Proceedings of the Estonian Academy of Sciences,
2017, **66**, 2, 109–123
https://doi.org/10.3176/proc.2017.2.01
Available online at www.eap.ee/proceedings

CONTROL SYSTEMS

Feedback linearization of possibly non-smooth systems

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Received 18 August 2016, accepted 24 October 2016, available online 20 March 2017

Abstract. The algebraic approach known as functions' algebra is used to develop the necessary and sufficient conditions for the existence of state transformation and static state feedback that linearize the system equations. The advantage of this method is that it allows considering also non-smooth systems. The main object in functions' algebra is the set of vector functions, divided into equivalence classes, which form a lattice. Both discrete- and continuous-time cases are considered. The solutions to the feedback linearization problem are expressed in terms of a finite sequence of vector functions, which contain all the independent functions having certain relative degrees. The theoretical results are illustrated by numerous examples.

Key words: nonlinear systems, feedback linearization, non-smooth functions, algebraic methods..

1. INTRODUCTION

The problem of linearization by static state feedback and change of coordinates is addressed in this paper. The problem has received a lot of attention [1,2,4,7,9,10,16–18]. In [16] only the state transformation was used. The first solution to the static state feedback linearizability problem was given in [10]. Since then numerous publications have addressed different aspects of the problem for different system classes using various mathematical tools. For the differential geometric approach see for instance [9] for continuous-time case and [7] for discrete-time case. For the algebraic methods see [4] for continuous-time case and [1] for discrete-time case. The papers [2,18] search for the largest feedback linearizable subsystem. An important point in all of these solutions is the assumption that the system equations are either smooth or analytic. To the authors' knowledge only papers [15,21] consider feedback linearization of non-smooth systems. Both these papers address only the single-input case and paper [21] allows non-smooth parts only in the feedback. Note that in the non-smooth case the methods of differential geometry and differential algebra are not applicable.

In this paper the (possibly) non-smooth systems are addressed and the algebraic approach, called 'functions' algebra' (see [15,24]), is used. This approach was developed in analogy of the algebra of partitions, see [8]. Its advantage is that one can consider also non-smooth systems. The main objects one operates with are vector functions that depend on the state and input variables. These vector functions are

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divided into the equivalence classes on the basis of preorder, which defines an equivalence relation. Namely, two vector functions are said to be equivalent if one can write the first vector function in terms of the second and vice versa. In what follows, one operates with the equivalence classes (or with their representatives) but not with the vector functions themselves. Since on the set of the equivalence classes the previously defined preorder becomes a partial order, the set becomes a lattice. Finally, this lattice is connected to the system equations by certain binary relation Δ. The described method is already used to solve problems such as disturbance decoupling [11] and fault diagnosis [19].

In this paper the feedback linearization problem is solved for discrete-time systems by a static state feedback and a state transformation. The paper generalizes the results of [15] for single-input systems to multi-input systems and then also extends the results for the continuous-time case. The extension of the linearizability conditions to the multi-input case is not direct. However, the specification of the new conditions for the single-input case is shown to be equivalent to those from [15]. Like in [15], the conditions for the existence of static feedback and coordinate transformation are given in terms of a certain finite sequence of vector functions δ^i. Additionally, compared to [15], we give a simpler formula for computing the sequence δ^i. In [15] a comparison of the solutions of the feedback linearization problem of [15] (single input case) and [1] is given. This comparison can be directly extended for the solutions given in this paper, i.e. for multiple input systems and for the continuous-time case.

The results of this paper are not global, but we study the generic case. This means that we do not fix a point and work in the neighbourhood of this point; instead, the results of this paper are valid locally around every point where the conditions for feedback linearizability are satisfied and all the necessary transformations can be defined uniquely. This allows us to simplify the presentation of the results compared to the local case where a working point and its neighbourhood have to be fixed.

The typical non-smooth functions for practical applications are those describing such phenomena as saturation, hysteresis, friction, and backlash. All these phenomena may be expressed or approximated mathematically with the help of the signum function. The other non-smooth functions are absolute value and functions described in different regions via different functions.

The paper is organized as follows. In Section 2, an overview of functions' algebra is given. Sections 3 and 4 present the main results. In Section 3 the discrete-time case and in Section 4 the continuous-time case are considered. The paper ends with conclusions.

2. FUNCTIONS' ALGEBRA

Consider the discrete-time system of the following form

$$x(t+1) = f(x(t), u(t)), \tag{1}$$

where $x(t) \in X \subseteq \mathbb{R}^n$ and $u(t) \in U \subseteq \mathbb{R}^m$. It is assumed that the inputs influence the system equations (1) independently, which means that one cannot eliminate input variables by a static state feedback. In the differentiable case this corresponds to matrix $\frac{\partial f}{\partial u}$, which generically has a full rank.

To simplify the presentation, we alternatively use the notations $x := x(t)$, $x^+ := x(t+1)$, $x^- := x(t-1)$, $x^{[j]} := x(t+j)$, $j \in \mathbb{Z}$, and similarly for the other variables and functions.

The mathematical approach called functions' algebra will be used in this paper (see [15,24]). Denote by $S_{X\times U}$ the set of vector functions with the domain $X \times U$. So, the elements of $S_{X\times U}$ are vectors with a finite dimension, whose elements are functions depending on variables x and u. We assume that these functions are piecewise smooth.

Note that some vector functions may contain information on the others. For example, if one knows the value of the vector function $\alpha = [x_1, x_2, x_3]^{\mathrm{T}}$, one knows also the value of the vector function $\beta = [x_1x_2, x_3]^{\mathrm{T}}$. This gives reason for considering the following definition. In the set $S_{X\times U}$, the relation of preorder $\leq$ is defined.

Definition 1. *Given $\alpha, \beta \in S_{X\times U}$, one says that $\alpha \leq \beta$ if there exists a vector function γ such that $\beta(x,u) = \gamma(\alpha(x,u))$ for $x \in X$, $u \in U$.*

Note that $\leq$ is not a partial order, i.e. there exist non-equal vector functions $\alpha, \beta \in S_{X\times U}$ such that $\alpha \leq \beta$ and $\beta \leq \alpha$. This does not agree with the intuition; moreover, this does not allow us to build up an algebraic structure for our purposes. To eliminate the problem, such functions are defined to be equivalent.

Definition 2. *If $\alpha \leq \beta$ and $\beta \leq \alpha$, then α and β are called equivalent, denoted by $\alpha \cong \beta$.*

Note that the relation $\cong$ is reflexive ($\alpha \cong \alpha$ for all $\alpha \in S_{X\times U}$), symmetric ($\alpha \cong \beta \Rightarrow \beta \cong \alpha$) and transitive ($\alpha \cong \beta$, and $\beta \cong \theta$ yield $\alpha \cong \theta$). Thus $\cong$ is an equivalence relation. The equivalence relation divides the set $S_{X\times U}$ into the equivalence classes containing the equivalent functions. If $S_{X\times U}/\cong$ is the set of all equivalence classes, then the relation $\leq$ becomes *a partial order* on this set. In the following we work on the set of equivalence classes $S_{X\times U}/\cong$ (or rather with their simplest representatives), i.e. one can always replace a vector function by an equivalent one. This also means that in the following the symbol '$=$' should be understood as '$\cong$'.

Note that there exist two special equivalence classes: $\mathbf{1}$ – the set of constant functions and $\mathbf{0} := [x,u]$. The equivalence class $\mathbf{1}$, which contains all the constant functions, satisfies $\alpha \leq \mathbf{1}$ for all $\alpha \in S_{X\times U}$. On the other hand, since every function in $S_{X\times U}$ can be written in terms of x and u, then the equivalence class $\mathbf{0}$ satisfies $\mathbf{0} \leq \alpha$ for all $\alpha \in S_{X\times U}$. Therefore, for any two equivalence classes in $S_{X\times U}/\cong$, represented respectively by vector functions α and β, there exist a minimal equivalence class, represented by a vector function γ that satisfies $\alpha \leq \gamma$, $\beta \leq \gamma$, and a maximal equivalence class, represented by a vector function ζ that satisfies $\zeta \leq \alpha$, $\zeta \leq \beta$.

Recall that a lattice is a set with a partial order where every two elements α and β have a unique supremum (least upper bound) $\sup(\alpha,\beta)$ and an infimum (greatest lower bound) $\inf(\alpha,\beta)$. Thus, $(S_{X\times U}/\cong, \leq)$ is a lattice. The equivalent definition of the lattice as an algebraic structure with two binary operations, $\times$ and $\oplus$, may be given if for every two elements both operations are commutative and associative and, moreover, $\alpha \times (\alpha \oplus \beta) = \alpha$, $\alpha \oplus (\alpha \times \beta) = \alpha$. The equivalence follows from the definition of the binary operations $\times$ and $\oplus$ as

$$\begin{aligned} \alpha \times \beta &= \inf(\alpha,\beta), \\ \alpha \oplus \beta &= \sup(\alpha,\beta). \end{aligned} \tag{2}$$

Therefore, the triple $(S_{X\times U}/\cong, \times, \oplus)$ can also be viewed as a lattice. With a slight abuse of notation, we use below for $S_{X\times U}/\cong$ also the notation $S_{X\times U}$.

In lattice theory it is customary not to operate with $\inf(\alpha,\beta)$ and $\sup(\alpha,\beta)$ but with binary operations $\times$ and $\oplus$, respectively.

Computation of $\oplus$. In the simple cases, (2) may be used to compute $\alpha \oplus \beta$.

Computation of $\times$. The rule for operation $\times$ is simple: $(\alpha \times \beta)(x) = [\alpha(x), \beta(x)]^{\mathrm{T}}$. However, the product may contain functionally dependent components that have to be found and removed, which just means finding the simplest representative in the equivalence class for $\alpha \times \beta$.

Example 1. Let $\alpha = [x_1, x_2, x_3]^{\mathrm{T}}$, $\beta = [x_1x_2, x_3]^{\mathrm{T}}$, $\eta = [x_1 + x_2, x_2, x_3]^{\mathrm{T}}$, and $\theta = [x_2, x_3]^{\mathrm{T}}$. Clearly, $\beta = [\alpha_1\alpha_2, \alpha_3]^{\mathrm{T}}$, where $\alpha = [\alpha_1, \alpha_2, \alpha_3]^{\mathrm{T}}$. Therefore, by Definition 1 $\alpha \leq \beta$. Since $\alpha = [\eta_1 - \eta_2, \eta_2, \eta_3]^{\mathrm{T}}$ and $\eta = [\alpha_1 + \alpha_2, \alpha_2, \alpha_3]^{\mathrm{T}}$, then by Definition 2, $\alpha \cong \eta$.

Note that when $\alpha \leq \beta$, then by the definitions of operations $\times$ and $\oplus$, one can yield $\alpha \times \beta = \alpha$, $\alpha \oplus \beta = \beta$. Also, one can compute $\beta \times \theta = \alpha$ and $\beta \oplus \theta = x_3$.

Next, the lattice $(S_{X\times U}/\cong, \times, \oplus)$ is connected with the system dynamics (1) through the following definition. Since (1) defines only the forward shift of x, not u, in the following definitions the vector functions must belong to S_X.

Definition 3. *Given $\alpha, \beta \in S_X$, one says that $(\alpha, \beta) \in \Delta$ if there exists a vector function f_* such that for all $(x,u) \in X \times U$,*

$$\beta(f(x,u)) = f_*(\alpha(x), u).$$

When $(\alpha, \beta) \in \Delta$, it is said that α and β form an ordered pair.

The binary relation Δ is mostly used to define the operators $\mathbf{m}$ and $\mathbf{M}$.

Definition 4. *The vector function $\mathbf{m}(\alpha) \in S_X$ is defined by the following two conditions:*
(i) $(\alpha, \mathbf{m}(\alpha)) \in \Delta$
(ii) if $(\alpha, \beta) \in \Delta$, then $\mathbf{m}(\alpha) \leq \beta$.

Definition 5. *The vector function $\mathbf{M}(\beta) \in S_X$ is defined by the following two conditions:*
(i) $(\mathbf{M}(\beta), \beta) \in \Delta$
(ii) if $(\alpha, \beta) \in \Delta$, then $\alpha \leq \mathbf{M}(\beta)$.

Remark 1. As mentioned before, in this approach one works with the equivalence classes $S_{X\times U}/\cong$ of the set $S_{X\times U}$ of vector functions. Therefore, $\mathbf{m}(\alpha)$ and $\mathbf{M}(\beta)$ are representatives of the respective equivalence classes and as such not unique. Any vector function equivalent to $\mathbf{m}(\alpha)$ (respectively $\mathbf{M}(\beta)$), satisfies also the conditions of Definition 4 (respectively Definition 5).

Computation of the operator $\mathbf{m}$. Note that by the definition of Δ, the condition

$$\mathbf{m}(\alpha)(f) \geq \alpha \times u$$

must be satisfied for the vector function $\alpha(x)$. Obviously, $\mathbf{m}(\alpha)(f) \geq f$. Therefore, by the definition of the operation $\oplus$

$$\mathbf{m}(\alpha)(f) = (\alpha \times u) \oplus f.$$

Finally, observe that $\mathbf{m}(\alpha)(x)$ can be computed by shifting the function $(\alpha \times u) \oplus f$ back once:

$$\mathbf{m}(\alpha) = [(\alpha \times u) \oplus f]^-, \tag{3}$$

which is possible, since $(\alpha \times u) \oplus f$ can be written in terms of f and $f^- = x$.

Computation of the operator $\mathbf{M}$. In the special case when $\beta(f(x,u))$ can be represented in the form

$$\beta(f(x,u)) = \sum_{i=1}^{d} a_i(x) b_i(u),$$

where a_1, a_2, ..., a_d are arbitrary functions and all the functions b_1, b_2, ..., b_d that are non-constant are linearly independent, $\mathbf{M}(\beta) := a_1 \times a_2 \times \cdots \times a_d$.

Example 2. Consider the system

$$\begin{aligned} x_1^+ &= x_2 u \\ x_2^+ &= x_1 + x_3 \\ x_3^+ &= x_3 + u \end{aligned}$$

and the vector function $\alpha = [x_1, x_2]^{\mathrm{T}}$. First, compute $\mathbf{m}(\alpha)$ by (3):

$$\begin{aligned} \alpha \times u &= [x_1, x_2, u]^{\mathrm{T}} \\ (\alpha \times u) \oplus f &= [x_1, x_2, u]^{\mathrm{T}} \oplus [x_2 u, x_1 + x_3, x_3 + u]^{\mathrm{T}} \\ &= [x_2 u, x_1 - u]^{\mathrm{T}} = [x_1^+, x_2^+ - x_3^+]^{\mathrm{T}} \\ \mathbf{m}(\alpha) &= [(\alpha \times u) \oplus f]^- = [x_1, x_2 - x_3]^{\mathrm{T}}. \end{aligned}$$

Now, compute $\mathbf{M}(\alpha)$ using the discussion above. Since $\alpha(f(x,u)) = [x_2 u, x_1 + x_3]^{\mathrm{T}}$, then $a_1(x) = x_2$, $b_1(u) = u$, $a_2(x) = x_1 + x_3$, $b_2(u) = 1$ and thus

$$\mathbf{M}(\alpha) = a_1 \times a_2 = x_2 \times (x_1 + x_3) = [x_2, x_1 + x_3]^{\mathrm{T}}.$$

3. FEEDBACK LINEARIZATION

In the previous section we defined the main tools that are used in this section to solve the feedback linearization problem. Note that while the partial order $\leq$ and the operations $\times$ and $\oplus$ do not depend on the system dynamics (1), the operators $\mathbf{m}$ and $\mathbf{M}$ are defined by the given system dynamics (1).

Problem statement. One searches for a generic state transformation[1] $z = \varphi(x)$ and a regular static state feedback $u = G(x, v)$ that transform the system (1) into the Brunovsky canonical form

$$\begin{aligned} z_{i,j}^{+} &= z_{i,j+1} \\ z_{i,k_i}^{+} &= v_i, \end{aligned} \tag{4}$$

where $z = [z_{i,j}]^{\mathrm{T}} \in Z \subseteq \mathbb{R}^n$, $i = 1, \ldots, m$, $j = 1, \ldots, k_i - 1$ and Z is an open and dense subset of the range of φ. The static state feedback $u = G(x, v)$ is said to be regular if G defines a bijective map between u and v, i.e. there exists a function G^{-1} such that $v = G^{-1}(x, u)$.

The solution will be given in terms of a sequence

$$\delta^0 \leq \delta^1 \leq \delta^2 \leq \cdots \leq \delta^i \leq \cdots$$

of the vector functions δ^i, defined in the following way, see also [15]. Let $\delta^0 = x$ and δ^1 be the minimal vector function such that its forward shift does not depend on the input u. For $i \geq 1$ define

$$\delta^{i+1} = \delta^i \oplus \mathbf{m}(\delta^i). \tag{5}$$

The sequence δ^i, $i \geq 1$, converges, see [15]. Denote the limit by δ and let k be such that $\delta^k \neq \delta$, $\delta^{k+1} = \delta$. Note that since $(\delta^1)^+$ does not depend on u and $\delta^i = \gamma_i(\delta^1)$, then $(\delta^i)^+$ does not depend on u for $i \geq 1$.

Lemma 1. *The vector functions δ^i satisfy the relations*

$$(\delta^{i+1})^+ = \delta^i \oplus (\delta^i)^+ \tag{6}$$

for $i \geq 0$.

Proof. Since $\mathbf{m}(\alpha) = ((\alpha \times u) \oplus f)^-$, one gets from (5)

$$\begin{aligned} (\delta^{i+1})^+ &= (\delta^i)^+ \oplus (\mathbf{m}(\delta^i))^+ \\ &= (\delta^i)^+ \oplus (\delta^i \times u) \oplus f \\ &= (\delta^i)^+ \oplus (\delta^i \times u). \end{aligned}$$

The last equivalence comes from the facts that $(\delta^i)^+ = \delta^i(f)$, $\delta^i(f) \geq f$, and thus $(\delta^i)^+ \oplus f = (\delta^i)^+$. Now

$$(\delta^i)^+ \oplus (\delta^i \times u) = (\delta^i)^+ \oplus \delta^i$$

by the properties of $\oplus$ and $\times$ and the fact that $(\delta^i)^+$ does not depend on u.

If $i = 0$, then $\delta^0 \oplus (\delta^0)^+ = x \oplus f$, which is exactly the shift of δ^1. □

Lemma 1 gives an alternative and a simpler way to define (compute) the vector functions δ^i, $i \geq 1$. Unlike (5), equality (6) can also be used to compute the vector function δ^1.

[1] Meaning that the state transformation holds on some open and dense subsets of X if it holds at some point of this domain.

Definition 6. *The relative degree of a vector function* $\alpha = [\alpha_1, \dots, \alpha_p]^{\mathrm{T}}$ *is defined as minimal number r such that* $\alpha_j(x^{[r]})$ *depends on system inputs for some* $j \in \{1, \dots, p\}$.

Note that in [15] one speaks about relative degrees of α_i's, for $i = 1, \dots, p$, not about the relative degree of a vector function α.

Another property of the sequence δ^i is the following.

Lemma 2. *The relative degree of* δ^i *is* $i+1$ *for* $i \geq 0$.

Proof. The proof is accomplished by induction over i. For $i = 0$, the relative degree of $\delta^0 = x$ is clearly 1. Now assume that the claim is also true for δ^i, $i = 0, \dots, p$. Then, by Lemma 1, $(\delta^{p+1})^{[1]} = \gamma(\delta^p)$ for some vector function γ. Since the relative degree of δ^p is $p+1$, the relative degree of δ^{p+1} is $p+2$. □

In the rest of this paper, $|\alpha|$ denotes the number of independent non-constant elements of the vector α. Note that in some cases this number may differ in various regions. For example, consider a vector function $\alpha = [x_1, F]$, where $F = x_2$ when $x_3 < 0$ and $F = x_1$ when $x_3 \geq 0$. Now, depending on whether $x_3 < 0$ or not, there are 2 or 1 independent non-constant elements in α. In this case, we say that $|\alpha|$ is the maximum of all the possibilities, thus for the example $|\alpha| = 2$.

Theorem 1. *System (1) can be transformed into the form (4) by a state transformation* $z = \varphi(x)$ *and static state feedback* $u = G(x, v)$ *if and only if* $\delta = \mathbf{1}$ *and*

$$\sum_{r=1}^{k+1} (|\delta^{r-1}| - |\delta^r \times (\delta^r)^+|) = m, \tag{7}$$

where k is defined as the minimal number such that $\delta^{k+1} = \delta$.

Proof. Necessity. First, note that the sequence of functions δ^i, $i \geq 1$, is invariant with respect to the state transformation and static state feedback [15].

Consider the ith subsystem of (4) and compute the vector functions δ_i^r, $r \geq 1$, defined by (5), for this subsystem of (4):

$$\begin{aligned} \delta_i^1 &= [z_{i,j}; j = 1, \dots, k_i - 1]^{\mathrm{T}} \\ &\vdots \\ \delta_i^r &= [z_{i,j}; j = 1, \dots, k_i - r]^{\mathrm{T}} \\ &\vdots \\ \delta_i^{k_i - 1} &= [z_{i,1}]^{\mathrm{T}} \\ \delta_i^{k_i} &= \mathbf{1}. \end{aligned}$$

Since

$$(\delta_i^r)^+ = [z_{i,j}; j = 2, \dots, k_i - r + 1]^{\mathrm{T}}$$

one gets

$$\begin{aligned} \delta_i^r \times (\delta_i^r)^+ &= [z_{i,j}; j = 1, \dots, k_i - r + 1]^{\mathrm{T}} = \delta_i^{r-1} \quad \text{if } r < k_i \\ \delta_i^r \times (\delta_i^r)^+ &= \mathbf{1} \quad \text{if } r \geq k_i. \end{aligned}$$

Thus,

$$\begin{aligned} |\delta_i^{r-1}| - |\delta_i^r \times (\delta_i^r)^+| &= 0 \quad \text{if } r < k_i \\ |\delta_i^{k_i - 1}| - |\delta_i^{k_i} \times (\delta_i^{k_i})^+| &= |[z_{i,1}]| = 1. \end{aligned}$$

Now $\delta^r = \delta_1^r \times \cdots \times \delta_m^r$ and

$$\sum_{r=1}^{k+1}(|\delta^{r-1}| - |\delta^r \times (\delta^r)^+|) = \sum_{i=1}^{m}\sum_{r=k_i}|[z_{i,1}]| = \sum_{i=1}^{m} 1 = m.$$

Sufficiency. Because $\delta^{r-1} \leq \delta^r$ and by (6), $\delta^{r-1} \leq (\delta^r)^+$, one gets $\delta^{r-1} \leq \delta^r \times (\delta^r)^+$. Then for every $r = 1,\ldots,k+1$ there exists a vector function φ_r (possibly equal to $\mathbf{1}$) such that

$$\delta^{r-1} = \delta^r \times (\delta^r)^+ \times \varphi_r, \tag{8}$$

where

$$|\varphi_r| = |\delta^{r-1}| - |\delta^r \times (\delta^r)^+|.$$

Let $|\varphi_r| = \rho_r$ for $r = 1,\ldots,k+1$ and

$$\varphi_r = [\varphi_{r,1},\ldots,\varphi_{r,\rho_r}]^{\mathrm{T}}.$$

Then, by (8)

$$\begin{aligned}
\delta^0 &= [\delta^1, (\delta^1)^+, \varphi_{1,1},\ldots,\varphi_{1,\rho_1}]^{\mathrm{T}}\\
\delta^1 &= [\delta^2, (\delta^2)^+, \varphi_{2,1},\ldots,\varphi_{2,\rho_2}]^{\mathrm{T}}\\
&\vdots\\
\delta^k &= [\delta^{k+1}, (\delta^{k+1})^+, \varphi_{k+1}]^{\mathrm{T}} = [\varphi_{k+1,1},\ldots,\varphi_{k+1,\rho_{k+1}}]^{\mathrm{T}}.
\end{aligned}$$

Substituting step by step δ^r and $(\delta^r)^+$ into δ^{r-1} for $r = 1,\ldots,k$, one gets

$$\delta^0 = [(\varphi_{i,l}(x))^{[j]}; i = 1,\ldots,k+1; j = 0,\ldots,i-1; l = 1,\ldots,\rho_i]^{\mathrm{T}}. \tag{9}$$

The elements $(\varphi_{i,l}(x))^{[j]}$ and $(\varphi_{i',l'}(x))^{[j']}$, $i \neq i'$, are independent by definition and since $\delta = \mathbf{1}$, the elements $(\varphi_{i,l}(x))^{[j]}$ and $(\varphi_{i,l}(x))^{[j']}$, $j \neq j'$, are also independent. Really, if $(\varphi_{i,l}(x))^{[j]}$ and $(\varphi_{i,l}(x))^{[j']}$ were dependent, then there would exist a function γ such that $(\varphi_{i,l}(x))^{[j]} = \gamma((\varphi_{i,l}(x))^{[j']})$ (assume that $j < j'$). This would mean that the relative degree of $(\varphi_{i,l}(x))^{[j]}$ is infinite and therefore $\delta \neq \mathbf{1}$.

Because of (7), $\sum_{r=1}^{k+1}|\varphi_r| = m$, and there exist exactly m independent functions $\varphi_{i,j}$, $i = 1,\ldots,k+1$, $j = 1,\ldots,\rho_i$. Let ϕ_i, $i = 1,\ldots,m$, be these functions. Then (9) becomes

$$\delta^0 = [(\phi_i(x))^{[j]}; i = 1,\ldots,m; j = 0,\ldots,k_i-1]^{\mathrm{T}} \tag{10}$$

for some k_i. Define the state transformation

$$\begin{aligned}
z_{i,1} &= \phi_i(x)\\
&\vdots\\
z_{i,k_i} &= \phi_i(x)^{[k_i-1]}
\end{aligned} \tag{11}$$

for $i = 1,\ldots,m$. Equations (11) really define a state transformation (i.e. a one-to-one correspondence) since by (10), $z = [z_{i,1},\ldots,z_{i,k_i}; i = 1,\ldots,m]^{\mathrm{T}}$ is equivalent to δ^0, which is equivalent to x.

Now, in the new coordinates, system (1) becomes

$$\begin{aligned}
z_{i,j}^+ &= z_{i,j+1}\\
z_{i,k_i}^+ &= K_i(z,u),
\end{aligned} \tag{12}$$

for $i = 1,\dots,m,\ j = 1,\dots,k_i - 1$ and where K_i is the forward-shift of $z_{i,k_i} = \phi_i(x)^{[k_i-1]}$, i.e. $K_i = \phi_i(x)^{[k_i]}$. Finally, since the inputs influence the system dynamics independently, which is also true for the transformed system (12), then $v = K(z,u) = [K_1(z,u),\dots,K_m(z,u)]^{\mathrm{T}}$ is solvable in u. This gives a static state feedback which takes the system into the form (4). □

Remark 2. The vector function $\phi = [\phi_1,\dots,\phi_m]^{\mathrm{T}}$ in the proof of Theorem 1 is the so-called vector of linearizing outputs, see [1], which are also called flat outputs. Condition (7) guarantees that there exist m linearizing outputs. Of course, condition (7) is only sufficient for the existence of linearizing outputs as static state feedback linearizability is only a special case of more general problem statement of dynamic feedback linearizability, which in turn is equivalent to the existence of linearizing outputs. Once the linearizing outputs (vector function ϕ in our case) are known, the computation of a state transformation (11) and a feedback to be computed from $v = K(z,u)$ follows the standard procedure.

Corollary 1. *In the single input case the conditions of Theorem 1 are equivalent to the conditions given in [15], i.e.* $\delta^i \neq \mathbf{1}$ *for* $i = 1,\dots,n-1$, *and* $\delta^n = \mathbf{1}$.

Proof. In the single input case, when the problem of feedback linearization is solvable, then $|\delta^{r-1}| - |\delta^r \times (\delta^r)^+| = 0$ for $r = 1,\dots,n-1$, see [15]. Then, assuming that $\delta^i \neq \mathbf{1}$ for $i = 1,\dots,n-1$, and $\delta^n = \mathbf{1}$, condition (7) is obviously satisfied, since $|\delta^{n-1}| = 1$ (there can be only one independent function with relative degree n). Conversely, when condition (7) is satisfied, then $\delta^i \neq \mathbf{1}$ for $i = 1,\dots,n-1$, because $|\delta^{r-1}| - |\delta^r \times (\delta^r)^+| = 0$ for $r = 1,\dots,n-1$. Also, note that always $\delta^n = \mathbf{1}$ (there cannot be functions with relative degree $n+1$). □

Example 3. Consider the discrete-time non-smooth system[2]

$$\begin{aligned} x_1^+ &= \frac{x_2}{x_3} \\ x_2^+ &= u_1 x_6 \\ x_3^+ &= u_1 \ \mathrm{abs}(x_1) \\ x_4^+ &= x_5 \ \mathrm{sign}(x_4) \\ x_5^+ &= u_2 x_2 \\ x_6^+ &= x_2 x_3 \ \mathrm{abs}(x_2), \end{aligned} \tag{13}$$

defined on $\mathbb{R}^6 \backslash \{x_1 = 0, x_3 = 0\}$. Next, we compute the sequence δ^i by Lemma 1. Since $\delta^0 = x$, then by (6)

$$(\delta^1)^+ = x \oplus f = \begin{pmatrix} \frac{x_2}{x_3} \\ \frac{x_6}{\mathrm{abs}(x_1)} \\ x_5 \ \mathrm{sign}(x_4) \\ x_2 x_3 \ \mathrm{abs}(x_2) \end{pmatrix}. \tag{14}$$

Therefore, we get $\delta^1 = [x_1, \frac{x_2}{x_3}, x_4, x_6]^{\mathrm{T}}$ by shifting the vector function in (14) back once. In the similar manner

$$(\delta^2)^+ \quad = \quad \delta^1 \oplus (\delta^1)^+ = \begin{pmatrix} x_1 \\ \frac{x_2}{x_3} \\ x_4 \\ x_6 \end{pmatrix} \oplus \begin{pmatrix} \frac{x_2}{x_3} \\ \frac{x_6}{\mathrm{abs}(x_1)} \\ x_5 \ \mathrm{sign}(x_4) \\ x_2 x_3 \ \mathrm{abs}(x_2) \end{pmatrix} = \begin{pmatrix} \frac{x_2}{x_3} \\ \frac{x_6}{\mathrm{abs}(x_1)} \end{pmatrix}.$$

[2] Since we use $|\delta|$ to denote the cardinality of vector δ, then for absolute value of x the following notation is used: $\mathrm{abs}(x)$.

Again, by shifting the last vector function back once, one gets $\delta^2 = [x_1, \frac{x_2}{x_3}]^{\mathrm{T}}$. Now, $(\delta^2)^+ = [\frac{x_2}{x_3}, \frac{x_6}{\mathrm{abs}(x_1)}]^{\mathrm{T}}$ and $(\delta^3)^+ = \delta^2 \oplus (\delta^2)^+ = \frac{x_2}{x_3}$. Therefore, $\delta^3 = x_1$. Finally, since $(\delta^3)^+$ cannot be written in terms of δ^3, then $\delta^4 = \mathbf{1}$.

We have computed the sequence δ^i. Next, condition (7) will be checked:

$$\sum_{r=1}^{4}(|\delta^{r-1}| - |\delta^r \times (\delta^r)^+|) \quad = \quad (6-6)+(4-3)+(2-2)+(1-0) = 2 = m.$$

Therefore, the conditions of Theorem 1 are satisfied. To find the linearizing outputs, first, as in the proof of Theorem 1, compute the vector functions φ_r, $r = 1,2,3,4$, satisfying (8). In this example, $\varphi_1 = \mathbf{1}$, $\varphi_2 = x_4$, $\varphi_3 = \mathbf{1}$, and $\varphi_4 = x_1$. Then the vector of linearizing outputs ϕ can be found by $\phi = \varphi_1 \times \varphi_2 \times \varphi_3 \times \varphi_4 = [x_1, x_4]^{\mathrm{T}}$.

Now, by (11), one gets the coordinate transformation

$$\begin{aligned}
z_{1,1} &= x_1 \\
z_{1,2} &= \frac{x_2}{x_3} \\
z_{1,3} &= \frac{x_6}{\mathrm{abs}(x_1)} \\
z_{1,4} &= x_2 x_3 \ \mathrm{abs}(x_3) \\
z_{2,1} &= x_4 \\
z_{2,2} &= x_5 \ \mathrm{sign}(x_4).
\end{aligned}$$

The static feedback that linearizes equations (13) after applying the change of coordinates is computed by solving the equations

$$\begin{aligned}
v_1 &= z_{1,4}^+ = K_1(z,u) \\
v_2 &= z_{2,2}^+ = K_2(z,u)
\end{aligned}$$

in u_1 and u_2. The expressions of functions $K_1(z,u)$, $K_2(z,u)$ and the feedback itself are rather long and complicated and thus omitted from the paper.

4. CONTINUOUS-TIME CASE

Consider the continuous-time system

$$\dot{x} = f_c(x,u), \tag{15}$$

where $x(t) \in X \subseteq \mathbb{R}^n$, $u(t) \in U \subseteq \mathbb{R}^m$. To simplify the presentation, we omit below the time argument t in many expressions, i.e. $x := x(t)$, $\dot{x} := \dot{x}(t)$, $x^{(j)} := x^{(j)}(t)$, $j \in \mathbb{Z}$, and similarly for the other variables and functions. In a similar manner as above, one can define

1. the preorder $\leq$;
2. the lattice $(S_{X\times U}/\cong, \leq)$;
3. operations $\times$ and $\oplus$.

Although we are able to define all the elements of functions' algebra as in the discrete-time case, the continuous-time case is more challenging. For instance, in general one cannot find the derivative of a non-smooth function, while forward-shifting of a non-smooth function is not a problem. Also, the integration of a vector function is not straightforward and requires the assumption of differentiability of vector functions.

In the continuous-time case the binary relation Δ is defined as follows [13,20].

Definition 7. *Given $\alpha, \beta \in S_X$, where β is differentiable, one says that $(\alpha, \beta) \in \Delta$ if there exists a function f_* such that for all $(x,u) \in X \times U$,*

$$\dot{\beta} = \frac{\partial \beta}{\partial x} f_c(x,u) = f_*(\alpha(x), u).$$

When $(\alpha, \beta) \in \Delta$, it is said that α and β form an ordered pair.

The definition of the binary relation Δ is given only for the differentiable vector function β. In particular, one has to be able to find the derivative of β. Note that this is sometimes also possible for non-smooth β. For instance, if $x \neq 0$, then one gets $\mathrm{d}/\mathrm{d}t \ \mathrm{abs}(x) = \mathrm{sign}(x) \cdot \dot{x}$.

Although the operators $\mathbf{m}$ and $\mathbf{M}$ may be defined as in the discrete-time case, one faces now difficulties in computations due to the remarks made before Definition 7. In particular, there is no general formula/algorithm to compute $\mathbf{m}(\alpha)$, and as a consequence, also the sequence δ^i, $i \geq 0$, which is defined similarly as in the discrete-time case. Let $\delta^0 = x$ and δ^1 be the minimal vector function such that its derivative does not depend on the input u. For $i \geq 1$ define

$$\delta^{i+1} = \delta^i \oplus \mathbf{m}(\delta^i). \tag{16}$$

The sequence δ^i, $i \geq 1$, converges. Denote the limit by δ and let k be such that $\delta^k \neq \delta$, $\delta^{k+1} = \delta$. Since the computing sequence δ^i requires finding the derivatives of vector functions δ^i, then from now on we assume that they exist.

Definition 8. *The relative degree of a vector function $\alpha = [\alpha_1, \ldots, \alpha_k]^{\mathrm{T}}$ is defined as the minimal number r such that $\alpha_j(x^{(r)})$ depends on system inputs for some $j \in \{1, \ldots, k\}$.*

To compute the sequence δ^i, the following Lemma may be useful.

Lemma 3. *The relative degree of δ^i is $i+1$ for $i \geq 0$.*

Proof. We prove the conjecture by induction on i. For $i = 1$, the conjecture is true by the definition. Assume that the conjecture is true for $i \leq p$. Next we show that it is also true for $i = p+1$.

Let

$$\frac{\partial \mathbf{m}(\delta^p)}{\partial x} f_c = f_*(\delta^p, u).$$

By (16) and the definition of $\oplus$, there exists a function ξ such that $\delta^{p+1} = \xi(\delta^p)$, and thus the relative degree of δ^{p+1} cannot be smaller than the relative degree of δ^p, i.e. $p+1$. On the other hand, (16) and the definition of $\oplus$ yield $\delta^{p+1} = \gamma(\mathbf{m}(\delta^p))$ for some function γ. Compute the derivative of δ^{p+1}:

$$\dot{\delta}^{p+1} = \frac{\partial \gamma}{\partial \mathbf{m}(\delta^p)} \frac{\partial \mathbf{m}(\delta^p)}{\partial x} f_c = \frac{\partial \gamma}{\partial \mathbf{m}(\delta^p)} f_*(\delta^p, u).$$

The relative degrees of $\frac{\partial \gamma}{\partial \mathbf{m}(\delta^p)}$ and δ^p are $p+1$. Since, as said above, the relative degree of δ^{p+1} is $p+1$ or higher, then $\dot{\delta}^{p+1}$ does not depend on u and the relative degree of $\dot{\delta}^{p+1}$ is $p+1$, which means that the relative degree of δ^{p+1} is $p+2$. □

To continue, we make the following assumption.

Assumption 1. The derivatives of vector functions δ^i, $i = 1, \ldots, k+1$, exist.

Using Lemma 3, one can prove for the continuous-time case the relation similar to (6) in the discrete-time case.

Corollary 2. *Under assumption 1, the vector functions* δ^i *satisfy the relations*

$$\dot{\delta}^{i+1} = \delta^i \oplus \dot{\delta}^i \tag{17}$$

for $i \geq 0$.

For continuous-time systems, the conditions for the existence of a state transformation and a static state feedback, which linearize the system, are similar as in the discrete-time case.

Theorem 2. *Under Assumption 1, system (15) can be transformed into the Brunovsky form by a state transformation* $z = \varphi(x)$ *and static state feedback* $u = G(x, v)$ *if and only if* $\delta = \mathbf{1}$ *and*

$$\sum_{r=1}^{k+1} (|\delta^{r-1}| - |\delta^r \times \dot{\delta}^r|) = m, \tag{18}$$

where k is defined as the minimal number such that $\delta^{k+1} = \delta$.

Proof. The proof of Theorem 2 is similar to that of Theorem 1; the only difference is that instead of shifts of vector functions, one has to take their derivatives. □

Example 4. Consider the DC-to-DC power converter model, given in [22]:

$$\begin{aligned} \dot{x}_1 &= -\frac{1}{L_1} u_1 x_2 + \frac{E}{L_1} \\ \dot{x}_2 &= \frac{1}{C_2} u_1 x_1 - \frac{1}{C_2} x_3 \\ \dot{x}_3 &= \frac{1}{L_3} x_2 - \frac{1}{L_3} u_2 x_4 \\ \dot{x}_4 &= \frac{1}{C_4} u_2 x_3 - \frac{1}{RC_4} x_4. \end{aligned} \tag{19}$$

One can find that $\delta^1 = [L_1 x_1^2 + c_2 x_2^2, L_3 x_3^2 + C_4 x_4^2]^{\mathrm{T}}$ and $\delta^2 = \mathbf{1}$. Then $\dot{\delta}^1 = [2(Ex_1 - x_2 x_3), 2(x_2 x_3 - 1/Rx_4^2)]^{\mathrm{T}}$ and

$$\begin{aligned} \sum_{r=1}^{2} (|\delta^{r-1}| - |\delta^r \times \dot{\delta}^r|) &= (|\delta^0| - |\delta^1 \times \dot{\delta}^1|) + (|\delta^1| - |\delta^2 \times \dot{\delta}^2|) \\ &= (4-4) + (2-0) = 2 = m. \end{aligned}$$

Thus, system (19) can be linearized by state transformation and a static feedback, as was also stated in [22].

Example 5. Consider the system (see [6, p. 313])

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \tfrac{1}{m}(u + F), \end{aligned} \tag{20}$$

where

$$F = \begin{cases} -F_1 \operatorname{sign} x_2, & x_2 \neq 0; \\ -u, & x_2 = 0, \operatorname{abs}(u) \leq F_0; \\ -F_0 \operatorname{sign} u, & x_2 = 0, \operatorname{abs}(u) > F_0. \end{cases}$$

The compensator

$$u = \begin{cases} mv + F_1 \operatorname{sign} x_2, & x_2 \neq 0; \\ [-F_0, F_0], & x_2 = 0, v = 0; \\ mv + F_0, & x_2 = 0, v > 0; \\ mv - F_0, & x_2 = 0, v < 0 \end{cases} \tag{21}$$

transforms system (20) into the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= v. \end{aligned}$$

Note that u in the static state feedback (21) is not uniquely defined when $v = 0$ and $x_2 = 0$. However, since any choice of u from $[-F_0, F_0]$ yields the same result, one can take, for example, $u = 0$ in (21) when $v = 0$ and $x_2 = 0$. Then one has a regular feedback.

Example 6. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1 u \operatorname{sign}(x_3) \\ \dot{x}_2 &= \operatorname{abs}(x_3) \\ \dot{x}_3 &= x_1 u. \end{aligned} \tag{22}$$

The vector function δ^1 is defined as a vector function whose derivative does not depend on the input u. To compute δ^1, we try to eliminate u on the right-hand side of equations (22) by combining the equations. Since there are 3 equations and 1 input, then the dimension of δ^1 is 2 or less. Obviously, $\dot{x}_2$ does not depend on u, thus $\delta^1 \leq x_2$. By using $\frac{d}{dt}\operatorname{abs}(x_3) = \operatorname{sign}(x_3) \cdot \dot{x}_3$, $x_3 \neq 0$, one can also check that $\mathrm{d}/\mathrm{d}t(x_1 + \operatorname{abs}(x_3)) = x_2$. Thus, we have

$$\delta^1 = [x_2, x_1 + \operatorname{abs}(x_3)]^{\mathrm{T}}.$$

Now, δ^2 is, by definition, a vector function of δ^1 whose derivative depends only on δ^1. Observe that the derivative of $x_1 + \operatorname{abs}(x_3)$ depends on $x_2 \geq \delta^1$ and therefore $\delta^2 = x_1 + \operatorname{abs}(x_3)$ and since the relative degree of δ^2 is 3, then $\delta^3 = \mathbf{1}$.

Condition (18) of Theorem 2 is satisfied, since

$$\sum_{r=1}^{3} (|\delta^{r-1}| - |\delta^r \times \dot{\delta}^r|) = (3-3) + (2-2) + (1-0) = 1 = m.$$

The vector of linearizing outputs ϕ is computed similarly as in the discrete-time case:

$$\phi = \varphi_1 \times \varphi_2 \times \varphi_3 = \mathbf{1} \times \mathbf{1} \times [x_1 + \operatorname{abs}(x_3)] = x_1 + \operatorname{abs}(x_3),$$

where φ_r, $r = 1, 2, 3$, satisfy $\delta^{r-1} = \delta^r \times \dot{\delta}^r \times \varphi_r$.

Following the standard procedure for computing the state transformation and the static state feedback, one gets the change of coordinates

$$\begin{aligned} z_1 &= x_1 + \operatorname{abs}(x_3) \\ z_2 &= x_2 \\ z_3 &= \operatorname{abs}(x_3) \end{aligned} \tag{23}$$

and a static feedback

$$u = \frac{v}{\operatorname{sign}(x_3) \cdot x_1}.$$

Note that (23) is a change of coordinates if and only if one specifies whether $x_3 \in (-\infty, 0)$ or $x_3 \in [0, \infty)$.

Example 7. Consider the following system (see [5]):

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \frac{1}{J}(M_0 \sin(x_1) + \frac{1}{2}\rho x_3 \operatorname{abs}(x_3)(m_1 \sin(2(x_1 - x_4)) + \frac{x_2}{x_3} m_2 V^{3/4})), \\
\dot{x}_3 &= \frac{1}{m_x}(-\frac{1}{2}\rho x_3 \operatorname{abs}(x_3)(r_{x0} + r_{x1}\cos(x_1 - x_4) + r_{x2}\sin(2(x_1 - x_4))) \\
&\quad + P\sin(x_4) + u_1 \cos(x_1 - x_4) - u_2 \sin(x_1 - x_4)), \\
\dot{x}_4 &= \frac{1}{m_y x_3}(\frac{1}{2}\rho x_3 \operatorname{abs}(x_3)(r_y \sin(2(x_1 - x_4)) + \frac{x_2}{x_3} CV) + P\cos(x_4) \\
&\quad + u_2 \cos(x_1 - x_4) - u_1 \sin(x_1 - x_4)), \\
\dot{x}_5 &= x_3 \sin(x_4),
\end{aligned} \tag{24}$$

defined on $\mathbb{R}^6 \backslash \{x_3 = 0\}$. These equations constitute a simplified model of the underwater vehicle moving on a vertical plane and developed under the assumption that the control moment is insignificant. Here x_1 and x_2 are the trim angle of the vehicle and its velocity, respectively, x_3 is the linear speed of the vehicle, x_4 is the bank angle, and x_5 is the depth of plunge; J, M_0, ρ, m_1, m_2, m_x, m_y, r_{x0}, r_{x1}, r_{x2}, r_y, V, P, and C are some constant coefficients characterizing the construction of the vehicle.

By Lemma 3 one can find that $\delta^1 = [x_1, x_2, x_5]^{\mathrm{T}}$, $\delta^2 = x_1$, $\delta^3 = \mathbf{1}$. Then

$$\begin{aligned}
\dot{\delta}^1 &= \begin{pmatrix} x_2 \\ \frac{1}{J}(M_0 \sin(x_1) + \frac{1}{2}\rho x_3 \operatorname{abs}(x_3)(m_1 \sin(2(x_1 - x_4)) + \frac{x_2}{x_3} m_2 V^{3/4})) \\ x_3 \sin(x_4), \end{pmatrix}, \\
\dot{\delta}^2 &= x_2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{r=1}^{3}(|\delta^{r-1}| - |\delta^r \times \dot{\delta}^r|) &= (|\delta^0| - |\delta^1 \times \dot{\delta}^1|) + (|\delta^1| - |\delta^2 \times \dot{\delta}^2|) \\
&\quad + (|\delta^2| - |\delta^3 \times \dot{\delta}^3|) \\
&= (5 - 5) + (3 - 2) + (1 - 0) = 2 = m.
\end{aligned}$$

Thus, by Theorem 2, system (24) can be linearized by the static state feedback and a change of coordinates. Set $z_{1,1} := x_1$, $z_{2,1} := x_5$, $z_{1,2} := \dot{x}_1 = x_2$, $z_{1,3} := \dot{x}_2 = \frac{1}{J}(M_0 \sin(x_1) + \frac{1}{2}\rho x_3 \operatorname{abs}(x_3)(m_1 \sin(2(x_1 - x_4)) + \frac{x_2}{x_3} m_2 V^{3/4}))$ and $z_{2,2} := \dot{x}_5 = x_3 \sin(x_4)$. These relations define the change of coordinates and by solving the equations

$$\begin{aligned}
v_1 &= \dot{z}_{1,3} =: K_1(x, u) \\
v_2 &= \dot{z}_{2,2} =: K_2(x, u)
\end{aligned}$$

in u_1 and u_2, one gets the static state feedback. The expressions for K_1 and K_2, as well as the feedback itself, are rather complex and are thus omitted.

5. CONCLUSIONS

The static state feedback linearization problem was solved for possibly non-smooth discrete- and continuous-time nonlinear systems. The algebraic method called ‘functions’ algebra’ allows also handling the non-smooth case. The solvability conditions were given in terms of the finite sequence of vector functions δ^i. Although the algebraic method used in this paper allows considering non-smooth functions, the computations with such functions are difficult and thus in many cases one has to find solutions intuitively. The solution obtained is generic, i.e. valid in open and dense subspaces of the state space. In the future one may try to extend the results so that they will be applicable also in singular points, like it is done, for example, in [3,23] for a problem of i/o linearization and output tracking through singularities.

The method used in this paper works better in the discrete-time case than in the continuous-time case, since shifting a function (even a non-smooth one) forward or backward is relatively easy, while in the continuous-time case one can not always find a derivative of a function.

A software has been developed to make the computations in functions' algebra, (see [12,14] and http://webmathematica.cc.ioc.ee/webmathematica/NLControl/funcalg). However, at the moment most of these functions work only for the smooth case. The bottleneck in implementing the non-smooth case is the absence of a method to check whether $\alpha \leq \beta$ for non-smooth α.

The future goal is to generalize Definition 7. Namely, one can say that $(\alpha, \beta) \in \Delta$ if the dynamics of β depends on x only through α. In this way, one does not have to assume that β is smooth.

ACKNOWLEDGEMENTS

The work of Ü. Kotta and A. Kaldmäe was supported by the Estonian Research Council, personal research funding grant PUT481. The work of A. Shumsky and A. Zhirabok was supported by the Russian Scientific Foundation (project 16-19-00046) and the Ministry of Education and Science of the Russian Federation (state task No. 1141). The publication costs of this article were covered by the Estonian Academy of Sciences.

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Teatud mittesiledate süsteemide tagasisidega lineariseerimine

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On kasutatud võreteoorial põhinevat meetodit, et uurida võimalusi mitme sisendi ja mitme väljundiga mittelineaarsete juhtimissüsteemide tagasisidega lineariseerimiseks. Tarvilikud ja piisavad tingimused on leitud olekuteisenduse ning staatilise olekutagasiside olemasoluks, mis lineariseerivad antud süsteemi kirjeldavad olekuvõrrandid. Kasutatava metoodika eeliseks teiste enam levinud meetodite ees on, et see võimaldab lineariseerida ka mittesiledaid süsteeme. Antud meetodi põhiline objekt on vektorfunktsioonide hulk, mis jagatakse ekvivalentsiklassideks, mis omakorda moodustavad algebralise struktuuri, mida nimetatakse võreks. Artiklis on kajastatud nii diskreetse kui ka pideva ajaga juhtumeid ja lahendus lineariseeruvuse probleemile on antud teatud vektorfunktsioonide jada kaudu. Saadud teoreetilised tulemused on illustreeritud mitme näitega.