As said above, unlike the method based on 1-forms, functions' algebra operates with vector functions. The functions' algebra is, in principal, not limited to analytic or smooth functions, but to make the comparisons with the 1-forms approach, we consider in the following the case of meromorphic functions. To every vector function (more precisely to every equivalence class in S_e) there corresponds a vector space of 1-forms. If A is a vector space of 1-forms corresponding to α and B corresponds to β, then the sum $A \cup B$ corresponds to $\alpha \times \beta$ and the largest integrable subspace of $A \cap B$ corresponds to $\alpha \oplus \beta$. Interpreting $\mathbf{m}(\alpha)$ and $\mathbf{M}(\beta)$ is more challenging; in fact, no direct counterparts exist in the method based on 1-forms. If $\mathscr{N}$ is the vector space corresponding to the vector function $\mathbf{m}(\alpha)$, then $\mathscr{N}^+$ is the largest integrable subspace belonging to $[A \cup \operatorname{span}_{\mathscr{K}}\{\mathrm{d}u\}] \cap \operatorname{span}_{\mathscr{K}}\{\mathrm{d}x^+\}$ (in the continuous-time case we take time derivatives instead of shifts). The vector space corresponding to $\mathbf{M}(\beta)$ is the minimal integrable vector space containing $B^+ \cap \operatorname{span}_{\mathscr{K}}\{\mathrm{d}x\}$.

In functions' algebra one assumes that the function f in (1) is surjective, which guarantees that all the operations are well defined, and the vector function $\mathbf{m}(\alpha)$ is uniquely defined. This assumption is equivalent to the submersivity assumption in the 1-forms approach by definition of a submersion. Compared to the 1-forms formalism, in functions' algebra there is no need to worry about integrability, since one works directly with functions. On the other hand, this makes computations more complex, limiting the use of the approach. For example, there is no general algorithm for computing $\mathbf{m}(\alpha)$ in the continuous-time case. Finally, note that functions' algebra has not been used to study systems described by the i/o equations. The method has been built for systems described by state equations, and to use it for systems described by the i/o equations, at least the

definition of binary relation Δ has to be modified. As the 1-forms approach, functions' algebra suits well for studying the generic properties of a system.

9.3. Comparison with the continuous-time case

The continuous-time case is easier to address since there is no need to construct the inversive closure, and because of this, there is no need for submersivity assumption. However, once the proper difference field is constructed, the solutions of many problems are easier in the discrete-time case. One such problem is system realization, where instead of Euclidean division or adjoint polynomials, extremely simple cut and shift operator may be applied [82].

Though many results for continuous- and discrete-time cases are similar, there are a number of important differences. For instance, unlike in the continuous-time case, there exist discrete-time systems, described either in terms of shift or difference operator, which cannot be decomposed into observable and unobservable subsystems due to the fact that the observable space, which is used to check observability instead of the well-known observability rank condition [98,99], is not necessarily integrable [70]. However, it has been proven that if the polynomial or analytic system is generically reversible, the observability decomposition can always be achieved (see [64] and [65], respectively). Another important difference is related to realization theory. If the i/o difference equation is not realizable in the state-space form, the compensated system can always be made realizable using a post-compensator [81], whereas this is impossible in the continuous-time case [96]. A third important difference is that continuous-time systems can be linearized by a dynamic feedback if and only if they can be linearized by the so-called dynamic endogenous feedback. In simple words, an endogenous feedback means that the variables of the compensator/feedback are functions of the given system variables and their derivatives/shifts. However, discrete-time systems may be linearizable by exogenous feedback even when they are not linearizable by endogenous feedback [5].

10. CONCLUSION

The approach has proven to be very useful for studying standard discrete- and continuous-time nonlinear control systems. The future studies are targeted to expand the classes of systems, for which the approach can be generalized. For example, we look to study non-uniformly sampled systems [65] and time-delay systems, for which some work has already been done. Also, a possible way to expand the number of control problems solvable by the method is to study the flatness-based nonlinear model predictive control [36,88].

In trying to apply nonlinear techniques, symbolic computation quickly becomes a dominant concern. Therefore, in Tallinn University a software package is created in the *Mathematica* environment for symbolic computations. The package contains functions, which help to make the necessary computations in the 1-forms approach, but also many of the control problems are implemented and added to the software package (see more in [49]).

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Mittelineaarsete juhtimissüsteemide globaalsel lineariseerimisel põhinev lähenemine: lühiülevaade

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On antud ülevaade diferentsiaalvormidel põhinevast algebralisest meetodist, mida kasutatakse mittelineaarsete juhtimissüsteemide uurimisel. Artikli eesmärk on kirjeldada antud meetodit, põhjendada vajalikke eeldusi ja näidata meetodi efektiivsust ning mõningaid puudusi. Meetodi kaks kõige tähtsamat iseloomulikku omadust on järgmised. Esiteks, mittelineaarsete funktsioonide asemel töötatakse nende diferentsiaalidega, mis tähendab, et arvutused on kuni integreerimiseni sarnased lineaarse juhuga. Teiseks, meetodit kasutatakse juhtimissüsteemide niisuguste omaduste uurimiseks, mis kehtivad vastava piirkonna peaaegu igas punktis. Esimene omadus tähendab, et lahendused leitakse 1-vormide kaudu, misjärel 1-vormide integreeruvus võimaldab lahendused esitada funktsioonide kaudu. Teise iseloomuliku omaduse tähtsus seisneb selles, et see võimaldab esitada lahendused kompaktsemalt ja selgemalt, sest ei teki vajadust täpsustada punkti, mille ümbruses töötatakse, ega antud punkti ümbrust. Lisaks nende aspektide üksikasjalikule kirjeldamisele on artiklis esitatud põhjalik ülevaade probleemidest, mida on antud meetodiga uuritud, ja meetodit on võrreldud mõningate teiste algebraliste meetoditega.