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MATHEMATICS

Representing the Banach operator ideal of completely continuous operators

Rauni Lillemets

Institute of Mathematics and Statistics, University of Tartu, J. Liivi 2, 50409 Tartu, Estonia; rauni.lillemets@ut.ee

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Abstract. Let $\mathscr{V}, \mathscr{W}_\infty$, and $\mathscr{W}$ be the operator ideals of completely continuous, weakly ∞-compact, and weakly compact operators, respectively. In a recent paper, William B. Johnson, Eve Oja, and the author proved that $\mathscr{V} = \mathscr{W}_\infty \circ \mathscr{W}^{-1}$ (Johnson, W. B., Lillemets, R., and Oja, E. Representing completely continuous operators through weakly ∞-compact operators. *Bull. London Math. Soc.*, 2016, **48**, 452–456). We show that this equality also holds in the context of Banach operator ideals.

Key words: mathematics, Banach operator ideals, completely continuous operators, weakly compact operators, weakly ∞-compact operators.

1. INTRODUCTION

Let $\mathscr{L}, \mathscr{K}, \mathscr{W}$, and $\mathscr{V}$ denote the operator ideals of bounded linear, compact, weakly compact, and completely continuous operators. Let X and Y be Banach spaces. Recall that a linear map $T: X \to Y$ is *completely continuous*, i.e. $T \in \mathscr{V}(X,Y)$, if T takes weakly null sequences in X to null sequences in Y. It is well known that operator ideals $\mathscr{K}$, $\mathscr{V}$, and $\mathscr{W}$ are Banach operator ideals with the usual operator norm.

Let $(x_n) \subset X$ be a bounded sequence. It is well known and easy to see that (x_n) defines an operator $\Phi_{(x_n)} \in \mathscr{L}(\ell_1, X)$ through the equality

$$\Phi_{(x_n)}(a_k) = \sum_{k=1}^{\infty} a_k x_k,\ (a_k) \in \ell_1.$$

Denote the classes of all null sequences and weakly null sequences in X by $c_0(X)$ and $c_0^w(X)$, respectively. Both of them are Banach spaces with the supremum norm. According to the Grothendieck compactness principle (see [3] or, e.g. [4, Proposition 1.e.2]), a subset $K \subset X$ is relatively compact if and only if for every $\varepsilon > 0$ there exists $(x_n) \in c_0(X)$, with $\sup_{n\in\mathbb{N}} \|x_n\| \leq \sup_{x\in K} \|x\| + \varepsilon$, such that $K \in \Phi_{(x_n)}(B_{\ell_1})$.

A subset K of X is *relatively weakly ∞-compact* if $K \subset \Phi_{(x_n)}(B_{\ell_1})$ for some sequence $(x_n) \in c_0^w(X)$. An operator $T \in \mathscr{L}(X,Y)$ is *weakly ∞-compact* if $T(B_X)$ is a relatively weakly ∞-compact subset of Y. Weakly ∞-compact (more generally, weakly p-compact) operators were considered by Sinha and Karn [7] in 2002 (for an even more general version of weakly (p,r)-compact operators, see [2]). Denote by $\mathscr{W}_\infty$ the class of all weakly ∞-compact operators acting between arbitrary Banach spaces. An easy straightforward verification (as in [1, Proposition 2.1]) shows that $\mathscr{W}_\infty$ is an operator ideal.

Recall that the *right-hand quotient* $\mathscr{A} \circ \mathscr{B}^{-1}$ of two operator ideals $\mathscr{A}$ and $\mathscr{B}$ is the operator ideal that consists of all operators $T \in \mathscr{L}(X,Y)$ such that $TS \in \mathscr{A}(X_0,Y)$ whenever $S \in \mathscr{B}(X_0,X)$ for some Banach space X_0 (see [6, 3.1.1]).

Let $(\mathscr{A}, \|\cdot\|_{\mathscr{A}})$ and $(\mathscr{B}, \|\cdot\|_{\mathscr{B}})$ be quasi-Banach operator ideals. The quotient $\mathscr{A} \circ \mathscr{B}^{-1}$ becomes a quasi-Banach operator ideal if for every operator $T \in \mathscr{A} \circ \mathscr{B}^{-1}(X,Y)$ one puts

$$\|T\|_{\mathscr{A} \circ \mathscr{B}^{-1}} = \sup\{\|TS\|_{\mathscr{A}} \mid S \in \mathscr{B}(X_0,X), \|S\|_{\mathscr{B}} \leq 1\},$$

where the supremum is taken over all Banach spaces X_0 (see [6, 7.2.1]).

In [5] Johnson, Oja, and the author proved that $\mathscr{V} = \mathscr{W}_\infty \circ \mathscr{W}^{-1}$ as operator ideals. Now, we will show that this equality holds in the context of Banach operator ideals. For this, we introduce a norm on the operator ideal $\mathscr{W}_\infty$. Let $T \in \mathscr{W}_\infty(X,Y)$ and put

$$\|T\|_{\mathscr{W}_\infty} = \inf\{\|(x_n)\|_{c_0^w(X)} \mid (x_n) \in c_0^w(Y), T(B_X) \subset \Phi_{(x_n)}(B_{\ell_1})\}.$$

As Proposition 1 below shows, $\mathscr{W}_\infty$ is a Banach operator ideal with this norm. The main result of this paper (Theorem 3) is that the equality $\mathscr{V} = \mathscr{W}_\infty \circ \mathscr{W}^{-1}$ indeed holds in the context of Banach operator ideals.

Throughout this paper, let $\mathbb{K}$ denote the scalar field $\mathbb{R}$ or $\mathbb{C}$.

2. BANACH OPERATOR IDEAL $\mathscr{W}_\infty$

In this section we verify that $\mathscr{W}_\infty$ is indeed a Banach operator ideal endowed with the norm $\|\cdot\|_{\mathscr{W}_\infty}$.

Proposition 1. *$\mathscr{W}_\infty$ is a Banach operator ideal with the norm $\|\cdot\|_{\mathscr{W}_\infty}$.*

Proof. It is easy to see that $\|I_{\mathbb{K}}\|_{\mathscr{W}_\infty} = 1$. Indeed, put $(\beta_n) = (1,0,0,\dots) \in c_0^w(\mathbb{K})$ and observe that $B_{\mathbb{K}} \subset \Phi_{(\beta_n)}(B_{\ell_1})$. Therefore $\|I_{\mathbb{K}}\|_{\mathscr{W}_\infty} \leq 1$. On the other hand, let $B_{\mathbb{K}} \subset \Phi_{(\beta_n)}(B_{\ell_1})$ for some $(\beta_n) \in c_0^w(\mathbb{K})$. Then there exists a sequence $(\alpha_n) \in B_{\ell_1}$ so that $1 = \sum_{n=1}^\infty \alpha_n \beta_n$. Therefore

$$1 \leq \left|\sum_{n=1}^\infty \alpha_n \beta_n\right| \leq \sum_{n=1}^\infty |\alpha_n \beta_n| \leq \sup_{n\in\mathbb{N}} |\beta_n| \sum_{n=1}^\infty |\alpha_n| \leq \sup_{n\in\mathbb{N}} |\beta_n|,$$

and we have shown that $\|I_{\mathbb{K}}\|_{\mathscr{W}_\infty} \geq 1$.

Let $S,T \in \mathscr{W}_\infty(X,Y)$. We need to prove that $\|S+T\|_{\mathscr{W}_\infty} \leq \|S\|_{\mathscr{W}_\infty} + \|T\|_{\mathscr{W}_\infty}$. For this, take $\varepsilon > 0$ and sequences $(x_n),(y_n) \in c_0^w(Y)$ such that $S(B_X) \subset \Phi_{(x_n)}(B_{\ell_1})$ and $T(B_X) \subset \Phi_{(y_n)}(B_{\ell_1})$ with $\|(x_n)\| \leq (1+\varepsilon)\|S\|_{\mathscr{W}_\infty}$ and $\|(y_n)\| \leq (1+\varepsilon)\|T\|_{\mathscr{W}_\infty}$.

Assume that $\sup_{n\in\mathbb{N}} \|x_n\| \neq 0$ and that $\sup_{n\in\mathbb{N}} \|y_n\| \neq 0$ (otherwise, either $S = 0$ or $T = 0$, and the proof is trivial). Put

$$q := \frac{\sup_{n\in\mathbb{N}} \|y_n\|}{\sup_{n\in\mathbb{N}} \|x_n\|}.$$

Define $(z_n) \in c_0^w(Y)$ by

$$z_n = \begin{cases} (q+1)x_k & \text{if } n = 2k-1, \\ \frac{q+1}{q} y_k & \text{if } n = 2k. \end{cases}$$

We check that

$$\sup_{n\in\mathbb{N}} \|z_n\| \leq \sup_{n\in\mathbb{N}} \|x_n\| + \sup_{n\in\mathbb{N}} \|y_n\|.$$

For this purpose, we use the fact that

$$(q+1)\sup_{n\in\mathbb{N}} \|x_n\| = \frac{q+1}{q} \sup_{n\in\mathbb{N}} \|y_n\|.$$

We have that

$$\begin{aligned}\sup_{n\in\mathbb{N}}\|z_n\| &= \max\left\{(q+1)\sup_{n\in\mathbb{N}}\|x_n\|,\ \frac{q+1}{q}\sup_{n\in\mathbb{N}}\|y_n\|\right\}\\ &= (q+1)\sup_{n\in\mathbb{N}}\|x_n\| = \sup_{n\in\mathbb{N}}\|x_n\| + q\sup_{n\in\mathbb{N}}\|x_n\| = \sup_{n\in\mathbb{N}}\|x_n\| + \sup_{n\in\mathbb{N}}\|y_n\|.\end{aligned}$$

It remains to show that $(S+T)(B_X)\subset\Phi_{(z_n)}(B_{\ell_1})$. Let $z\in B_X$, let $Sz=\sum_{n=1}^{\infty}\alpha_n x_n$, and let $Tz=\sum_{n=1}^{\infty}\beta_n y_n$. Define $(\gamma_n)\in B_{\ell_1}$ by

$$\gamma_n=\begin{cases}\frac{1}{q+1}\alpha_k & \text{if } n=2k-1,\\ \frac{q}{q+1}\beta_k & \text{if } n=2k.\end{cases}$$

Then

$$(S+T)(x)=\sum_{n=1}^{\infty}\alpha_n x_n+\sum_{n=1}^{\infty}\beta_n y_n=\sum_{\substack{n=2k-1,\\ k\in\mathbb{N}}}\gamma_n z_n+\sum_{\substack{n=2k,\\ k\in\mathbb{N}}}\gamma_n z_n=\sum_{n=1}^{\infty}\gamma_n z_n.$$

Let $T\in\mathscr{L}(X_0,X)$, $S\in\mathscr{W}_\infty(X,Y)$, and $R\in\mathscr{L}(Y,Y_0)$ be given. We prove that $\|RST\|_{\mathscr{W}_\infty}\le\|R\|\,\|S\|_{\mathscr{W}_\infty}\|T\|$. Let $\varepsilon>0$ and let $(y_n)\in c_0^w(Y)$ be given such that $\sup_{n\in\mathbb{N}}\|y_n\|\le\|S\|_{\mathscr{W}_\infty}+\varepsilon$ and $S(B_X)\subset\Phi_{(y_n)}(B_{\ell_1})$. Put $(z_n):=(\|T\|Ry_n)$. Then $(z_n)\in c_0^w(Y_0)$ because the operator R (as every bounded linear operator) is weakly-weakly continuous. Therefore

$$RST(B_{X_0})\subset\|T\|RS(B_X)\subset\|T\|R(\Phi_{(y_n)}(B_{\ell_1}))=\|T\|\Phi_{(Ry_n)}(B_{\ell_1})=\Phi_{(z_n)}(B_{\ell_1}).$$

Since $RST(B_{X_0})\subset\Phi_{(z_n)}(B_{\ell_1})$, we have $\|RST\|_{\mathscr{W}_\infty}\le\|(z_n)\|_{c_0^w(Y_0)}$. This gives us that

$$\|(z_n)\|_{c_0^w(Y_0)}=\sup_{n\in\mathbb{N}}\|z_n\|\le\|T\|\,\|R\|\sup_{n\in\mathbb{N}}\|y_n\|\le\|R\|\,(\|S\|_{\mathscr{W}_\infty}+\varepsilon)\,\|T\|$$

and therefore $\|RST\|_{\mathscr{W}_\infty}\le\|R\|\,\|S\|_{\mathscr{W}_\infty}\|T\|$.

We have shown that $(\mathscr{W}_\infty,\|\cdot\|_{\mathscr{W}_\infty})$ is a normed operator ideal. To prove that it is a Banach operator ideal, we need to verify that $\sum_{k=1}^{\infty}R_k\in\mathscr{W}_\infty(X,Y)$ whenever $\sum_{k=1}^{\infty}\|R_k\|_{\mathscr{W}_\infty}<\infty$. Clearly,

$$\sum_{k=1}^{\infty}\|R_k\|\le\sum_{k=1}^{\infty}\|R_k\|_{\mathscr{W}_\infty}<\infty.$$

Therefore we may define $R=\sum_{k=1}^{\infty}R_k\in\mathscr{L}(X,Y)$. It remains to show that $R\in\mathscr{W}_\infty(X,Y)$. Put $S_1:=\sum_{k=1}^{m_1}R_k$, $S_2:=\sum_{k=m_1+1}^{m_2}R_k$, etc., so that $\|S_m\|_{\mathscr{W}_\infty}<\frac{1}{4^m}$ for every $m\ge 2$. Notice that

$$R=\sum_{k=1}^{\infty}R_k=\sum_{k=1}^{\infty}S_k.$$

Since $S_1\in\mathscr{W}_\infty(X,Y)$, there exists a sequence $(y_k^1)\in c_0^w(Y)$ such that $S_1(B_X)\subset\Phi_{(y_k^1)}(B_{\ell_1})$. Furthermore, for every $m\ge 2$ there exists a sequence $(y_k^m)_{k\in\mathbb{N}}\in c_0^w(Y)$ so that $\sup_{k\in\mathbb{N}}\left\|y_k^m\right\|\le\frac{1}{4^m}$ and $S_m(B_X)\subset\Phi_{(y_k^m)}(B_{\ell_1})$.

We define the sequence (z_n) as any permutation of the following elements:

$$\begin{gathered}2y_1^1,2y_2^1,\dots,2y_n^1,\dots,\\ 4y_1^2,4y_2^2,\dots,4y_n^2,\dots,\\ \dots,\\ 2^m y_1^m,2^m y_2^m,\dots,2^m y_n^m,\dots,\\ \dots,\end{gathered}$$

where $z_n = 2^{j_n} y_{i_n}^{j_n}$. To prove that $(z_n) \in c_0^w(Y)$, we take any $f \in Y^*$, let $\varepsilon > 0$, and show that the set $\{n \in \mathbb{N} \mid |f(z_n)| > \varepsilon\}$ is finite. It is so because $2^m \sup_{k\in\mathbb{N}} \|y_k^m\| \xrightarrow[m\to\infty]{} 0$ and each of the sequences $(y_k^m)_{k\in\mathbb{N}}$ contains only a finite number of elements such that $\left|2^m f(y_k^m)\right| > \varepsilon$.

We claim that $R(B_X) \subset \Phi_{(z_n)}(B_{\ell_1})$. Let $x \in B_X$. For every $m \in \mathbb{N}$ we have that $S_m x = \sum_{k\in\mathbb{N}} \alpha_k^m y_k^m$ for some sequence $(\alpha_k^m)_{k\in\mathbb{N}} \in B_{\ell_1}$. Put

$$\beta_n := \frac{1}{2^{j_n}} \alpha_{i_n}^{j_n}.$$

Notice that $(\beta_n) \in B_{\ell_1}$, because

$$\sum_{n=1}^{\infty} |\beta_n| = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2^m} |\alpha_k^m| \leq 1.$$

We complete the proof by observing that

$$Rx = \sum_{m=1}^{\infty} S_m x = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \alpha_k^m y_k^m = \sum_{n=1}^{\infty} \frac{\alpha_{i_n}^{j_n}}{2^{j_n}} \left(2^{j_n} y_{i_n}^{j_n}\right) = \sum_{n=1}^{\infty} \beta_n z_n.$$ □

3. THE MAIN RESULT

Proposition 2. *Let $T \in \mathscr{K}(X,Y)$. Then $T \in \mathscr{W}_\infty(X,Y)$ and $\|T\|_{\mathscr{W}_\infty} = \|T\|$.*

Proof. Clearly, $T \in \mathscr{W}_\infty(X,Y)$. The Grothendieck compactness principle allows us to write

$$\|T\| = \inf\{\sup_{n\in\mathbb{N}} \|x_n\| \mid (x_n) \in c_0(Y), T(B_X) \subset \Phi_{(x_n)}(B_{\ell_1})\}.$$

Therefore $\|T\|_{\mathscr{W}_\infty} \leq \|T\|$, since infimum in the definition of $\|T\|_{\mathscr{W}_\infty}$ is taken over a larger set than in the previous formula. On the other hand, $\|T\| \leq \|T\|_{\mathscr{W}_\infty}$ because $\mathscr{W}_\infty$ is a Banach operator ideal. □

For the proof of the next theorem, recall that $\mathscr{K} = \mathscr{V} \circ \mathscr{W}$ and $\mathscr{V} = \mathscr{K} \circ \mathscr{W}^{-1}$ as Banach operator ideals (see [6, 3.1.3] and [6. 3.2.3], respectively).

Theorem 3. *The equality $\mathscr{V} = \mathscr{W}_\infty \circ \mathscr{W}^{-1}$ holds in the context of Banach operator ideals.*

Proof. Fix an operator $T \in \mathscr{V}(X,Y) = \mathscr{W}_\infty \circ \mathscr{W}^{-1}(X,Y)$. By definition,

$$\|T\|_{\mathscr{W}_\infty \circ \mathscr{W}^{-1}} = \sup\{\|TW\|_{\mathscr{W}_\infty} \mid W \in \mathscr{W}(X_0,X), \|W\|_{\mathscr{W}} \leq 1\},$$

where the supremum is taken over all Banach spaces X_0.

Therefore $TW \in \mathscr{V} \circ \mathscr{W}(X_0,Y) = \mathscr{K}(X_0,Y)$ for any $W \in \mathscr{W}(X_0,X)$. According to Proposition 2,

$$\|TW\|_{\mathscr{W}_\infty} = \|TW\| = \|TW\|_{\mathscr{K}}.$$

Therefore

$$\begin{aligned}\|T\|_{\mathscr{W}_\infty \circ \mathscr{W}^{-1}} &= \sup\{\|TW\|_{\mathscr{W}_\infty} \mid W \in \mathscr{W}(X_0,X), \|W\|_{\mathscr{W}} \leq 1\} \\ &= \sup\{\|TW\|_{\mathscr{K}} \mid W \in \mathscr{W}(X_0,X), \|W\|_{\mathscr{W}} \leq 1\} = \|T\|_{\mathscr{K} \circ \mathscr{W}^{-1}} = \|T\|_{\mathscr{V}}.\end{aligned}$$ □

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Täielikult pidevate operaatorite Banachi operaatorideaali kirjeldus

Rauni Lillemets

Olgu $\mathscr{V}, \mathscr{W}_\infty$ ja $\mathscr{W}$ operaatorideaalid, mis koosnevad vastavalt täielikult pidevatest, nõrgalt ∞-kompaktsetest ning nõrgalt kompaktsetest operaatoritest. Hiljutises artiklis [5], mille autoriteks on William B. Johnson, Eve Oja ja käesoleva artikli autor, tõestasime, et kehtib võrdus $\mathscr{V} = \mathscr{W}_\infty \circ \mathscr{W}^{-1}$. On teada, et $\mathscr{V}$ ja $\mathscr{W}$ on Banachi operaatorideaalid tavalise operaatornormi suhtes. Antud artiklis varustame ka operaatorideaali $\mathscr{W}_\infty$ normiga ja veendume, et see on selle normi suhtes Banachi operaatorideaal. Seejärel näitame, et võrdus $\mathscr{V} = \mathscr{W}_\infty \circ \mathscr{W}^{-1}$ kehtib ka Banachi operaatorideaalide kontekstis.