Proceedings of the Estonian Academy of Sciences,
2016, **65**, 3, 220–236
doi: 10.3176/proc.2016.3.02
Available online at www.eap.ee/proceedings

CONTROL
SYSTEMS

On the transformation of a nonlinear discrete-time input–output system into the strong row-reduced form

Zbigniew Bartosiewicz[a], Juri Belikov[b*], Ülle Kotta[b], Maris Tõnso[b], and Małgorzata Wyrwas[a]

[a] Faculty of Computer Science, Białystok University of Technology, Wiejska 45a, 15-351 Białystok, Poland; {z.bartosiewicz, m.wyrwas}@pb.edu.pl

[b] Institute of Cybernetics at Tallinn University of Technology, Akadeemia tee 21, 12618 Tallinn, Estonia; {jbelikov, kotta, maris}@cc.ioc.ee

Received 6 July 2015, accepted 16 November 2015, available online 20 June 2016

Abstract. The paper addresses the problem of the transformation of nonlinear discrete-time systems, described by implicit higher-order difference equations, into the strong row-reduced form. The motivating example illustrates the phenomenon that sometimes equations in the row-reduced form may contain higher-order shifts of output variables than the corresponding row degrees. This means that, in general, linear transformations of equations are not enough for transforming equations into the strong row-reduced form. Therefore, in this paper we study the possibility of using local nonlinear transformations to reduce the order of a system. A constructive (up to the solution of a system of partial differential equations) step-by-step algorithm is provided. It is followed by several illustrative examples.

Key words: discrete-time systems, input–output models, input–output equivalence transformations, strong row-reduced form.

1. INTRODUCTION

The transformation of a set of higher-order nonlinear input–output (i/o) equations into a row-reduced form is an important problem in control theory for several reasons. First, the row-reduced form may be seen as an intermediate step towards a so-called doubly-reduced (i.e., both row- and column-reduced) or Popov form (see [1,2,6,7]). Thus, an algorithm for transforming arbitrary system equations into the row-reduced form is necessary to obtain the double-reduced or Popov form. Second, this form can serve as a good starting point for the application of realization procedures (see, for example, [4] and the references therein). The realization problem is a fundamental research topic in nonlinear control theory, which studies the possibility of transforming a set of higher-order i/o difference equations into a classical state-space form. Moreover, the sum of row degrees of the system in the row-reduced form defines the order of the realization, i.e., the number of state variables. In addition, the form under study also shows explicitly when not all of the inputs are free (independent) variables, or, when the system is not right invertible since certain functions of outputs are not affected by controls.

The problem has been studied by various authors, both in continuous-time [10] and discrete-time cases [3,5]. The results of this paper can be understood as an extension of those presented in [3,5,9].

* Corresponding author, jbelikov@cc.ioc.ee

In [3] a specific version of a leading coefficient matrix was assumed with $m_1 = \cdots = m_p = 0$ (see Eq. (4) below). A particular solution, based on *linear* i/o equivalence transformations, was proposed in [5]. In such a case the original and transformed equations are related to each other through the transformation over the field of meromorphic functions in system variables by applying to the original set of equations an operator defined in terms of a unimodular polynomial matrix whose indeterminate may be interpreted as a forward-shift operator. The transformation in [5] was found from the variational (i.e., globally linearized) system description, presented in terms of the polynomial matrices and differentials of system variables. This (polynomial) description was used to calculate a unimodular matrix by means of elementary matrix operations. Finally, the unimodular transformation matrix was applied to the original system of equations to find its row-reduced form. This approach works well in many situations. However, there exist numerous examples when this method leads to equations of row-reduced form, which contain higher-order shifts of output variables than their row degrees. The row-reducedness property as defined in [5] is actually a property of linearized equations (differential one-forms describing the linearized equations) and not the property of equations themselves. In particular, the set of i/o equations is called row-reduced if and only if their linearization is row-reduced. However, in general, the row-reducedness property cannot be easily translated back to the original system equations. This fact motivated us to introduce a new stronger definition of a row-reduced system based on row orders (see Definitions 4 and 6). In [9] local nonlinear transformations were used to transform a continuous-time system to the row-reduced form. Note that the continuous-time case is different from its discrete counterpart, since the verification whether a system is in the row-reduced form or not is done in a slightly different manner (see Example 1 below). In addition, the results for the continuous-time case from [9] rely on a special case of the rank theorem and are valid locally under certain constant rank assumptions. Finally, it should be mentioned that in this paper we work with system equations and not with linearized description as in [5].

Based on the reasons mentioned above, the main goals of this paper are: to present a new definition of the strong row-reducedness property of a system and to specify a larger class of local *nonlinear* i/o equivalence transformations. For the first purpose we combine two ideas from [3] and [5]. Moreover, we adopt some ideas from [9] to achieve the second goal. Of course, even though the existence of such nonlinear transformations can be proven, their computation may be a difficult task.

The paper is organized as follows. Section 2 recalls the main notions and definitions regarding the i/o equivalence and row-reduced form. It is followed by a motivating example that illustrates the difficulties one may face when applying the results of [5]. Section 3 is devoted to local nonlinear i/o equivalence transformations and presents also the algorithm allowing one to transform a set of i/o equations into the strong row-reduced form. A number of illustrative examples are given in Section 4. Concluding remarks are drawn in the final section.

2. INPUT–OUTPUT EQUIVALENCE AND ROW-REDUCED FORM

Consider a nonlinear discrete-time multi-input multi-output (MIMO) control system, described by the set of implicit higher-order i/o difference equations

$$\phi_i\left(y(t), y(t+1), \ldots, y(t+n), u(t), u(t+1), \ldots, u(t+n)\right) = 0, \quad i = 1, \ldots, p, \tag{1}$$

where $t \in \mathbb{Z}$, $u(t) \in \mathbb{R}^m$ is a vector of input variables, $y(t) \in \mathbb{R}^p$ is a vector of output variables, and ϕ_i is a meromorphic function. Sometimes, to simplify the exposition, the abridged notations are used. In particular, if a time-dependent variable is denoted as $\xi(t)$, then $\xi^{[k]}(t)$ stands for the kth-step forward time shift $\xi(t+k)$ and $\xi^{[-l]}(t)$ for the lth-step backward time shift $\xi(t-l)$ with $k, l \in \mathbb{Z}^+$. Furthermore, we may leave the time argument t to make the notation even more compact, i.e., $\xi := \xi(t)$.

Recall briefly the algebraic formalism from [5] that is used in this paper. Let $\mathscr{A}$ be the ring of analytic functions in a finite number of variables from the sets $\mathscr{Y} = \{y_i^{[k]}, k \in \mathbb{Z}, i = 1, \ldots, p\}$

and $\mathscr{U} = \{u_j^{[l]}, l \in \mathbb{Z},\ j = 1,\dots,m\}$, where $y_i^{[0]} := y_i$ and $u_j^{[0]} := u_j$. For the function F, depending on variables from $\mathscr{Y}$ and $\mathscr{U}$, the forward-shift operator $\sigma : \mathscr{A} \to \mathscr{A}$ is defined as follows:

$$\sigma(F)\left(\dots,y^{[-1]},y,y^{[1]},\dots,u^{[-1]},u,u^{[1]},\dots\right) := F\left(\dots,y,y^{[1]},y^{[2]},\dots,u,u^{[1]},u^{[2]},\dots\right),$$

and the backward-shift operator $\sigma^{-1} : \mathscr{A} \to \mathscr{A}$ is given by

$$\sigma^{-1}(F)\left(\dots,y^{[-1]},y,y^{[1]},\dots,u^{[-1]},u,u^{[1]},\dots\right) := F\left(\dots,y^{[-2]},y^{[-1]},y,\dots,u^{[-2]},u^{[-1]},u,\dots\right).$$

Then, $\sigma(y_i^{[k]}) = y_i^{[k+1]}$, $\sigma(u_j^{[l]}) = u_j^{[l+1]}$, and $\sigma^{-1}(y_i^{[k]}) = y_i^{[k-1]}$, $\sigma^{-1}(u_j^{[l]}) = u_j^{[l-1]}$ for $i = 1,\dots,p$, $j = 1,\dots,m$, and $k,l \in \mathbb{Z}$. Note that $\mathscr{A}$ is a difference ring with the shift operator, being an automorphism.

Let $\mathscr{S}$ be a multiplicative subset of the ring $\mathscr{A}$, meaning that $1 \in \mathscr{S}, 0 \notin \mathscr{S}$ and if $\alpha \in \mathscr{S}$ and $\beta \in \mathscr{S}$, then $\alpha\beta \in \mathscr{S}$. Assume that $\mathscr{S}$ is invariant with respect to both σ and σ^{-1}. Then, $\mathscr{A}_{\mathscr{S}} := \mathscr{S}^{-1}\mathscr{A} = \{\alpha/\beta \mid \alpha \in \mathscr{A} \text{ and } \beta \in \mathscr{S}\}$ defines the localization of the ring $\mathscr{A}$ with respect to $\mathscr{S}$. Observe that $\mathscr{A}_{\mathscr{S}}$ is an inversive difference ring with the shift operator σ given by $\sigma(\alpha/\beta) := \sigma(\alpha)/\sigma(\beta)$ and $\mathscr{S}$ may be interpreted as a subset of $\mathscr{A}_{\mathscr{S}}$ due to the natural injection $\alpha \mapsto \alpha/1$.

Let $\Phi = \{\phi_1,\dots,\phi_p\}$ be a finite subset of $\mathscr{A}_{\mathscr{S}}$. Note that Φ may be interpreted as a system of implicit i/o equations. Let $\mathscr{I}_{\mathscr{S}} := \langle\Phi\rangle_{\mathscr{S}}$ be the smallest ideal of $\mathscr{A}_{\mathscr{S}}$ that contains all forward and backward shifts of ϕ_i, i.e., $\mathscr{I}_{\mathscr{S}}$ is generated by $\{\sigma^k(\phi_i) \mid i = 1,\dots,p, k \in \mathbb{Z}\}$. Note that $\mathscr{I}_{\mathscr{S}}$ is a difference ideal, since it is closed with respect to all shifts of ϕ_i. Observe that Φ may be considered as a subset of $\widetilde{\mathscr{S}}^{-1}\mathscr{A}$ for some other multiplicative set $\widetilde{\mathscr{S}}$. For that reason we put $\mathscr{S}$ in the notation of the ideal $\mathscr{I}_{\mathscr{S}}$.

Assumption 1. $\mathscr{I}_{\mathscr{S}}$ is prime, i.e., if $\alpha,\beta \in \mathscr{A}_{\mathscr{S}}$ and $\alpha\beta \in \mathscr{I}_{\mathscr{S}}$, then $\alpha \in \mathscr{I}_{\mathscr{S}}$ or $\beta \in \mathscr{I}_{\mathscr{S}}$.

Assumption 2. $\mathscr{I}_{\mathscr{S}}$ is proper, i.e., different from the entire ring.

Properness of the ideal $\mathscr{I}_{\mathscr{S}}$ is equivalent to the condition $\mathscr{S} \cap \mathscr{I}_{\mathscr{S}} = \emptyset$. In particular, numerators of ϕ_i do not belong to $\mathscr{S}$.

Observe that $\mathscr{S}$ is constructed for system (1). However, when applying equivalence transformations with Eqs (1), $\mathscr{S}$ may have to be extended to $\widetilde{\mathscr{S}}$ by including possible expressions that do not equal zero, restricting in this way the domain of definition. When we start, some functions ϕ_i in (1) may have denominators that, together with their forward/backward shifts and powers, should be included in the set $\mathscr{S}$. If the functions are analytic, one may set $\mathscr{S} := \{1\}$, meaning that $\mathscr{S}^{-1}\mathscr{A} = \mathscr{A}$. Of course, additional denominators that show up in the row-reduction should also be included in $\mathscr{S}$ together with their shifts and powers. That is, we extend our initial $\mathscr{S}$ by adding an infinite number of elements. The infinite $\mathscr{S}$ can be briefly described by its generator $\mathscr{S}_0$. The set $\mathscr{S}_0$ generates $\mathscr{S}$ if each element of $\mathscr{S}$ can be obtained from a finite number of elements of $\mathscr{S}_0$ by applying a finite number of multiplications and backward/forward shifts to these elements.

Let $\mathscr{A}_{\mathscr{S}}/\mathscr{I}_{\mathscr{S}}$ be the quotient ring. It consists of cosets $\bar{\varphi} = \varphi + \mathscr{I}_{\mathscr{S}}$ for $\varphi \in \mathscr{A}_{\mathscr{S}}$. We define addition and multiplication in this new ring by $\bar{\varphi} + \bar{\psi} := \overline{\varphi + \psi}$ and $\bar{\varphi} \cdot \bar{\psi} := \overline{\varphi \cdot \psi}$. These definitions do not depend on the choice of a representative in a coset. Since $\mathscr{I}_{\mathscr{S}}$ is a prime ideal, $\mathscr{A}_{\mathscr{S}}/\mathscr{I}_{\mathscr{S}}$ is an integral ring. Now we can redefine σ on $\mathscr{A}_{\mathscr{S}}/\mathscr{I}_{\mathscr{S}}$ (denoted by σ_Φ to indicate the dependence on Φ) as $\sigma_\Phi(\bar{\varphi}) = \overline{\sigma(\varphi)}$. The operator σ_Φ is well defined and bijective, so σ_Φ^{-1} is well defined on $\mathscr{A}_{\mathscr{S}}/\mathscr{I}_{\mathscr{S}}$. Let $\mathscr{Q}_{\mathscr{S}}^{\Phi}$ denote the field of fractions of the ring $\mathscr{A}_{\mathscr{S}}/\mathscr{I}_{\mathscr{S}}$. Since σ_Φ can be naturally extended to the field of fractions, $\mathscr{Q}_{\mathscr{S}}^{\Phi}$ is an inversive difference field.

Definition 1. *The sequence of pairs* $\{(u(t),y(t)), t \geq 0\}$ *is called a solution of* (1) *if, for any* $t \geq 0$, $u(t)$ *and* $y(t)$ *satisfy the equations*

$$\phi_i\left(y(t),\dots,y(t+n),u(t),\dots,u(t+n)\right) = 0, \quad i = 1,\dots,p.$$

Definition 2. *Two systems of the form* (1) *are called i/o equivalent if their solutions coincide.*

Definition 3. *An i/o equivalence transformation for system* (1) *is an invertible transformation of the system equations to another set of equations of the form* (1), *being i/o equivalent with the original system equations.*

Definition 4. *The row orders*[1] *of* $\Phi = \begin{bmatrix}\phi_1 & \cdots & \phi_p\end{bmatrix}^{\mathrm{T}}$, *in the output* y, *denoted by* μ_i, *are the largest integers such that in* (1)

$$\frac{\partial \phi_i}{\partial y_j^{[\mu_i]}} \not\equiv 0 \tag{2}$$

for $i = 1,\dots,p$ *and some* $j \in \{1,\dots,p\}$.

In other words, μ_i is the highest forward-shift of the output component[2], appearing nontrivially in ϕ_i. Next, we set $\mu := (\mu_1,\dots,\mu_p)$ and define the $p\times p$-dimensional matrix $\mathscr{M}_\mu$ as the matrix with the (i,j)th element given by $\partial\phi_i/\partial y_j^{[\mu_i]}$, i.e.,

$$\mathscr{M}_\mu := \left[\frac{\partial \phi_i}{\partial y_j^{[\mu_i]}}\right]_{ij} \tag{3}$$

for $i,j = 1,\dots,p$.

In the continuous-time case the matrix $\mathscr{M}_\mu$ is enough to verify whether the original system is in the row-reduced form or not (see [10]). However, in the case of discrete-time systems one has to multiply $\mathscr{M}_\mu$ by certain diagonal matrix (as defined below) from the left (see the explanation in Example 1). Let $N_\mu = \max \mu_i$ and $m = (m_1,\dots,m_p)$ with $m_i = N_\mu - \mu_i$ for $i = 1,\dots,p$. Define $\sigma^0 := \mathrm{id}_{\mathscr{A_S}}$ and $\sigma^{m_i}\varphi := \sigma^{m_i}(\varphi) = (\sigma\circ\cdots\circ\sigma)(\varphi)$ for $\varphi \in \mathscr{A_S}$ as m_i-fold composition of operator σ.

Definition 5. *The matrix*

$$L_\mu = \mathrm{diag}\{\sigma^{m_1},\dots,\sigma^{m_p}\}\mathscr{M}_\mu \tag{4}$$

is called the leading coefficient matrix of system (1).

Definition 6. *The set of i/o difference equations* (1) *is called strongly row-reduced if the leading coefficient matrix* L_μ *has full rank over* $\mathscr{A_S}$. *If* L_μ *contains zero rows, then Eqs* (1) *are in the strong row-reduced form if the submatrix of* L_μ *consisting of non-zero rows is strongly row-reduced.*

Example 1. Consider the set of i/o equations

$$\begin{aligned}\phi_1 &:= u_1 y_1^{[1]} + y_2^{[1]} + u_2 = 0,\\ \phi_2 &:= u_1^{[1]} y_1^{[2]} + y_2^{[2]} + u_2^{[1]} = 0,\end{aligned} \tag{5}$$

in which the second equation is a forward shifted version of the first. Obviously, we do not want to call this set of equations to be in a row-reduced form. Find, according to Definition 4, the row orders of system (5) as $\mu = (1,2)$. Then, by (3)

$$\mathscr{M}_\mu = \begin{bmatrix} u_1 & 1\\ u_1^{[1]} & 1\end{bmatrix}$$

and $\mathrm{rank}_{\mathscr{A_S}}\mathscr{M}_\mu = 2$ indicating that Eqs (5) are independent. However, for (5), $N_\mu = \max\{\mu_1,\mu_2\} = 2$ and $m = (1,0)$. Compute, according to (4),

$$L_\mu = \mathrm{diag}\{\sigma,1\}\mathscr{M}_\mu = \begin{bmatrix} u_1^{[1]} & 1\\ u_1^{[1]} & 1\end{bmatrix}.$$

Observe that $\mathrm{rank}_{\mathscr{A_S}} L_\mu = 1$ as expected.

---

[1] The notion of row degrees is kept for the indices ρ_i, defined in [5], as the largest integers such that $\partial\phi_i/\partial y_j^{[\rho_i]} \notin \mathscr{I_S}$. Observe that the indices μ_i are greater than or equal to ρ_i.

[2] If ϕ_i does not depend on y or $\partial\phi_i/\partial y_j^{[\mu_i]}$ does not exist, we set $\mu_i = -1$. Recall that $\mu_i = 0$ corresponds to the case when ϕ_i depends on y_j only and not on its shifts.

2.1. Motivating example

Let us study the motivating example that illustrates difficulties one may face when applying the approach proposed in [5]. Recall that the elements of the field $\mathscr{Q}_{\mathscr{S}}^{\Phi}$ are not fractions of functions but abstract fractions (equivalence classes of functions) since the construction of $\mathscr{Q}_{\mathscr{S}}^{\Phi}$ is based on the quotient ring $\mathscr{A}_{\mathscr{S}}/\mathscr{I}_{\mathscr{S}}$. In the following example we use the (simplest) representatives of these equivalence classes.

Example 2. Consider the set of i/o equations

$$\begin{aligned} \phi_1 &:= y_1^{[3]} - y_1^{[2]} - u_1 = 0, \\ \phi_2 &:= \sin\left(y_1^{[2]} - u_2 + y_2^{[1]}\right) = 0 \end{aligned} \tag{6}$$

and perform the calculations according to the approach from [5]. Since there are no denominators in (6), we set $\mathscr{S} := \{1\}$. Next, one can find the row degrees for system (6) as $\rho = \{\rho_1, \rho_2\} = \{3, 2\}$. Then we have to reorder equations $\Phi := \begin{bmatrix}\phi_1 & \phi_2\end{bmatrix}^{\mathrm{T}}$ with respect to the row degrees starting from the lowest that can be done by means of multiplication by the permutation matrix

$$\tilde{\Phi} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Phi = \begin{bmatrix} \phi_2 \\ \phi_1 \end{bmatrix}.$$

Obviously, we have to multiply ρ by the same permutation matrix. Next, define $N := \max \rho$ and $M := (m_1, m_2) = (1, 0)$. By Definition 4 from [5] the leading coefficient matrix is

$$L_\rho = \begin{bmatrix} \cos\beta^{[1]} & 0 \\ 1 & 0 \end{bmatrix},$$

where $\beta := y_1^{[2]} - u_2 + y_2^{[1]}$. Obviously, L_ρ is not of full rank over $\mathscr{Q}_{\mathscr{S}}^{\Phi}$, meaning that Eqs (6) are not in the row-reduced form[3] (see Definition 5 in [5]). One can easily check that rows L_ρ^1 and L_ρ^2 of the matrix L_ρ are linearly dependent, resulting in $\alpha_1 L_\rho^1 + L_\rho^2 = 0$. It thus follows that $\alpha_1 \cos\beta^{[1]} + 1 = 0$ and so $\alpha_1 = -1/\cos\beta^{[1]}$. Set $\gamma := N_\rho - \rho_2 = 0, \nu_1 := \rho_2 - \rho_1 = 1$ and construct the matrix

$$E_0(z) = \begin{bmatrix} 1 & 0 \\ \sigma^{-\gamma}(\alpha_1) z^{\nu_1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{\cos\beta^{[1]}} z & 1 \end{bmatrix}.$$

Note that we have to extend the set $\mathscr{S}$ as $\widetilde{\mathscr{S}} := \{1, \sigma^k(\cos\beta^{[1]}) \mid k \in \mathbb{Z}\}$ and $\sigma^k(\cos\beta^{[1]}) \notin \mathscr{I}_{\mathscr{S}}$. Now the transformation matrix $U(z)$ can be found as

$$U(z) = E_0(z) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -\frac{1}{\cos\beta^{[1]}} z \end{bmatrix}.$$

Finally, compute the row-reduced form of the system by applying the transformation operator $U(z)$, denoted by the symbol $\Rsh$, to functions in the original system description as follows:

$$U(z) \Rsh \begin{bmatrix} \widetilde{\phi}_1 \\ \widetilde{\phi}_2 \end{bmatrix} = \begin{bmatrix} \phi_2 \\ \phi_1 - \frac{1}{\cos\beta^{[1]}} \sigma(\phi_2) \end{bmatrix}, \tag{7}$$

[3] Note that in Definition 6 we used the word *strongly* to avoid confusion with Definition 4 from [5].

where the application of z to a function is defined as $z \,\lrcorner\, \xi = \sigma(\xi)$. It is easy to observe that the second (transformed) function still depends on $y_1^{[3]}$ since

$$\phi_1 - \frac{1}{\cos\beta^{[1]}}\sigma(\phi_2) = y_1^{[3]} - y_1^{[2]} - u_1 - \frac{1}{\cos\left(y_1^{[3]} - u_2^{[1]} + y_2^{[2]}\right)} \cdot \sin\left(y_1^{[3]} - u_2^{[1]} + y_2^{[2]}\right). \tag{8}$$

Observe that on the level of linearized system equations (in terms of the one-forms) the transformation $U(z)$ results in the row-reduced form. In fact, the multiplication of the linearized system description $\begin{bmatrix} \mathrm{d}\widetilde{\phi}_1 & \mathrm{d}\widetilde{\phi}_2 \end{bmatrix}^{\mathrm{T}}$ by the transformation matrix $U(z)$ from the left yields

$$\underbrace{\begin{bmatrix} \cos(\beta)z^2 & \cos(\beta)z \\ -z^2 & -z^2 \end{bmatrix}}_{P(z)} \mathrm{d}y + \underbrace{\begin{bmatrix} 0 & -\cos\beta \\ -1 & z \end{bmatrix}}_{Q(z)} \mathrm{d}u = 0. \tag{9}$$

One may also observe that the leading coefficient matrix of $P(z)$ has the full rank, i.e.,

$$\operatorname{rank}_{\mathscr{Q}^{\Phi}_{\mathscr{S}}} L_\rho = \operatorname{rank}_{\mathscr{Q}^{\Phi}_{\mathscr{S}}} \begin{bmatrix} \cos(\beta^{[1]}) & 0 \\ -1 & -1 \end{bmatrix} = 2.$$

However, according to Definition 6, the transformed system (7) is not in the strong row-reduced form since

$$L_\mu = \begin{bmatrix} \cos(\beta^{[1]}) & 0 \\ -\tan^2(\beta^{[1]}) & 0 \end{bmatrix},$$

and therefore, $\operatorname{rank}_{\mathscr{A}_{\mathscr{S}}} L_\mu = 1$. Note that the difference stems from the difference between the row degrees and the row orders: the row orders μ_i, defined by (4), are either greater than or equal to the row degrees ρ_i. This comes from the fact that in computation of ρ_i we take the values of the elements from $\mathscr{I}_{\mathscr{S}}$ equal to zero. Note that, according to the definition of row degrees, we have to find such ρ_i for which the derivative of a function does not belong to the ideal, i.e., $\partial\phi_i/\partial y_j^{[\rho_i]} \notin \mathscr{I}_{\mathscr{S}}$. For example, the partial derivatives of (8) with respect to $y_1^{[2]}$ and $y_1^{[3]}$ are -1 and $-\tan^2(\beta^{[1]})$, respectively. Since $\tan(\beta^{[1]}) = \sin(\beta^{[1]})/\cos(\beta^{[1]})$ and $\sin(\beta^{[1]}) \in \mathscr{I}_{\mathscr{S}}$ (the shifted version of the second equation in (6)), for Eq. (8) we have $\mu_2 = 3$ and $\rho_2 = 2$.

To conclude, this example points to the fact that sometimes the linear transformations from [5] cannot transform the system equations into the strong row-reduced form.

3. NONLINEAR INPUT–OUTPUT EQUIVALENCE TRANSFORMATIONS

The following lemma has been proved in [9, Lemma 6.2] and corrected in [8] in the smooth (C^∞) case. It will be used to construct nonlinear equivalence transformations.

Lemma 1. *Let $f_1, \ldots, f_p$ be analytic functions on an analytic manifold Ω and depending analytically on the parameter $\xi \in \mathbb{R}^k$ for some k such that the dimension of the codistribution* $\operatorname{span}\{\mathrm{d}f_1, \ldots, \mathrm{d}f_p\}$ *is constant. Suppose there exist analytic functions $\lambda_2, \ldots, \lambda_p$ on Ω depending on ξ such that*

$$\mathrm{d}f_1 = \lambda_2 \mathrm{d}f_2 + \cdots + \lambda_p \mathrm{d}f_p.$$

Then for fixed ξ there exists an analytic function $F_\xi : \mathscr{O} \to \mathbb{R}^p$ with $\mathscr{O} \subset \mathbb{R}^p$ an open neighbourhood of $0 \in \mathbb{R}^p$ such that

$$\mathrm{d}[F_\xi \circ (f_1, \ldots, f_p)] = 0$$

and

$$F_\xi(0, \ldots, 0) = 0, \quad \frac{\partial F_\xi}{\partial x_1}(0, \ldots, 0) \neq 0.$$

The functions F_ξ can be chosen in such a way that they depend analytically on ξ.

Proof. The proof in the analytic case is exactly the same as in the smooth case [9]. □

Procedure of transforming the system equations to the strong row-reduced form:
Consider the case when for system (1) the matrix L_μ does not have full rank over $\mathscr{A}_\mathscr{S}$, meaning that the system is not in the strong row-reduced form. Assume that the rows in L_μ are ordered with respect to row orders starting from the lowest and denote the rows by L^i_μ for $i = 1,\dots,p$. Since $\operatorname{rank}_{\mathscr{A}_\mathscr{S}} L_\mu < p$, the rows of the matrix L_μ are linearly dependent over the ring $\mathscr{A}_\mathscr{S}$. Then there exists an integer i such that the first i rows of L_μ are independent, and the first $i+1$ rows are dependent. Therefore there exist $\lambda_1,\dots,\lambda_{i+1} \in \mathscr{A}_\mathscr{S}$ such that $\lambda_{i+1} \not\equiv 0$ and

$$\lambda_1 L^1_\mu + \lambda_2 L^2_\mu + \cdots + \lambda_i L^i_\mu + \lambda_{i+1} L^{i+1}_\mu = 0. \tag{10}$$

Using Definition 5, rewrite the relation (10) as follows:

$$\lambda_1 \sigma^{m_1}\left(\frac{\partial \phi_1}{\partial y^{[\mu_1]}}\right) + \cdots + \lambda_i \sigma^{m_i}\left(\frac{\partial \phi_i}{\partial y^{[\mu_i]}}\right) + \lambda_{i+1}\sigma^{m_{i+1}}\left(\frac{\partial \phi_{i+1}}{\partial y^{[\mu_{i+1}]}}\right) = 0. \tag{11}$$

Assumption 3. Assume $\lambda_{i+1} \notin \mathscr{I}_\mathscr{S}$.

Let $\gamma := N_\mu - \mu_{i+1}$ and $\lambda^*_\upsilon := \sigma^{-\gamma}(\lambda_\upsilon/\lambda_{i+1}) \in \mathscr{A}_{\widetilde{\mathscr{S}}}$, for $\upsilon = 1,\dots,i$, with $\widetilde{\mathscr{S}}$ being the smallest multiplicative shift invariant set containing $\mathscr{S} \cup \{\lambda_{i+1}\}$. Note that $m_k = N_\mu - \mu_k$, for $k = 1,\dots,p$, and $\sigma^k(\partial\phi/\partial y^{[l]}) = \partial\sigma^k(\phi)/\partial y^{[l+k]}$. Applying the operator $\sigma^{-\gamma}$ to (11), we get

$$\lambda^*_1 \frac{\partial \sigma^{\mu_{i+1}-\mu_1}(\phi_1)}{\partial y^{[\mu_{i+1}]}} + \cdots + \lambda^*_i \frac{\partial \sigma^{\mu_{i+1}-\mu_i}(\phi_i)}{\partial y^{[\mu_{i+1}]}} + \frac{\partial \phi_{i+1}}{\partial y^{[\mu_{i+1}]}} = 0. \tag{12}$$

Define $\nu_1 := \mu_{i+1} - \mu_1, \dots, \nu_i := \mu_{i+1} - \mu_i$. Our goal is to eliminate the highest shift, which is, according to the procedure above, μ_{i+1}. Then, we get that the functions $\sigma^{\nu_1}(\phi_1),\dots,\sigma^{\nu_i}(\phi_i),\phi_{i+1}$ depend only on the variables $y^{[\mu_{i+1}]} = [y_1^{[\mu_{i+1}]},\dots,y_p^{[\mu_{i+1}]}]$ and the parameters ξ consisting of $[y_1^{[\mu_{i+1}-j]},\dots,y_p^{[\mu_{i+1}-j]}]$ for $j = 1,\dots,\mu_{i+1}$ and u_i, $i = 1,\dots,m$ with a finite number of their possible shifts. Apply Lemma 1 to functions ϕ_{i+1}, $\sigma^{\nu_1}(\phi_1),\dots,\sigma^{\nu_i}(\phi_i)$ seen as functions of $y^{[\mu_{i+1}]}$, depending on the parameter ξ, yielding the existence of functions F_ξ depending analytically on ξ such that

$$F_\xi(0,\dots,0) = 0, \tag{13}$$

$$\frac{\partial F_\xi}{\partial x_1}(0,\dots,0) \neq 0, \tag{14}$$

and

$$\frac{\partial}{\partial y_j^{[\mu_{i+1}]}} F_\xi\left(\phi_{i+1},\sigma^{\nu_1}(\phi_1),\dots,\sigma^{\nu_i}(\phi_i)\right) = 0, \quad j = 1,\dots,p$$

hold for every ξ. We can assume that $x_1 \mapsto F(x_1,0,\dots,0,\xi)$ is injective. Now define

$$F\left(\phi_{i+1},\sigma^{\nu_1}(\phi_1),\dots,\sigma^{\nu_i}(\phi_i),\xi\right) := F_\xi\left(\phi_{i+1},\sigma^{\nu_1}(\phi_1),\dots,\sigma^{\nu_i}(\phi_i)\right)$$

and replace the set of equations (1) by a set of equations of the form

$$\phi_j(\cdot) = 0, \quad j = 1,\dots,p, \quad j \neq i+1 \tag{15}$$

and

$$F\left(\phi_{i+1},\sigma^{\nu_1}(\phi_1),\dots,\sigma^{\nu_i}(\phi_i),\xi\right) = 0. \tag{16}$$

We restrict the domains of the functions $\phi_{i+1},\sigma^{\nu_1}(\phi_1),\dots,\sigma^{\nu_i}(\phi_i)$ in such a way that the composition (16) is well defined. This means that we restrict also the common domain of all $\phi_1,\dots,\phi_p$. Since the domain of F_ξ is a neighbourhood of 0 and we solve the equations $\phi_i = 0, i = 1,\dots,p$, such a restriction of the domain of $\phi_1,\dots,\phi_p$ does not change the solutions of the system $\phi_i = 0, i = 1,\dots,p$.

The function F_ξ can be found by solving a certain system of partial differential equations.

Proposition 1. *Let F_ξ be a function of $i+1$ variables and $\zeta := (\phi_{i+1}, \sigma^{\nu_1}(\phi_1), \ldots, \sigma^{\nu_i}(\phi_i))$. If*

$$\begin{cases} \dfrac{\partial F_\xi}{\partial x_1}(\zeta) = 1, \\ \dfrac{\partial F_\xi}{\partial x_2}(\zeta) = \lambda_1^*, \\ \qquad \vdots \\ \dfrac{\partial F_\xi}{\partial x_{i+1}}(\zeta) = \lambda_i^*, \end{cases} \tag{17}$$

then

$$\frac{\partial F_\xi(\zeta)}{\partial y^{[\mu_{i+1}]}} = 0. \tag{18}$$

Proof. Note that

$$\frac{\partial F_\xi(\zeta)}{\partial y^{[\mu_{i+1}]}} = \frac{\partial F_\xi}{\partial x_1}(\zeta)\frac{\partial \phi_{i+1}}{\partial y^{[\mu_{i+1}]}} + \frac{\partial F_\xi}{\partial x_2}(\zeta)\frac{\sigma^{\nu_1}(\phi_1)}{\partial y^{[\mu_{i+1}]}} + \cdots + \frac{\partial F_\xi}{\partial x_{i+1}}(\zeta)\frac{\sigma^{\nu_i}(\phi_i)}{\partial y^{[\mu_{i+1}]}}. \tag{19}$$

Then, using (12) and (17), we get (18). □

Remark 1. Note that if λ_j^*, $j = 1, \ldots, i$ do not depend on $y^{[\mu_{i+1}]}$, the system of partial differential equations (17) has always the linear solution:

$$F_\xi(x_1, \ldots, x_i, x_{i+1}) = x_1 + \lambda_1^* x_2 + \cdots + \lambda_i^* x_{i+1}.$$

Otherwise, one has to rely on nonlinear solutions.

Now define $F(x_1, \ldots, x_i, x_{i+1})(\xi) := F_\xi(x_1, \ldots, x_i, x_{i+1})$. Note that F is defined on some subset $\mathscr{V}$ of $\mathbb{R}^{i+1}$ and to each point $(x_1, \ldots, x_i, x_{i+1})$ it assigns a function depending on ξ. To define the transformation, we substitute $\phi_{i+1}, \sigma^{\nu_1}(\phi_1), \ldots, \sigma^{\nu_i}(\phi_i)$ for $x_1, \ldots, x_i, x_{i+1}$. Then the equivalence transformation of system (1) can be found by solving the system of partial differential equations (17), resulting in the new system having the same row orders, except the $(i+1)$th one which, by (18), is strictly less than μ_{i+1}.

Proposition 2. *Solutions of* (15), (16) *equal to the solutions of* (1).

Proof. Indeed, let $\{(u(t), y(t)), t \geq 0\}$ satisfy (1). Then, trivially, $\{(u(t), y(t)), t \geq 0\}$ satisfies (15). Furthermore, $\{(u(t), y(t)), t \geq 0\}$ satisfies $\phi_j(y^{[\nu_j]}, \ldots, y^{[n+\nu_j]}, u^{[\nu_j]}, \ldots, u^{[n+\nu_j]}) = 0$ for $j = 1, \ldots, i$ and $\nu_j \geq 0$. By (13), $\{(u(t), y(t)), t \geq 0\}$ satisfies (16).

Conversely, let $\{(u(t), y(t)), t \geq 0\}$ satisfy Eqs (15), (16). Then, by (15) we get $\phi_j(y^{[\nu_j]}, \ldots, y^{[n+\nu_j]}, u^{[\nu_j]}, \ldots, u^{[n+\nu_j]}) = 0$ for $j = 1, \ldots, i$, $\nu_j \geq 0$ and we see that $\{(u(t), y(t)), t \geq 0\}$ satisfies $F(\phi_{i+1}, 0, \ldots, 0, \xi) = 0$ for every ξ. By (14) it follows that $\phi_{i+1}(\cdot) = 0$, and so $\{(u(t), y(t)), t \geq 0\}$ satisfies (1). □

If the rank of the matrix L_μ of the new i/o system equals p, we have transformed the system equations into the i/o equivalent strong row-reduced form. Otherwise, we may repeat the above procedure. Note that at each step the sum of row degrees decreases, converging this way to some constant number greater than $-p$. After a finite number of steps we either obtain matrix L_μ with rank p or obtain matrix L_μ for which (possibly after permutation of the rows) the first p' rows are independent, while the last $p - p''$ rows are zero. In the latter case we obtain the i/o equations of the form

$$\phi_i^\star\left(y, \ldots, y^{[n]}, u, \ldots, u^{[n]}\right) = 0, \quad i = 1, \ldots, p',$$

$$\begin{aligned} \phi_i^{\star}\left(y,\ldots,y^{[n]}\right) &= 0, \quad i = p'+1,\ldots,p'', \\ \phi_i^{\star}\left(u,\ldots,u^{[n]}\right) &= 0, \quad i = p''+1,\ldots,p \end{aligned} \tag{20}$$

with

$$\operatorname{rank}_{\mathscr{A}_{\widetilde{\mathscr{S}}}}\{\operatorname{diag}\{\sigma^{m_1},\ldots,\sigma^{m_p}\}\mathscr{M}_\mu\} = p'',$$

where $\widetilde{\mathscr{S}}$ is an extended multiplicative set obtained during the transformation procedure. Note that $\phi_{p''+1}^{\star},\ldots,\phi_p^{\star}$ depend only on input variables or are zeros.

The above considerations give the proof of the following theorem.

Theorem 1. *Consider a set of higher-order difference equations* (1). *Under Assumption* 3 *there exists a* (*local*) *equivalence transformation that allows one to transform the set of equations* (1) *into a strong row-reduced form*, *possibly together with some equations which are trivially satisfied*, *or define restrictions on input or output signals* (20).

Using the theoretical considerations given above, we are ready to present an algorithm for transforming the set of i/o equations into the strong row-reduced form.

Step 0. Start of Algorithm.

Step 1. Let $\Phi = \begin{bmatrix}\phi_1 & \cdots & \phi_p\end{bmatrix}^{\mathrm{T}}$.

Step 2. According to Definition 4, compute the row orders $\mu = (\mu_1,\ldots,\mu_p)$ of Φ.

Step 3. Reorder the elements in Φ with respect to the row orders starting from the lowest. This operation corresponds to the multiplication of the matrix Φ by a permutation matrix R from the left that can be obtained by (repeated) swapping of the ith and the jth rows of the identity matrix I_p, resulting in a new matrix $\widetilde{\Phi}$. Then, reorder the elements of μ by multiplying it by the same permutation matrix from the right, i.e., $\widetilde{\mu} = \mu R$.

Step 4. Set $N_\mu := \max \widetilde{\mu}_i$ and calculate $m = (m_1,\ldots,m_p)$, where $m_i = N_\mu - \widetilde{\mu}_i$ for $i = 1,\ldots,p$.

Step 5. Find the matrix

$$A_\mu = \left[\frac{\partial \widetilde{\phi}_i}{\partial y_j^{[\widetilde{\mu}_i]}}\right]_{ij}$$

for $i = 1,\ldots,p$ and $j = 1,\ldots,p$. Compute its leading coefficient matrix as $L_\mu = \operatorname{diag}\{\sigma^{m_1},\ldots,\sigma^{m_p}\}A_\mu$.

Step 6. Check whether $\operatorname{rank}_{\widetilde{A}} L_\mu = p$. In case of an affirmative answer go to Step 9; otherwise, go to Step 7.

Step 7. Check whether Assumption 3 holds or not. If $\lambda_{i+1} \notin \mathscr{I}_{\mathscr{S}}$, solve Eq. (12) to find λ_υ^* and go to Step 8; otherwise it is not possible to complete the algorithm.

Step 8. From Lemma 1 and Proposition 1 it follows that there exists a function F_ξ satisfying (17) and (18), which defines the transformation. Apply the obtained transformation F to $\widetilde{\Phi}$, resulting in the new system and proceed to Step 1.

Step 9. The system is in the strong row-reduced form. End of the algorithm.

4. EXAMPLES

Several illustrative examples are presented in this section. The first example shows that the approach proposed in this paper in some cases yields the same linear i/o equivalence transformation as the method from [5]. The next two examples address the different aspects of the motivating example. The first of them shows how to calculate a local nonlinear transformation for system (6) from Example 2 which transforms equations into the strong row-reduced form. Recall that this is impossible using the linear transformation, since the equations obtained after the application of the method from [5] are in the row-reduced form (as expected), but not in the strong row-reduced form. In the next example we take these transformed equations (obtained after the application of the linear transformation) as a starting point and explain how to find a suitable nonlinear transformation. The final example is again intended to illustrate the applicability of a nonlinear transformation. The key moment here is that sometimes it is necessary to shift the elements of the leading coefficient matrix back to get the system of partial differential equations (17). Moreover, in this example the transformation depends on the parameter ξ.

Example 3. Consider the set of i/o equations

$$\begin{aligned}\phi_1 &:= y_2^{[1]}y_3 - u_3 = 0,\\ \phi_2 &:= u_2 y_3^{[3]} + y_1^{[2]}y_2^{[3]} - \frac{u_1^{[2]}u_2}{y_2^{[2]}} - \frac{u_3^{[2]}y_1^{[2]}}{y_3^{[2]}} + y_2^{[2]} - y_1^{[1]} + u_1 = 0,\\ \phi_3 &:= y_2 y_3^{[1]} - u_1 = 0.\end{aligned} \tag{21}$$

Since ϕ_2 contains the denominators $y_2^{[2]}$ and $y_3^{[2]}$, we set $\mathscr{S}_0 = \{1, y_2, y_3\}$. Then $\mathscr{A}_{\mathscr{S}} = \mathscr{S}^{-1}\mathscr{A}$ is a localization of the ring $\mathscr{A}$ with respect to the multiplicative subset $\mathscr{S}$ generated by $\mathscr{S}_0$. Let $\Phi = \begin{bmatrix}\phi_1 & \phi_2 & \phi_3\end{bmatrix}^{\mathrm{T}}$ and compute, according to Definition 4, the row orders as $\mu = (\mu_1, \mu_2, \mu_3) = (1, 3, 1)$. In order to permute the second and third elements of the vector Φ, it has to be multiplied by the permutation matrix as follows:

$$\widetilde{\Phi} := \begin{bmatrix}\widetilde{\phi}_1\\ \widetilde{\phi}_2\\ \widetilde{\phi}_3\end{bmatrix} = \underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}}_{R}\Phi = \begin{bmatrix}\phi_1\\ \phi_3\\ \phi_2\end{bmatrix}.$$

Hence, we have $\widetilde{\mu} = \mu R = (1, 1, 3)$. Set $N_\mu := \max \widetilde{\mu}_i = 3$, yielding $m = (2, 2, 0)$. Using (3), find the matrix $\mathscr{M}_\mu$ as

$$\mathscr{M}_\mu = \begin{bmatrix}\dfrac{\partial\widetilde{\phi}_1}{\partial y_1^{[1]}} & \dfrac{\partial\widetilde{\phi}_1}{\partial y_2^{[1]}} & \dfrac{\partial\widetilde{\phi}_1}{\partial y_3^{[1]}}\\[2ex] \dfrac{\partial\widetilde{\phi}_2}{\partial y_1^{[1]}} & \dfrac{\partial\widetilde{\phi}_2}{\partial y_2^{[1]}} & \dfrac{\partial\widetilde{\phi}_2}{\partial y_3^{[1]}}\\[2ex] \dfrac{\partial\widetilde{\phi}_3}{\partial y_1^{[3]}} & \dfrac{\partial\widetilde{\phi}_3}{\partial y_2^{[3]}} & \dfrac{\partial\widetilde{\phi}_3}{\partial y_3^{[3]}}\end{bmatrix} = \begin{bmatrix}0 & y_3 & 0\\ 0 & 0 & y_2\\ 0 & y_1^{[2]} & u_2\end{bmatrix}$$

and the leading coefficient matrix L_μ, according to (4), as follows:

$$L_\mu = \operatorname{diag}\{\sigma^2, \sigma^2, 1\}\mathscr{M}_\mu = \begin{bmatrix}0 & y_3^{[2]} & 0\\ 0 & 0 & y_2^{[2]}\\ 0 & y_1^{[2]} & u_2\end{bmatrix}.$$

One can easily check that $\operatorname{rank}_{\mathscr{A}_{\mathscr{S}}} L_\mu = 2$. Rows of the matrix L_μ are linearly dependent. Moreover, the third row is a linear combination of the first and second rows, yielding

$$\lambda_1 L_\mu^1 + \lambda_2 L_\mu^2 + \lambda_3 L_\mu^3 = 0. \tag{22}$$

It is easy to see that $\lambda_3 = y_2^{[2]}y_3^{[2]} \notin \mathscr{I}_{\mathscr{S}}$. Solving (22) with respect to λ_k, for $k = 1, 2, 3$ and taking into account that $\gamma = 0$, we get $\lambda_1^* = -y_1^{[2]}/y_3^{[2]}, \lambda_2^* = -u_2/y_2^{[2]}, \lambda_3^* = 1$. Note that the set $\mathscr{S}$ remains the same and λ_i^* do not depend on $y^{[3]}$. Thus, we get the following system of partial differential equations:

$$\begin{cases} \frac{\partial F}{\partial x_1} = 1, \\ \frac{\partial F}{\partial x_2} = -\frac{y_1^{[2]}}{y_3^{[2]}}, \\ \frac{\partial F}{\partial x_3} = -\frac{u_2}{y_2^{[2]}} \end{cases}$$

the solution of which, according to Remark 1, can be given as a linear function of the form

$$F(x_1, x_2, x_3) = x_1 - \frac{y_1^{[2]}}{y_3^{[2]}} x_2 - \frac{u_2}{y_2^{[2]}} x_3,$$

which leads to

$$F\left(\widetilde{\phi}_3, \sigma^2(\widetilde{\phi}_1), \sigma^2(\widetilde{\phi}_2)\right) = \widetilde{\phi}_3 - \frac{y_1^{[2]}}{y_3^{[2]}} \sigma^2(\widetilde{\phi}_1) - \frac{u_2}{y_2^{[2]}} \sigma^2(\widetilde{\phi}_2).$$

Then, applying the transformation F to $\widetilde{\Phi}$ yields

$$\begin{aligned} y_2^{[1]} y_3 - u_3 &= 0, \\ y_2 y_3^{[1]} - u_1 &= 0, \\ y_2^{[2]} - y_1^{[1]} + u_1 &= 0. \end{aligned} \tag{23}$$

Next we repeat all the steps in the same manner as above. The leading coefficient matrix of system (23) is then given as

$$L_\mu = \begin{bmatrix} 0 & y_3^{[1]} & 0 \\ 0 & 0 & y_2^{[1]} \\ 0 & 1 & 0 \end{bmatrix},$$

which is clearly not of full rank. Then system (23) can be transformed, via transformation

$$F(\phi_3, \sigma(\phi_1)) = \phi_3 - \frac{1}{y_3^{[1]}} \sigma(\phi_1),$$

into the form

$$\begin{aligned} y_2^{[1]} y_3 - u_3 &= 0, \\ y_2 y_3^{[1]} - u_1 &= 0, \\ -y_1^{[1]} + u_1 + \frac{u_3^{[1]}}{y_3^{[1]}} &= 0. \end{aligned} \tag{24}$$

Repeating again the above procedure, one can find that $\operatorname{rank}_{\mathscr{A}_\mathscr{S}} L_\mu = 3$. Therefore, according to Definition 6, system (24) is in the strong row-reduced form. Note that transformations, found in this example using the approach of this paper, coincide with those constructed using the algorithm from [5].

Example 4. (Continuation of Example 2). Recall that $\widetilde{\Phi} := \begin{bmatrix} \widetilde{\phi}_1 & \widetilde{\phi}_2 \end{bmatrix}^{\mathrm{T}}$, where

$$\widetilde{\phi}_1 := \sin\left(y_1^{[2]} - u_2 + y_2^{[1]}\right) = 0,$$
$$\widetilde{\phi}_2 := y_1^{[3]} - y_1^{[2]} - u_1 = 0.$$

Hence, we have $\widetilde{\mu} = (2,3), N_\mu = 3$, and $m = (1,0)$. Furthermore,

$$L_\mu = \begin{bmatrix} \cos\beta^{[1]} & 0 \\ 1 & 0 \end{bmatrix},$$

where $\beta^{[1]} = y_1^{[3]} - u_2^{[1]} + y_2^{[2]}$. One can easily check that the rows of L_μ are not independent, yielding

$$\lambda_1 L_\mu^1 + \lambda_2 L_\mu^2 = 0. \tag{25}$$

Observe that $\lambda_2 = \cos\beta^{[1]} \notin \mathscr{I}_{\mathscr{S}}$. After solving (25) with respect to λ_1, λ_2 and using the fact that $\gamma = 0$, we get $\lambda_1^* = -1/\cos\beta^{[1]}, \lambda_2^* = 1$. Observe that λ_1^* depends on $y_1^{[3]}$ meaning that, according to Remark 1, the linear solution cannot be used. Recall from Example 2 that $\mathscr{S}_0 = \mathscr{S} = \{1\}$. Now, $\mathscr{S}_0$ has to be extended as $\widetilde{\mathscr{S}_0} := \{1, \cos\beta\}$. We continue with the system of partial differential equations

$$\begin{cases} \dfrac{\partial F}{\partial x_1}\left(\widetilde{\phi}_2, \sigma(\widetilde{\phi}_1)\right) = 1, \\ \dfrac{\partial F}{\partial x_2}\left(\widetilde{\phi}_2, \sigma(\widetilde{\phi}_1)\right) = -\dfrac{1}{\sqrt{1-\left(\sigma(\widetilde{\phi}_1)\right)^2}}. \end{cases} \tag{26}$$

Note that in (26) we used[4] Pythagorean trigonometric identity to express the dependency of λ_1^* on $\sigma(\widetilde{\phi}_1)$ in the explicit form. Then, the solution of (26) can be found as

$$F(x_1, x_2) = x_1 - \arcsin x_2$$

that yields

$$F\left(\widetilde{\phi}_2, \sigma(\widetilde{\phi}_1)\right) = \widetilde{\phi}_2 - \arcsin\sigma(\widetilde{\phi}_1).$$

Observe that F is a nonlinear function defined on $\mathscr{V} = \mathbb{R} \times (-1,1)$. It is easy to verify that $F(0,0) = 0$ and $F(\cdot,0)$ is a diffeomorphism. Therefore, the i/o transformation is defined by

$$\widehat{\phi}_1 = \widetilde{\phi}_1,$$
$$\widehat{\phi}_2 = \widetilde{\phi}_2 - \arcsin\sigma(\widetilde{\phi}_1)$$

that yields the i/o equivalent description of the original system (6) in the form

$$\begin{aligned} \sin\left(y_1^{[2]} - u_2 + y_2^{[1]}\right) &= 0, \\ -y_1^{[2]} - u_1 + u_2^{[1]} - y_2^{[2]} &= 0. \end{aligned} \tag{27}$$

Note that the second equation of (27) does not depend on $y_1^{[3]}$. Therefore, the nonlinear transformation based on F allowed us to transform Eqs (6) into the strong row-reduced form for which $\operatorname{rank}_{\mathscr{A}_{\widetilde{\mathscr{S}}}} L_\mu = 2$.

[4] See also Section 5.

Example 5. Suppose we are given the following system that, in fact, is the system obtained in Example 2 after applying the linear transformation:

$$\begin{aligned} \sin\left(y_1^{[2]} - u_2 + y_2^{[1]}\right) &= 0, \\ y_1^{[3]} - y_1^{[2]} - u_1 - \tan\left(y_1^{[3]} - u_2^{[1]} + y_2^{[2]}\right) &= 0. \end{aligned} \tag{28}$$

Recall that for (28) the set $\mathscr{S}$ is defined as $\widetilde{\mathscr{S}} = \{1, \cos\beta^{[k]} \mid k \in \mathbb{Z}\}$ with $\beta = y_1^{[2]} - u_2 + y_2^{[1]}$. Though system (28), according to Definition 4 from [5], is in the row-reduced form, it is not in the strong row-reduced form (see Definition 6), since $\mu = (2,3)$, $N_\mu = \max \mu_i = 3, m = (1,0)$, and

$$A_\mu = \begin{bmatrix} \cos\left(y_1^{[2]} - u_2 + y_2^{[1]}\right) & 0 \\ -\tan^2\left(y_1^{[3]} - u_2^{[1]} + y_2^{[2]}\right) & 0 \end{bmatrix},$$

yielding

$$L_\mu = \mathrm{diag}\{\sigma, 1\} A_\mu = \begin{bmatrix} \cos\left(y_1^{[3]} - u_2^{[1]} + y_2^{[2]}\right) & 0 \\ -\tan^2\left(y_1^{[3]} - u_2^{[1]} + y_2^{[2]}\right) & 0 \end{bmatrix},$$

which is obviously not of full rank. Note that the rows of L_μ are linearly dependent, yielding $\lambda_1 L_\mu^1 + \lambda_2 L_\mu^2 = 0$. Since $\gamma = 0$, the respective solution is $\lambda_1^* = \sin^2(\beta^{[1]})/\cos^3(\beta^{[1]}), \lambda_2^* = 1$. Observe that λ_1^* can be rewritten as

$$\lambda_1^* = \frac{\sin^2(\beta^{[1]})}{\cos^3(\beta^{[1]})} = \frac{\sin^2(\beta^{[1]})}{(\cos^2(\beta^{[1]}))^{\frac{3}{2}}} = \frac{[\sigma(\phi_1)]^2}{(1 - [\sigma(\phi_1)]^2)^{\frac{3}{2}}}$$

that yields the system of partial differential equations

$$\begin{cases} \dfrac{\partial F}{\partial x_1}\left(\widetilde{\phi}_2, \sigma(\widetilde{\phi}_1)\right) = 1 \\ \dfrac{\partial F}{\partial x_2}\left(\widetilde{\phi}_2, \sigma(\widetilde{\phi}_1)\right) = \dfrac{[\sigma(\phi_1)]^2}{(1 - [\sigma(\phi_1)]^2)^{\frac{3}{2}}}. \end{cases}$$

The solution of the system of equations above can be found as

$$F(x_1, x_2) = x_1 + \frac{x_2}{\sqrt{1 - x_2^2}} - \arcsin(x_2)$$

that yields

$$F\left(\phi_2, \sigma(\phi_1)\right) = \phi_2 + \frac{\sigma(\phi_1)}{\sqrt{1 - [\sigma(\phi_1)]^2}} - \arcsin(\sigma(\phi_1)).$$

It is easy to verify that $F(0,0) = 0$ and $F(\cdot, 0)$ is a diffeomorphism. Therefore, the local nonlinear transformation is defined by

$$\begin{aligned} \widetilde{\phi}_1 &= \phi_1, \\ \widetilde{\phi}_2 &= \phi_2 + \frac{\sigma(\phi_1)}{\sqrt{1 - [\sigma(\phi_1)]^2}} - \arcsin(\sigma(\phi_1)), \end{aligned}$$

whose application to system (28) yields the i/o equivalent description of the original system (6) in the strong row-reduced form

$$\begin{aligned} \sin\left(y_1^{[2]} - u_2 + y_2^{[1]}\right) &= 0, \\ -y_1^{[2]} - u_1 + u_2^{[1]} - y_2^{[2]} &= 0. \end{aligned} \tag{29}$$

Finally, observe that (29) coincides with (27) obtained in Example 4.

Example 6. Consider the set of i/o equations

$$\begin{aligned} \phi_1 &:= \left(y_1^{[2]}\right)^3 - u_2 = 0, \\ \phi_2 &:= y_1^{[3]} - y_2^{[2]} + u_1 = 0, \\ \phi_3 &:= y_3^{[4]} + y_2^{[1]} + u_2 = 0. \end{aligned} \tag{30}$$

Since there are no denominators in (30), we set $\mathscr{S}_0 = \mathscr{S} := \{1\}$. Compute the row orders as $\mu = (2,3,4)$. Hence, we have $N_\mu = 4$ and $m = (2,1,0)$. Next, calculate the leading coefficient matrix L_μ as follows:

$$L_\mu = \operatorname{diag}\{\sigma^2, \sigma, 1\}\mathscr{M}_\mu = \begin{bmatrix} 3\left(y_1^{[4]}\right)^2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Clearly L_μ is not of full rank: the second row depends on the first, yielding $\lambda_1 L_\mu^1 + \lambda_2 L_\mu^2 = 0$. Note that $\gamma = N_\mu - \mu_2 = 4 - 3 = 1$ and $\lambda_2 = 3(y_1^{[4]})^2 \notin \mathscr{I}_{\mathscr{S}}$. Then $\lambda_1^* = \sigma^{-1}\left(-1/[3(y_1^{[4]})^2]\right) = -1/[3(y_1^{[3]})^2], \lambda_2^* = 1$. The set $\mathscr{S}_0$ has to be extended as $\widetilde{\mathscr{S}_0} := \{1, y_1\}$. Observing that $y_1^{[3]} = \left(\sigma(\phi_1) + u_2^{[1]}\right)^{1/3}$, we continue with the system of partial differential equations

$$\begin{cases} \dfrac{\partial F_\xi}{\partial x_1}(\phi_2, \sigma(\phi_1)) = 1, \\ \dfrac{\partial F_\xi}{\partial x_2}(\phi_2, \sigma(\phi_1)) = -\dfrac{1}{3\left(\sigma(\phi_1) + u_2^{[1]}\right)^{\frac{2}{3}}}. \end{cases}$$

Observe that the second equation depends on the parameter $u_2^{[1]}$. The solution of the above system of equations is given by

$$F_\xi(x_1, x_2) = x_1 - \left(x_2 + u_2^{[1]}\right)^{\frac{1}{3}},$$

where $\xi = u_2^{[1]}$. Then the transformation is

$$\begin{aligned} \widetilde{\phi}_1 &= \phi_1, \\ \widetilde{\phi}_2 &= \phi_2 - \left(\sigma(\phi_1) + u_2^{[1]}\right)^{\frac{1}{3}} \end{aligned}$$

that yields the following equations:

$$\begin{aligned} (y_1^{[2]})^3 - u_2 &= 0, \\ -y_2^{[2]} + u_1 &= 0, \\ y_3^{[4]} + y_2^{[1]} + u_2 &= 0. \end{aligned}$$

Finally, we check that $\operatorname{rank}_{\mathscr{A}_{\widetilde{\mathscr{S}}}} L_\mu = 3$. Therefore, the transformed system is in the strong row-reduced form.

5. DISCUSSION AND CONCLUDING REMARKS

The problem of transforming a set of nonlinear i/o equations, described by implicit higher-order difference equations, into the strong row-reduced form is studied. The theory, presented and developed in this paper, extends the results from [5], where the so-called *linear* i/o equivalence transformations were used. The main idea of the algorithm, based on the linear i/o transformations, requires the application of a specific operator related to the unimodular matrix, whose entries are skew polynomials in the forward-shift operator, to a set of i/o equations. In [5], it was proved that any system of the form (1) is linearly i/o equivalent to the row-reduced form. In principle, this statement always holds, but because of the definition of row-reducedness, this only guarantees that the globally linearized (variational) system equations (described in terms of one-forms) can be transformed into the row-reduced form. When coming back to the level of equations (integrating one-forms), although the transformed equations are in the row-reduced form, they may depend on higher-order shifts of output variables than the respective row degrees, which is certainly undesirable. To overcome the above inconsistency, in this paper we provided a new definition of the strong row-reduced form and introduced the local nonlinear i/o equivalence transformations to enlarge the class of i/o equations transformable into the strong row-reduced form. To conclude, whereas row-reducedness is the property of a linearized system, strong row-reducedness is the property of i/o equations.

Recall some facts from the motivating Example 2. It was shown that the application of the transformation matrix $U(z)$ from [5] to the original system equations resulted in the equations

$$\sin\left(y_1^{[2]} - u_2 + y_2^{[1]}\right) = 0,$$
$$y_1^{[3]} - y_1^{[2]} - u_1 - \tan\left(y_1^{[3]} - u_2^{[1]} + y_2^{[2]}\right) = 0.$$

Observe that the second equation depends on the third-order shift of the output variable y_1, i.e., $\mu_1 = 2$ and $\mu_2 = 3$, whereas $\rho_1 = 2$ and $\rho_2 = 2$. The application of $U(z)$ to the globally linearized equations (polynomial system description) yields the one-forms

$$\omega_1 = \cos\beta \mathrm{d}y_1^{[2]} + \cos\beta \mathrm{d}y_2^{[1]} - \cos\beta \mathrm{d}u_2,$$
$$\omega_2 = -\mathrm{d}y_1^{[2]} - \mathrm{d}y_2^{[2]} - \mathrm{d}u_1 + \mathrm{d}u_2^{[1]},$$

which do not depend explicitly on $\mathrm{d}y_1^{[3]}$. Hence, the transformed system is in the row-reduced form; however, it is not in the strong row-reduced form according to Definition 6.

Note that even though the existence of nonlinear transformations can be proven, one cannot always express the solution in terms of elementary functions, or the transformation may be difficult to find. Moreover, to find the nonlinear transformation, one may need to perform certain replacements in the solution of (12) to eliminate the dependence of λ_υ^* on the highest shift $y^{[\mu_{i+1}]}$. Recall from Example 4 that the solution of (25) is given as $\lambda_1^* = -1/\cos\beta^{[1]}, \lambda_2^* = 1$. One can easily observe that $\sigma(\widetilde{\phi}_1) = \sin\beta^{[1]}$ (with $\beta = y_1^{[2]} - u_2 + y_2^{[1]}$) and use the well-known Pythagorean trigonometric identity to transform λ_1^* as

$$\lambda_1^* = -\frac{1}{\sqrt{1-\left(\sigma(\widetilde{\phi}_1)\right)^2}},$$

yielding the nonlinear transformation $F\left(\widetilde{\phi}_2, \sigma(\widetilde{\phi}_1)\right) = \widetilde{\phi}_2 - \arcsin\sigma(\widetilde{\phi}_1)$. It is important to stress that we consider the neighbourhood of 0 and take the positive square root as $\cos\beta^{[1]} = \sqrt{1-\sin^2\beta^{[1]}}$. In addition, in this particular example the goal of replacement was to modify λ_1^* to be dependent on $\widetilde{\phi}_1$ and not on y, u. Hence, one may see that in some cases replacements are obvious (as in Example 6), whereas in other cases they are not (as in Examples 4 and 5).

Finally, it is interesting to observe that when λ_υ^*, $\upsilon = 1, \ldots, i$ do not depend on $y^{[\mu_{i+1}]}$, then (17) yields the linear solution (see Remark 1). However, this solution does not necessarily coincide with the one obtained via the approach from [5], since we rely on a different definition of row-reducedness involving row orders. Though, this observation points to a link between approaches presented in this paper and that from [5], also, it raises a problem of finding a more general set of nonlinear i/o equivalence transformations, including transformations from [5] as a special case. This is the subject for future research.

ACKNOWLEDGEMENTS

The work of Z. Bartosiewicz and M. Wyrwas was supported by the Białystok University of Technology grant S/WI/2/2011. Ü. Kotta and M. Tõnso were additionally supported by the Estonian Research Council, personal research funding grant PUT481. We are grateful to V. Kaparin and the anonymous referee for helpful reviews of the manuscript. The publication costs of this article were covered by the Estonian Academy of Sciences.

REFERENCES

1. Beckermann, B., Cheng, H., and Labahn, G. Fraction-free row reduction of matrices of Ore polynomials. *J. Symb. Comput.*, 2006, **41**, 513–543.
2. Davies, P., Cheng, H., and Labahn, G. Computing Popov form of general Ore polynomial matrices. In *Milestones in Computer Algebra, Stonehaven Bay, Trinidad and Tobago*. 2008, 149–156.
3. Kotta, Ü. Towards solution of the state-space realization problem of a set of multi-input multi-output nonlinear difference equations. In *European Control Conference*. Karlsruhe, Germany, August–September 1999, 1–6.
4. Kotta, Ü. and Tõnso, M. Realization of discrete-time nonlinear input-output equations: Polynomial approach. *Automatica*, 2012, **48**, 255–262.
5. Kotta, Ü., Bartosiewicz, Z., Nõmm, S., and Pawłuszewicz, E. Linear input-output equivalence and row reducedness of discrete-time nonlinear systems. *IEEE Trans. Autom. Contr.*, 2011, **56**, 1421–1426.
6. Middeke, J. *A Computational View on Normal Forms of Matrices of Ore Polynomials*. PhD thesis, Research Institute for Symbolic Computation, Johannes Kepler University, Linz, Austria, 2011.
7. Popov, V. M. Some properties of the control systems with irreducible matrix – Transfer functions. In *Seminar on Differential Equations and Dynamical Systems* (Yorke, J. A., ed.), *Lecture Notes Math.*, 1970, **144**, 169–180.
8. Van der Schaft, A. J. Correction. On realization of nonlinear systems described by higher-order differential equations. *Math. Syst. Theory*, 1987, **20**, 305–306.
9. Van der Schaft, A. J. On realization of nonlinear systems described by higher-order differential equations. *Math. Syst. Theory*, 1987, **19**, 239–275.
10. Van der Schaft, A. J. Transformations of nonlinear systems under external equivalence. In *New Trends in Nonlinear Control Theory* (Descusse, Y., Fliess, M., Isidori, A., and Leborgne, D., eds), *Lecture Notes Control Inform. Sci.*, 1989, **122**, 33–43.

Diskreetsete mittelineaarsete sisend-väljundvõrrandite teisendamine tugevale reapõhiselt taandatud kujule

Zbigniew Bartosiewicz, Juri Belikov, Ülle Kotta, Maris Tõnso ja Małgorzata Wyrwas

On käsitletud ilmutamata diferentsiaalvõrranditega esitatud mittelineaarse süsteemi teisendamist tugevale reapõhiselt taandatud kujule, mis on ekvivalentne esialgse kirjeldusega. On näidatud, et varasemas artiklis [5] leitud lineaarsetest teisendustest üle teatud polünoomide ringi alati ei piisa. Kuigi lineaarteisendused teisendavad globaalselt lineariseeritud süsteemi võrrandid sobivale kujule, ei ole alati võimalik lineariseeritud võrranditest saada (tagasi) vastavat kuju mittelineaarsete võrrandite endi jaoks. Nimelt, pärast integreerimist võivad võrrandid sisaldada väljundite kõrgemat järku nihkeid kui vastavad väljundite diferentsiaalid lineariseeritud võrrandites. Artiklis on uuritud võimalust laiendada ekvivalentsiteisenduste hulka, tuues sisse (lokaalsed) mittelineaarsed teisendused, ja esitatud algoritm võrrandite teisendamiseks. Kui lineaarteisendusest piisab, on algoritmi tulemuseks vastav lineaarteisendus; kui mitte, on algoritm konstruktiivne, v.a samm, mis nõuab osatuletistega diferentsiaalvõrrandite süsteemi lahendamist (tüüpiline paljude mittelineaarsete juhtimisprobleemide lahenduste korral). Teoreetilisi tulemusi ja algoritmi on illustreeritud mitme näitega.