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CONTROL THEORY

# Model matching problem for discrete-time nonlinear systems

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**Abstract.** This paper addresses the model matching problem (MMP) for nonlinear single-input single-output discrete-time systems. The approach is based on the infinitesimal system description in terms of the one-forms that is converted into polynomial system representation by interpreting the polynomial indeterminate as the forward shift operator acting on the one-forms. The polynomial description is then used to derive the generalized transfer function. The problem statement of the MMP (both for the feedforward and feedback cases) is given in terms of the generalized transfer function. In general, the feedforward solution exists under restrictive conditions. Therefore, the subclass of nonlinear control systems is specified for which the solution is guaranteed to exist. The feedback solution exists always. The additional restrictions are specified for the existence of a proper compensator (in both cases). The results of the paper are illustrated by numerous examples, and the feedback solution is compared to the earlier results.

**Key words:** nonlinear control systems, discrete-time, model matching problem.

## 1. INTRODUCTION

The model matching problem (MMP) is of both theoretical and practical importance since it accommodates various other problems such as input–output (i/o) linearization, disturbance decoupling, noninteractive control, model tracking, model reference adaptive control, etc. The main idea may be illustrated on the basis of linear control systems. More specifically, for a given plant and a model the problem is to find a compensator such that the transfer functions of the reference model and that of the compensated system coincide [21].

The MMP has been extensively studied for linear time-invariant systems. A precise formulation and the first solution in terms of the static state feedback were given in [42], followed by a similar paper [41]. The authors of [39] proposed an approach in which the Markov parameters of the closed-loop system are equated to those of the model. A dynamical feedback solution for the MMP was presented in [34] relying on the structural algorithm and in [35] relying on the geometric approach. The case of the combination of dynamic output feedback with feedforward reference compensation (also referred to as two-degee-of-freedom dynamic compensation) was studied in [22] and [29]. A more general case of the MMP for linear time-varying systems was addressed in [31] and time delay systems were considered in [38]. The interested reader is referred to [30] for a more detailed review on the exact model matching.

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For the nonlinear case the MMP has mainly been studied within the state-space approach, see, for example, [9,19,24,33]. For the i/o representation of a system the problem has been studied in [13,18,23,25]. One reason to state and solve the MMP for the i/o model is the fact that nonlinear systems are frequently modelled as i/o difference equations resulting from identification [37]. In addition, i/o equations are not always realizable in the state-space form [40]. The existing contributions developed for nonlinear state-space models are valid for initialized systems as in [9] or are stated as a generic problem as in [33]. Those problems are not equivalent, and it is unclear whether they are good candidates for the equality of transfer function matrices. That open problem motivates the search for a transparent solution stated in terms of the recently introduced generalized transfer function formalism [15,18] for the class of nonlinear systems. The generalized transfer function can be constructed simply from the polynomial system description obtained from the infinitesimal linearized system.

Note that in [18] the transfer function formalism was applied for solving the MMP of nonlinear continuous-time systems. Herein, the discrete-time case is considered. Conceptually, the results are similar to those of [18], the main difference being that the derivative and shift operators define the different noncommutative polynomial rings with different multiplication tools.

Two types of solutions, feedforward and feedback compensators, are typically looked for within the MMP. We extend the results presented in our conference paper [5]. First, more detailed proofs are given. Second, several illustrative real-life examples are added. Finally, a brief comparison with the earlier results of [25] is given.

The paper is organized as follows. Section 2 recalls the essential notions of the algebraic framework of differential forms. It is followed by the polynomial formalism which allows us to construct the main mathematical tools used in the paper. In Sections 3 and 4 the feedforward and feedback compensators are considered. Corresponding proofs can be found in Appendix. Section 5 provides brief concluding remarks.

## 2. PRELIMINARIES

Hereinafter, for a time-dependent variable $\xi(t)$, $\xi^{[k]}$ stands for its $k$th-step forward time shift $\xi(t+k)$ and $\xi^{[-l]}$ for the $l$th-step backward time shift $\xi(t-l)$ with $k,l$ being nonnegative integers, implying that $\xi^{[0]} = \xi(t)$. Consider a nonlinear discrete-time single-input single-output (SISO) system, described by the difference equation

$$y^{[n]} = \phi\left(y, y^{[1]}, \ldots, y^{[n-1]}, u, u^{[1]}, \ldots, u^{[s]}\right), \tag{1}$$

where $u = u(t) \in \mathscr{U} \subset \mathbb{R}$ is the input, $y = y(t) \in \mathscr{Y} \subset \mathbb{R}$ is the output, and $\phi$ is a real analytic function defined on $\mathscr{Y}^n \times \mathscr{U}^{s+1}$. Moreover, we assume that $s \leq n$ are nonnegative integers.

### 2.1. Algebraic framework

Recall briefly the algebraic formalism from [28] that we use in this paper. Denote by $\mathscr{K}$ the field of meromorphic functions in a finite number of (independent) variables from the set $\mathscr{C} = \left\{y, \ldots, y^{[n-1]}, u^{[k]}, k \geq 0\right\}$, and introduce the forward-shift operator $\sigma : \mathscr{K} \to \mathscr{K}$. In particular, $\sigma(y^{[n-1]}) := \phi(\cdot)$, meaning that $y^{[n]}$ as a dependent variable has to be replaced by $\phi(\cdot)$ from (1). For the remaining elements of $\mathscr{C}$, the forward shift is defined in a standard manner, i.e., $\sigma(y^{[i]}) := y^{[i+1]}$, $i = 0, \ldots, n-2$, $\sigma(u^{[j]}) := u^{[j+1]}$, $j \geq 0$, where $y^{[0]} = y$ and $u^{[0]} = u$. Moreover, the application of $\sigma$ to $\varphi \in \mathscr{K}$ is defined by shifting arguments of the function according to the rules described above, i.e.,

$$\sigma\left[\varphi\left(y, \ldots, y^{[n-1]}, u, \ldots, u^{[l]}\right)\right] := \varphi\left(y^{[1]}, \ldots, \phi(\cdot), u^{[1]}, \ldots, u^{[l+1]}\right).$$

The assumption below is a standard assumption made in most papers and is not restrictive as it is the necessary condition for system accessibility.

**Assumption 1.** We assume the system (1) to be submersive, i.e., the map $\phi$ satisfies generically the condition

$$\frac{\partial \phi}{\partial (y,u)} \not\equiv 0. \tag{2}$$

Under Assumption 1 there exists an inversive difference overfield[1] of $(\mathscr{K},\sigma)$ such that $\sigma$, when extended to this overfield, becomes an automorphism [2,10]. Therefore, $\sigma$ has an inverse operator $\sigma^{-1}$, interpreted as a backward-shift operator. For a detailed description of the backward-shift operator $\sigma^{-1}$ see [2,6].

Consider the infinite set of differentials $\mathrm{d}\mathscr{C} = \{\mathrm{d}y, \mathrm{d}y^{[1]}, \ldots, \mathrm{d}y^{[n-1]}, \mathrm{d}u^{[l]}, l \geq 0\}$ and denote by $\mathscr{E}$ the vector space over the field $\mathscr{K}$ spanned by the elements of $\mathrm{d}\mathscr{C}$, i.e., $\mathscr{E} = \operatorname{span}_{\mathscr{K}} \mathrm{d}\mathscr{C}$. For $F \in \mathscr{K}$, define the operator $\mathrm{d} : \mathscr{K} \to \mathscr{E}$ as follows:

$$\mathrm{d}F := \sum_{i=0}^{n-1} \frac{\partial F}{\partial y^{[i]}} \mathrm{d}y^{[i]} + \sum_{j=0}^{k} \frac{\partial F}{\partial u^{[j]}} \mathrm{d}u^{[j]}.$$

One says that $\mathrm{d}F$ is a *total differential* (or simply the differential) of the function $F$. Note that any element in $\mathscr{E}$ is a vector of the form $\omega = \sum_i \alpha_i \mathrm{d}\zeta_i$ with $\mathrm{d}\zeta_i \in \mathrm{d}\mathscr{C}$ and $\alpha_i \in \mathscr{K}$. Then the operators $\sigma : \mathscr{K} \to \mathscr{K}$ and $\sigma^{-1} : \mathscr{K} \to \mathscr{K}$ induce, respectively, the operators $\sigma : \mathscr{E} \to \mathscr{E}$ and $\sigma^{-1} : \mathscr{E} \to \mathscr{E}$ by $\sigma(\omega) := \sum_i \sigma(\alpha_i)\mathrm{d}(\sigma(\zeta_i))$ and $\sigma^{-1}(\omega) := \sum_i \sigma^{-1}(\alpha_i)\mathrm{d}(\sigma^{-1}(\zeta_i))$.

An arbitrary element of $\mathscr{E}$ is called a *one-form*. One says that $\omega \in \mathscr{E}$ is an *exact* one-form if $\omega = \mathrm{d}\zeta$ for some $\zeta \in \mathscr{K}$. A one-form $\omega$ for which $\mathrm{d}\omega = 0$ is said to be *closed*. Note that exact one-forms are closed, whereas closed one-forms are only locally exact.

**Lemma 2** [11, Poincaré's Lemma]**.** *Let $\omega$ be a closed one-from in $\mathscr{E}$. Then there exists $\varphi \in \mathscr{K}$ such that locally $\omega = \mathrm{d}\varphi$.*

A one-form is called integrable if there exists an integrating factor $\lambda \in \mathscr{K}$ such that $\lambda\omega$ is an exact one-form. The integrability of a one-form can be checked by the theorem below, where the symbol $\mathrm{d}\omega$ denotes the exterior derivative of the one-form $\omega$ and $\wedge$ means the exterior or wedge product.

**Theorem 3** [11]. *Given $\omega \in \mathscr{E}$ there exists a function $\zeta$ such that* $\operatorname{span}_{\mathscr{K}}\{\omega\} = \operatorname{span}_{\mathscr{K}}\{\mathrm{d}\zeta\}$ *if and only if* $\mathrm{d}\omega \wedge \omega = 0$.

## 2.2. Polynomial framework

A *left* polynomial can be uniquely written in the form

$$a(z) = \alpha_0 z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + \alpha_n \tag{3}$$

for $\alpha_i \in \mathscr{K}$, $i = 0, \ldots, n$, where $z$ is a formal variable (polynomial indeterminate) and $a(z) \neq 0$ if and only if at least one of the functions $\alpha_i$, for $i = 0, \ldots, n$, is nonzero. The highest power $n$ in the polynomial $a(z)$ is called the degree of the left polynomial $a(z)$, denoted by $\deg a(z)$, if $\alpha_0 \not\equiv 0$. In addition, we use convention $\deg 0 = -\infty$.

**Definition 4.** *The left skew polynomial ring, induced by $(\mathscr{K}, z)$, is the ring $\mathscr{K}[z;\sigma]$ of polynomials in the indeterminate $z$ with usual addition and multiplication satisfying the relation*

$$z \cdot \alpha = \sigma(\alpha) z \tag{4}$$

*for any $\alpha \in \mathscr{K}$.*

[1] With a slight abuse of notation, for the field extension, we use the same symbol.

The term *skew* means that the coefficients of the polynomial do not necessarily commute with the indeterminate. The skew polynomial ring $\mathcal{K}[z;\sigma]$ is proved to satisfy the following left Ore condition.

**Proposition 5** [12]. *For all nonzero $a,b \in \mathcal{K}[z;\sigma]$ there exist nonzero $a_1,b_1 \in \mathcal{K}[z;\sigma]$ such that $a_1 b = b_1 a$.*

If the left condition holds, the skew polynomial ring is called the left Ore ring. Thus, the ring $\mathcal{K}[z;\sigma]$ can be embedded into the field of left fractions, denoted as $\mathcal{K}(z;\sigma)$, see [36]. In $\mathcal{K}(z;\sigma)$ one can define the sum of two quotients as

$$b_1^{-1}a_1 + b_2^{-1}a_2 = (\beta_2 b_1)^{-1}(\beta_2 a_1 + \beta_1 a_2),$$

where $\beta_2 b_1 = \beta_1 b_2$ satisfy the left Ore condition and the product as

$$b_1^{-1}a_1 b_2^{-1}a_2 = (\beta_2 b_1)^{-1}\alpha_1 a_2, \tag{5}$$

where $\beta_2 a_1 = \alpha_1 b_2$ again satisfy the left Ore condition.

A ring is called an *integral domain* if it does not contain any zero divisors. This means that for any two elements $a$ and $b$ of the ring, $ab = 0$ implies either $a = 0$ or $b = 0$ or both.

**Proposition 6** [32].
1. *The ring $\mathcal{K}[z;\sigma]$ is an integral domain.*
2. *If $a(z)$ and $b(z)$ are non-zero polynomials*, *then* $\deg(a(z)b(z)) = \deg a(z) + \deg b(z)$.

Define

$$z^i \mathrm{d}y := \mathrm{d}y^{[i]}, \quad z^j \mathrm{d}u := \mathrm{d}u^{[j]} \tag{6}$$

for $i, j \geq 0$ to represent the globally linearized system in terms of two polynomials. Differentiate (1) to obtain the infinitesimal system description

$$\mathrm{d}y^{[n]} - \sum_{i=0}^{n-1} \frac{\partial \phi}{\partial y^{[i]}} \mathrm{d}y^{[i]} - \sum_{j=0}^{s} \frac{\partial \phi}{\partial u^{[j]}} \mathrm{d}u^{[j]} = 0 \tag{7}$$

and use the relations (6) to rewrite (7) as

$$p(z)\mathrm{d}y + q(z)\mathrm{d}u = 0 \tag{8}$$

with $p(z) = z^n - \sum_{i=0}^{n-1} p_i z^i$, $q(z) = -\sum_{j=0}^{s} q_j z^j$ and $p_i = \partial\phi/\partial y^{[i]} \in \mathcal{K}$, $q_j = \partial\phi/\partial u^{[j]} \in \mathcal{K}$. Equation (8) describes the globally linearized system, corresponding to Eq. (1).

*Example 1.* Consider the discrete-time model of the controlled van der Pol oscillator, derived in [1]

$$y^{[2]} = \theta_1 y^{[1]} - \theta_2 y + \theta_3 y^2 y^{[1]} + \theta_4 y^3 + \theta_5 u, \tag{9}$$

where $\theta_i \in \mathbb{R}$ for $i = 1, \ldots, 5$. In (9), $n = 2$ and $s = 0$. Applying operator d to (9) and using relations (6) yields the polynomial system description (8) with

$$\begin{aligned} p_0 &= \theta_2 - 3\theta_4 y^2 - 2\theta_3 y y^{[1]}, \\ p_1 &= -(\theta_1 + \theta_3 y^2), \\ q_0 &= \theta_5. \end{aligned} \tag{10}$$

Next, we recall several definitions from [15], see also [14].

**Definition 7.** *An element of the form $F(z) := p^{-1}(z)q(z) \in \mathcal{K}(z;\delta)$, such that* $\mathrm{d}y = F(z)\mathrm{d}u$, *is said to be a generalized transfer function*[2] *of the nonlinear system* (1).

[2] Note that there exists an algorithm which allows us to obtain the transfer function from nonlinear state equations, see [14].

Note that in the linear case each proper rational function may be interpreted as a transfer function corresponding to some i/o equation of a control system. However, things are different in the nonlinear case. Though every system can be described by the rational function called the generalized transfer function of the nonlinear system, the converse is not always true. It means that not every quotient of skew polynomials necessarily represents a control system, since the corresponding one-form may be non-integrable, see [15] for details. Further in this paper we omit the word generalized, referring to the 'generalized transfer function' just as the 'transfer function' of the nonlinear system.

It follows from (8) that the transfer function of (1) can be represented as

$$F(z) = (z^n + \cdots + p_1 z + p_0)^{-1} (q_s z^s + \cdots + q_1 z + q_0) \,. \tag{11}$$

**Definition 8.** *The transfer function $F(z)$ is said to be proper if $s = \deg q(z) \leq n = \deg p(z)$.*

**Definition 9.** *For a proper transfer function, the difference $n - s$, denoted as $\mathrm{rel\,deg}\,F(z)$, is called the relative degree of the system* (1).

*Example 2* (continuation of Example 1). By (10) and (11), compute the transfer function of (9)

$$F(z) = \left(z^2 - (\theta_1 + \theta_3 y^2)z + \theta_2 - 3\theta_4 y^2 - 2\theta_3 y y^{[1]}\right)^{-1} \theta_5,$$

which is strictly proper, and the relative degree of the system equals 2.

## 3. FEEDFORWARD COMPENSATOR

Consider a nonlinear system $F$ and a model $G$ described by their transfer functions

$$F(z) = p_F^{-1}(z) q_F(z) \tag{12}$$

and

$$G(z) = p_G^{-1}(z) q_G(z), \tag{13}$$

respectively. The goal is to find a (proper) feedforward compensator $R$ described by its transfer function

$$R(z) = p_R^{-1}(z) q_R(z)$$

such that the transfer function of the compensated system coincides with that of the model $G$, i.e.,

$$G(z) = F(z) R(z),$$

or equivalently

$$R(z) = F^{-1}(z) G(z), \tag{14}$$

as depicted in Fig. 1.

**Proposition 10.** *Given $F(z) \neq 0$ and $G(z)$, the feedforward model matching problem is solvable if the one-form $p_R(z)\mathrm{d}u - q_R(z)\mathrm{d}v$ is integrable.*

*Proof.* The proof is a direct consequence from (14) yielding

$$R(z) = q_F^{-1}(z) p_F(z) p_G^{-1}(z) q_G(z). \tag{15}$$

Alternatively, $R(z)$ is described by the relationship

$$p_R(z)\mathrm{d}u = q_R(z)\mathrm{d}v, \tag{16}$$

where $p_R(z)$ and $q_R(z)$ are defined by Ore condition (5), applied to (15). Thus, the i/o equation of the compensator $R$ can be obtained if the one-form (16) is integrable. □

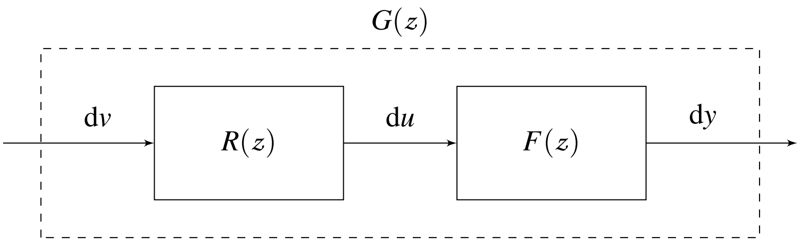


**Fig. 1.** Compensated system.

Usually, one is interested in finding a solution in a class of proper compensators. Therefore, to guarantee the existence of the solution, one has to introduce the restriction on the relative degree of the model $G$.

**Proposition 11.** *The transfer function of compensator* (14) *is proper* (*causal*) *if and only if*

$$\operatorname{rel\,deg} G(z) \geq \operatorname{rel\,deg} F(z). \tag{17}$$

*Proof.* See Appendix. □

*Example 3.* Consider the system

$$y^{[2]} = y + uu^{[1]}$$

and compute its transfer function

$$F(z) = \left(z^2 - 1\right)^{-1} \left(uz + u^{[1]}\right),$$

which is strictly proper. Suppose that the reference model is

$$G(z) = z^{-2},$$

satisfying the condition (17). By (14) and (5), we can find the transfer function of the compensator

$$R(z) = \left(uz + u^{[1]}\right)^{-1} \left(z^2 - 1\right) z^{-2} = \left(u^{[2]} z^3 + u^{[3]} z^2\right)^{-1} \left(z^2 - 1\right), \tag{18}$$

where the Ore condition $\beta(z)\left(z^2 - 1\right) = \alpha(z) z^2$ is satisfied for $\alpha(z) = z^2 - 1$ and $\beta(z) = z^2$. Note that $R(z)$ results in the integrable one-form, yielding the compensator given by the equation

$$u^{[2]} u^{[3]} = v^{[2]} - v. \tag{19}$$

Moreover, this compensator has a classical state-space realization of the form

$$\begin{aligned}
u &= \eta_1, \\
\eta_1^{[1]} &= \eta_2 + \frac{v}{\eta_1}, \\
\eta_2^{[1]} &= \frac{\eta_3}{v + \eta_1 \eta_2}, \\
\eta_3^{[1]} &= -v\left(\eta_2 + \frac{v}{\eta_1}\right).
\end{aligned} \tag{20}$$

The algorithm for constructing (20) from (18) or (19) can be found in [16,17] or [8,27], respectively. However, one may easily check that (20) yields (18) by direct computations. This can be done by shifting the first equation in (20) three times, eliminating the variable $\eta$, and calculating a transfer function for the obtained i/o equation.

*Example 4* (continuation of Example 2). Recall that the transfer function of system (9) is

$$F(z) = \left(z^2 - (\theta_1 + \theta_3 y^2)z + \theta_2 - 3\theta_4 y^2 - 2\theta_3 y y^{[1]}\right)^{-1} \theta_5.$$

Suppose that the reference model is

$$G(z) = z^{-2}.$$

By (14) and (5) one can find

$$R(z) = \left(\theta_5 z^2\right)^{-1} \left(z^2 - (\theta_1 + \theta_3 (y^{[2]})^2)z + \theta_2 - 3\theta_4 (y^{[2]}) - 2\theta_3 y^{[2]} y^{[3]}\right). \tag{21}$$

The transfer function $R(z)$ results in the one-form

$$\theta_5 \mathrm{d}u^{[3]} = \mathrm{d}v^{[2]} - \left(\theta_1 + \theta_3 (y^{[2]})^2\right) \mathrm{d}v^{[1]} + \left(\theta_2 - 3\theta_4 (y^{[2]}) - 2\theta_3 y^{[2]} y^{[3]}\right) \mathrm{d}v,$$

which, according to Lemma 2 and Theorem 3, is not integrable. The latter means that $R(z)$ in (21) does not correspond to any compensator $R$.

Thus, unlike the linear time-invariant case, a class of nonlinear systems for which the solution in terms of a feedforward compensator exists is, due to the integrability condition, quite restricted. Propositions 10 and 11 give weak results, because they do not define the class of nonlinear systems for which the feedforward compensator exists. In Proposition 12 below we specify one such subclass.

**Proposition 12.** *The one-form* $p_R(z)\mathrm{d}u - q_R(z)\mathrm{d}v$ *is integrable if the system* $F$ *and the model* $G$ *are given by*

$$y^{[n_F]} = f_1\left(y, y^{[1]}, \ldots, y^{[n_F-1]}\right) + f_2\left(u, u^{[1]}, \ldots, u^{[s_F]}\right) \tag{22}$$

*and*

$$y^{[n_G]} = g_1\left(y, y^{[1]}, \ldots, y^{[n_G-1]}\right) + g_2\left(u, u^{[1]}, \ldots, u^{[s_G]}\right), \tag{23}$$

*respectively, such that*

$$\begin{aligned} p_F(z) &= \gamma_F(z)\rho(z), \\ p_G(z) &= \gamma_G(z)\rho(z), \end{aligned} \tag{24}$$

*where* $\gamma_F(z)$ *and* $\gamma_G(z)$ *are polynomials with real coefficients, and* $\rho(z) = \sum_{i=0}^{m} \rho_i z^{m-i}$ *with* $\rho_i \in \mathscr{K}$.

*Proof.* See Appendix. □

Observe that in Proposition 12 we require polynomials $p_F(z), p_G(z)$ to be in the form (24), but do not impose additional restrictions on polynomials $q_F(z), q_G(z)$. This is due to the specific structure of the feedforward controller that, according to (14), can be represented as $R(z) = q_F^{-1}(z)p_F(z)p_G^{-1}(z)q_G(z)$. Observe that condition (24) makes it possible to get rid of the variable $y$ and/or its successive shifts in the final expression. Hence, the remaining polynomials $\alpha(z), \beta(z)$ with real coefficients, defined by the left Ore condition, do not influence the integrability of the one-form $\beta(z)q_F(z)\mathrm{d}u - \alpha(z)q_G(z)\mathrm{d}v$, since $q_F(z)$ and $q_G(z)$ are obtained directly by differentiating (22) and (23). More technical details can be found in the proof of the proposition. The next example illustrates why under the conditions of Proposition 12 the solution always exists.

*Example 5.* Consider the system $F$

$$y^{[2]} = -\frac{\left(y^{[1]}\right)^2}{2} - y^{[1]} - \frac{y^2}{2} - y + u^{[1]}u \tag{25}$$

and the model $G$

$$y^{[2]} = -\frac{\left(y^{[1]}\right)^2}{2} + v. \tag{26}$$

Observe that (25) and (26) are in the forms (22) and (23), respectively, required by Proposition 12. Apply the operator d to (25) and (26) to obtain the one-forms (7). Then, using the relations (6) for $i, j = 1, 2$, one can represent the form (7) for the system $F$ and the model $G$ in terms of two skew polynomials as in (8)

$$p_F(z) = z^2 + \left(y^{[1]} + 1\right) z + y, \qquad q_F(z) = uz + u^{[1]},$$
$$p_G(z) = z^2 + y^{[1]} z, \qquad q_G(z) = 1.$$

Using (14), the transfer function of the compensator can be described as

$$R(z) = q_F^{-1}(z) p_F(z) p_G^{-1}(z) q_G(z) = \left(uz + u^{[1]}\right)^{-1} \left(z^2 + \left(y^{[1]} + 1\right) z + y\right) \left(z^2 + y^{[1]} z\right)^{-1}. \tag{27}$$

Observe that polynomials $p_F(z), p_G(z)$ can be represented in the form (24) as $p_F(z) = (z+1)(z+y)$ and $p_G(z) = z(z+y)$, where $\rho(z) = z + y$. Therefore, the expression (27) can be simplified by cancelling $\rho(z)$ as

$$R(z) = \left(uz + u^{[1]}\right)^{-1} (z+1) z^{-1}.$$

Next, using (5), the description of the compensator in terms of the one-forms can be alternatively presented as

$$\beta(z) \left(uz + u^{[1]}\right) \mathrm{d}u = \alpha(z) \mathrm{d}v,$$

where $\alpha(z), \beta(z)$ are defined by the left Ore condition $\beta(z)(z+1) = \alpha(z)z$, which is trivially satisfied for $\alpha(z) = z + 1$ and $\beta(z) = z$. Thus, the relation in terms of the one-form, describing the compensator $R(z)$, can be represented as

$$u^{[1]} \mathrm{d}u^{[2]} + u^{[2]} \mathrm{d}u^{[1]} = \mathrm{d}v^{[1]} + \mathrm{d}v. \tag{28}$$

One may easily observe that $\alpha(z), \beta(z)$ are polynomials with real coefficients. In general, it may not be the case and $\alpha(z), \beta(z)$ may have coefficients from $\mathscr{K}$, in particular, they may depend on variable $y$ and its successive shifts. Then, the integrability would be questionable. See, for instance, Example 4 in which despite the fact that the system and model are in the forms (22), (23), respectively, condition (24) is not satisfied as variable $y$ remains in the expressions of $\gamma_F(z), \gamma_G(z)$ and causes non-integrability of the one-form corresponding to the compensator. Integrating the one-form (28) yields

$$u^{[1]} u^{[2]} + v^{[1]} + v = 0.$$

## 4. FEEDBACK COMPENSATOR

Consider a nonlinear system $F$ and a model $G$ described by their transfer functions (12) and (13), respectively. Find a (proper) feedback compensator $R$

$$\mathrm{d}u = R_v(z) \mathrm{d}v + R_y(z) \mathrm{d}y, \tag{29}$$

described by the transfer functions from d$v$ to d$u$ and d$y$ to d$u$, i.e., by

$$R_v(z) = p_R^{-1}(z) q_{R_v}(z), \tag{30}$$

$$R_y(z) = p_R^{-1}(z) q_{R_y}(z), \tag{31}$$

respectively, such that the transfer function of the compensated system coincides with that of the model $G$:

$$G(z) = (1 - F(z)R_y(z))^{-1} F(z) R_v(z) \tag{32}$$

as depicted in Fig. 2.

**Assumption 13.** $\deg p_G(z) \geq \deg p_F(z)$.

**Theorem 14.** *Given* $F(z) \neq 0$ *and* $G(z)$ *satisfying Assumption* 13, *the model matching problem by feedback* (29) *is always solvable.*

*Proof.* See Appendix. □

Assumption  in the proof of Theorem 14 is clearly necessary to get a reasonable solution by the left Euclidean division algorithm of $p_G(z)$ and $p_F(z)$. However, this assumption is not restrictive, since instead of model (13) with $\deg p_G(z) < \deg p_F(z)$ one can always, without loss of generality, use the transfer function $G'(z) = \left[z^k p_G(z)\right]^{-1} z^k q_G(z)$ being transfer equivalent to $G(z)$, such that $\deg\left(z^k p_G(z)\right) \geq \deg p_F(z)$. Roughly speaking, modulo transfer equivalence there always exists a feedback compensator which solves the model matching problem for given $F(z)$ and $G(z)$.

If one is looking for a solution within a class of proper compensators, the situation is similar to that of the case of a feedforward solution.

**Proposition 15.** $R(z)$ *is proper* (*causal*) *if and only if*

$$\operatorname{rel}\deg G(z) \geq \operatorname{rel}\deg F(z). \tag{33}$$

*Proof.* See Appendix. □

*Example 6.* Consider the model of neutron kinetics [3], described via the state equations as

$$\begin{aligned} x_1^{[1]} &= x_2 + b_1 u x_1, \\ x_2^{[1]} &= a_2 x_1 + (b_2 + a_1 b_1) u x_1, \\ y &= x_1, \end{aligned}$$

where $x_1$ denotes the population of neutrons, $x_2$ denotes the average population of precursor groups, $u$ is the reactivity, and $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Using the approach proposed in [14], one can compute the transfer function as

$$F(z) = \left(z^2 - b_1 u^{[1]} z - a_2 - (a_1 b_1 + b_2) u\right)^{-1} \left(b_1 y^{[1]} z + (a_1 b_1 + b_2) y\right).$$

Suppose that the reference model is

$$G(z) = z^{-2}.$$

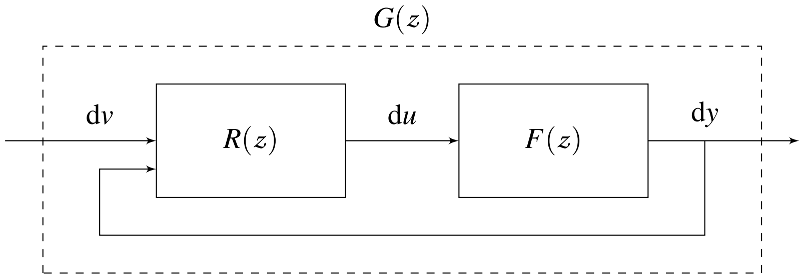


**Fig. 2.** Compensated system.

One can check that the feedforward solution does not exist. However, the problem is solvable via a feedback compensator. Indeed,

$$p_F(z) = z^2 - b_1 u^{[1]} z - a_2 - (a_1 b_1 + b_2)u, \qquad p_G(z) = z^2,$$
$$q_F(z) = b_1 y^{[1]} z + (a_1 b_1 + b_2)y, \qquad q_G(z) = 1.$$

The compensator $R(z)$ is determined by the polynomials

$$q_{R_v}(z) = 1,$$
$$q_{R_y}(z) = -b_1 u^{[1]} z - a_2 - (a_1 b_1 + b_2)u,$$
$$p_R(z) = b_1 y^{[1]} z + (a_1 b_1 + b_2)y.$$

Thus, the one-form, corresponding to the compensator $R$, is

$$b_1 y^{[1]} \mathrm{d}u^{[1]} + (a_1 b_1 + b_2)y\mathrm{d}u = \mathrm{d}v - b_1 u^{[1]} \mathrm{d}y^{[1]} - (a_2 + (a_1 b_1 + b_2)u)\mathrm{d}y. \tag{34}$$

Integrating (34) yields

$$u^{[1]} = \frac{v - a_2 y - (a_1 b_1 + b_2)uy}{b_1 y^{[1]}}.$$

*Example 7.* Recall that the feedforward solution did not exist in Example 4, where the transfer function was

$$F(z) = \left(z^2 - (\theta_1 + \theta_3 y^2)z + \theta_2 - 3\theta_4 y^2 - 2\theta_3 y y^{[1]}\right)^{-1} \theta_5.$$

Suppose that the reference model is

$$G(z) = z^{-2}$$

and calculate

$$p_F(z) = z^2 - \left(\theta_1 + \theta_3 y^2\right) z + \theta_2 - 3\theta_4 y^2 - 2\theta_3 y y^{[1]}, \qquad p_G(z) = z^2,$$
$$q_F(z) = \theta_5, \qquad q_G(z) = 1.$$

Using the left Euclidean division algorithm, we get $\gamma(z) = 1$ and $q_{R_y}(z) = -\left(\theta_1 + \theta_3 y^2\right) z + \theta_2 - 3\theta_4 y^2 - 2\theta_3 y y^{[1]}$ such that $p_G(z) = \gamma(z) p_F(z) - q_{R_y}(z)$. Then, the integrable one-form, corresponding to the compensator $R$, is

$$\theta_5 \mathrm{d}u = \mathrm{d}v - \left(\theta_1 + \theta_3 y^2\right) \mathrm{d}y^{[1]} + \left(\theta_2 - 3\theta_4 y^2 - 2\theta_3 y y^{[1]}\right) \mathrm{d}y,$$

yielding

$$u = \frac{1}{\theta_5} \left(v - (\theta_1 + \theta_3 y^2) y^{[1]} + \theta_2 y - \theta_4 y^3\right).$$

*Example 8.* Consider the model of a grain drying process by a column-type grain dryer [26]

$$y^{[3]} = 1.6332 y^{[2]} - 0.4567 y^{[1]} - 0.1751 y - 0.0081 u^{[2]} y^{[2]} - 0.0045 u^{[1]} y^{[1]} - 0.0073 uy,$$

where $y$ is the temperature in the uppermost layer of the dryer and $u$ is the productivity of the grain exhaust mechanism. Compute the transfer function as

$$F(z) = \left(z^3 + \left(-1.6332 + 0.0081 u^{[2]}\right) z^2 + \left(0.4567 + 0.0045 u^{[1]}\right) z + 0.0073u + 0.1751\right)^{-1}$$
$$\times \left(-0.0081 y^{[2]} z^2 - 0.0045 y^{[1]} z - 0.0073 y\right)$$

and suppose that the reference model is

$$G(z) = z^{-3}.$$

In the same manner as in the previous example, we can find the compensator described by the following equation:

$$u^{[2]} = -\frac{v + 0.1751y + 0.0073uy + 0.4567y^{[1]} + 0.0045u^{[1]}y^{[1]} - 1.6332y^{[2]}}{0.0081y^{[2]}}.$$

Next, we provide a brief comparison of the feedback solution with the results from [25]. Compared to our case, the paper [25] addresses only the case of a proper feedback compensator. Moreover, the solution (for the multi-input multi-output, MIMO, case) is based on the application of the implicit function theorem, and therefore, is constructive only up to the application of this theorem. However, this is not a problem in the SISO case. Next, we consider a simple example to compare both approaches.

*Example 9.* Consider the system and the model from Example 3. Following the proof of Theorem 14, we can find $\gamma(z) = 1$, $q_{R_y}(z) = -1$, $q_{R_v}(z) = q_G(z) = 1$, $p_R(z) = \gamma(z)q_F(z) = uz + u^{[1]}$, and

$$\left(uz + u^{[1]}\right) \mathrm{d}u = \mathrm{d}v - \mathrm{d}y,$$

yielding the equation of the compensator

$$uu^{[1]} = v - y. \tag{35}$$

Finally, note that the compensator has the following state-space realization:

$$\begin{aligned} u &= \eta, \\ \eta^{[1]} &= \frac{v - y}{\eta}. \end{aligned}$$

Now, we illustrate how to use the approach from [25]. Note that the reference model is described by the equation $y^{[2]} = v$ that corresponds to the transfer function $G(z) = 1/z^2$. Therefore, equating the right-hand sides of the system and that of the model yields $v = y + uu^{[1]}$, i.e., (35), which is exactly the same obtained above based on the transfer function approach.

## 5. CONCLUSIONS

The paper addresses the model matching problem for nonlinear SISO discrete-time systems. The recently developed formalism based on the generalization of the notion of a transfer function to the case of nonlinear systems is applied. Both feedforward and feedback solutions are studied. In the case of a feedforward compensator, the solvability of the problem depends critically on the integrability of a certain one-form exactly like in the continuous-time case [18]. Since, in general, the problem is not solvable, we single out a subclass of discrete-time nonlinear systems for which it is always possible to construct a feedforward compensator. In contrast to the feedforward solution, it is shown that the feedback solution always exists. In the majority of cases one is interested in finding proper compensators. Therefore, additional restrictions are specified, depending on the degrees of the polynomials corresponding to the system and model under which the proper solution exists.

One possible direction for the future extension of this work is to solve the MMP for MIMO systems. Like in the linear time-varying case [31], the procedure to derive the structure at the infinity of a transfer matrix has to be introduced. For that purpose one may use the special form of the matrix with rational elements, called the Jacobson–Teichmüller[3] form. This is a three-step procedure. First, one has to transform a matrix into the form with elements from the skew polynomial ring. Then, using the basic algorithm from [10], one

[3] Note that in the linear control theory this form is known as the Smith–McMillan form, see [20].

can transform a matrix into the Jacobson form (being the special case of the Jacobson–Teichmüller form). A software implementation is discussed in [4]. On the one hand, the matrix in the Jacobson form becomes very complex even for simple low-order systems and computations become tedious as it was shown in [4] and [7]. On the other hand, this approach mimics the linear case, and therefore, provides a formalism intuitively understandable by many engineers. The last step requires the transformation of the Jacobson form into the Jacobson–Teichmüller form. It should be mentioned that it is still unclear how to perform this step. To conclude, it requires further study which approach to use in the MIMO case – whether [14] or [25].

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APPENDIX

## PROOF OF PROPOSITION 11

*Proof. Necessity*: Assume that there exists a proper transfer function $R(z)$ of the compensator $R$ that solves the MMP. By Definition 8, this means that

$$\deg p_R(z) \geq \deg q_R(z). \tag{36}$$

Next, using the relation

$$G(z) = p_G^{-1}(z)q_G(z) = p_F^{-1}(z)q_F(z)p_R^{-1}(z)q_R(z) = F(z)R(z)$$

and condition (2) of Proposition 6, we get

$$\begin{aligned} \deg q_G(z) &= \deg q_F(z) + \deg q_R(z), \\ \deg p_G(z) &= \deg p_F(z) + \deg p_R(z). \end{aligned} \tag{37}$$

Substituting (37) into (36), we obtain

$$\deg p_G(z) - \deg p_F(z) \geq \deg q_G(z) - \deg q_F(z)$$

or

$$\deg p_G(z) - \deg q_G(z) \geq \deg p_F(z) - \deg q_F(z).$$

Finally, according to Definition 9,

$$\begin{aligned} \operatorname{rel}\deg G(z) &= \deg p_G(z) - \deg q_G(z), \\ \operatorname{rel}\deg F(z) &= \deg p_F(z) - \deg q_F(z) \end{aligned}$$

that yields (17).

*Sufficiency*: Assume that (17) holds. Since all the previous steps can be done in the reverse order, we get that the transfer function $R(z)$ is proper. □

## PROOF OF PROPOSITION 12

*Proof.* By differentiating Eqs (22), (23) and using relations (6) together with $z^k \mathrm{d}v = \mathrm{d}v^{[k]}$, we get (12), where

$$p_F(z) = z^{n_F} - \sum_{i=0}^{n_F-1} p_i^F z^i, \qquad p_i^F = \frac{\partial f_1}{\partial y^{[i]}},$$
$$q_F(z) = \sum_{j=0}^{s_F} q_j^F z^j, \qquad q_j^F = \frac{\partial f_2}{\partial u^{[j]}}$$

and (13), where

$$p_G(z) = z^{n_G} - \sum_{i=0}^{n_G-1} p_i^G z^i, \qquad p_i^G = \frac{\partial g_1}{\partial y^{[i]}},$$
$$q_G(z) = \sum_{j=0}^{s_G} q_j^G z^j, \qquad q_j^G = \frac{\partial g_2}{\partial v^{[j]}},$$

respectively. Note that now in (16), $p_R(z) = \beta(z)q_F(z)$ and $q_R(z) = \alpha(z)q_G(z)$, where $\alpha(z)$, $\beta(z)$ are polynomials defined by the left Ore condition as $\beta(z)p_F(z) = \alpha(z)p_G(z)$. According to condition (24), the previous equality can be rewritten as $\beta(z)\gamma_F(z)\rho(z) = \alpha(z)\gamma_G(z)\rho(z)$ or $\beta(z)\gamma_F(z) = \alpha(z)\gamma_G(z)$, where $\gamma_F(z), \gamma_G(z)$ can be represented as $\gamma(z) = \sum_{i=0}^{\tau} \gamma_i z^{\tau-i}$, $\gamma_i \in \mathbb{R}$. So, it follows that $\alpha(z)$ and $\beta(z)$ are also polynomials with real coefficients.

Next, the relationship (16) can be rewritten as

$$\beta(z)q_F(z)\mathrm{d}u = \alpha(z)q_G(z)\mathrm{d}v. \tag{38}$$

Note that the coefficients of the polynomials $q_F(z)$ and $q_G(z)$ do not depend on $y$, proving the exactness of the one-form (38). □

## PROOF OF THEOREM 14

*Proof.* By (32),

$$G(z) = (1 - F(z)R_y(z))^{-1} F(z)R_v(z).$$

Next, using (12), (30), and (31), $G(z)$ may be rewritten in the form

$$G(z) = \left(1 - p_F^{-1}(z)q_F(z)p_R^{-1}(z)q_{R_y}(z)\right)^{-1} \left(p_F^{-1}(z)q_F(z)p_R^{-1}(z)q_{R_v}(z)\right).$$

By multiplying the numerator and denominator of $G(z)$ by the expression $p_R(z)q_F^{-1}(z)p_F(z)$ from the left one gets

$$G(z) = \left(p_R(z)q_F^{-1}(z)p_F(z) - q_{R_y}(z)\right)^{-1} q_{R_v}(z).$$

Matching the latter to (13) results in

$$q_G(z) = q_{R_v}(z), \quad p_G(z) = p_R(z)q_F^{-1}(z)p_F(z) - q_{R_y}(z).$$

One may choose $p_R(z)$ to be $\gamma(z)q_F(z)$, yielding

$$p_G(z) = \gamma(z)p_F(z) - q_{R_y}(z).$$

Under Assumption 13, $\gamma(z)$ and $-q_{R_y}(z)$ may be interpreted as the right quotient and remainder of skew polynomials $p_G(z)$ and $p_F(z)$, respectively. Thus, from given $p_G(z)$ and $p_F(z)$ one can, by the left Euclidean division algorithm, determine the infinitesimal description of the compensator

$$\mathrm{d}u = R_v(z)\mathrm{d}v + R_y(z)\mathrm{d}y,$$

written alternatively as

$$p_R(z)\mathrm{d}u = q_{R_v}(z)\mathrm{d}v + q_{R_y}(z)\mathrm{d}y \tag{39}$$

with $p_G(z) = \gamma(z)p_F(z) - q_{R_y}(z)$, $p_R(z) = \gamma(z)q_F(z)$, $q_{R_v}(z) = q_G(z)$.

Unlike the case of feedforward solution, now the one-form (39) is always integrable. Indeed, the relation (39) can be rewritten as

$$\gamma(z)q_F(z)\mathrm{d}u = q_G(z)\mathrm{d}v + (\gamma(z)p_F(z) - p_G(z))\mathrm{d}y$$

or as

$$\gamma(z)(q_F(z)\mathrm{d}u - p_F(z)\mathrm{d}y) = q_G(z)\mathrm{d}v - p_G(z)\mathrm{d}y.$$

Observe that both one-forms $q_F(z)\mathrm{d}u - p_F(z)\mathrm{d}y$ and $q_G(z)\mathrm{d}v - p_G(z)\mathrm{d}y$ are exact, since they correspond to the system $F$ and model $G$, described by (12) and (13), respectively. Finally, applying $\gamma(z)$ to an exact one-form results again in an exact one-form. □

## PROOF OF PROPOSITION 15

*Proof. Necessity*: Assume that the transfer function $R(z)$ of the feedback compensator is proper, which by Definition 8 means

$$\deg p_R(z) \geq \deg q_{R_v}(z).$$

Next, taking into account that $p_R(z) = \gamma(z)q_F(z)$, $q_G(z) = q_{R_v}(z)$ (see the proof of Proposition 11), and using the condition (2) of Proposition 6, the previous inequality can be rewritten as

$$\deg \gamma(z) + \deg q_F(z) \geq \deg q_G(z).$$

Add $\deg p_F(z)$ to both sides and regroup the terms to obtain

$$\deg \gamma(z) + \deg p_F(z) - \deg q_G(z) \geq \deg p_F(z) - \deg q_F(z). \tag{40}$$

Since $p_G(z) = \gamma(z)p_F(z) - q_{R_y}(z)$ and $\deg q_{R_y}(z) = 0$, from (40) we get

$$\deg p_G(z) - \deg q_G(z) \geq \deg p_F(z) - \deg q_F(z).$$

Finally, according to Definition 9,

$$\begin{aligned}\operatorname{rel}\deg G(z) &= \deg p_G(z) - \deg q_G(z),\\ \operatorname{rel}\deg F(z) &= \deg p_F(z) - \deg q_F(z)\end{aligned}$$

that yields (33).

*Sufficiency*: The fact that all the steps in the necessity part of the proof can be done in the reverse order proves the sufficiency. □

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## Diskreetsete mittelineaarsete juhtimissüsteemide mudeliga sobitamise ülesanne

Juri Belikov, Miroslav Halás, Ülle Kotta ja Claude H. Moog

On uuritud mudeliga sobitamise ülesannet mittelineaarsete diskreetsete ühe sisendi ja ühe väljundiga juhtimissüsteemide jaoks. Meetod põhineb süsteemi nn globaalsel lineariseerimisel, kus süsteemile seatakse vastavusse diferentsiaalne 1-vorm, mis seejärel esitatakse kahe polünoomi abil. Viimane on võimalik, kui polünoomi muutujat interpreteerida edasinihkeoperaatorina 1-vormide vektorruumis. Seejärel leitakse viimasest esitusest süsteemi üldistatud ülekandefunktsioon. Uuritava probleemi püstitus ja lahendus on antud üldistatud ülekandefunktsiooni abil. Lahendit otsitakse kahel kujul: otsesidena ja tagasisidena. Lahend otseside kujul eksisteerib rangetel tingimustel. Seetõttu on artiklis leitud süsteemide alamklass, mille korral lahend eksisteerib garanteeritult. On tõestatud, et lahend tagasiside kujul eksisteerib alati. Mõlema juhu jaoks on leitud lisatingimused, mille täidetuse korral leitud kompensaator on kausaalne. Artikli tulemusi on illustreeritud arvukate näidetega ja tagasiside kujul leitud lahendit on võrreldud varem teisel meetodil saaduga.