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NONLINEAR WAVES

Multisoliton interactions for the Manakov system under composite external potentials

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Abstract. The soliton interactions of Manakov soliton trains subjected to composite external potentials are modelled by the perturbed complex Toda chain (PCTC). The model is applied to several classes of potentials, such as: (i) harmonic, (ii) periodic, (iii) 'wide well'-type potentials, and (iv) inter-channel interactions. We demonstrate that the potentials can change the asymptotic regimes of the soliton trains. Our results can be implemented, e.g., in experiments on Bose–Einstein condensates and can be used to control the soliton motion. In general, our numerical experiments demonstrate that the predictions of complex Toda chain (CTC) (respectively PCTC) match very well the Manakov (respectively perturbed Manakov) model numerics for long-time evolution, often much longer than expected. This means that both CTC and PCTC are reliable *dynamical* models for predicting the dynamics of the multisoliton trains of the Manakov model in adiabatic approximation. This extends our previous results on scalar soliton trains to the Manakov trains with compatible initial parameters.

Key words: perturbed complex Toda chain, Manakov system, adiabatic interaction, breathing of multisoliton trains.

1. INTRODUCTION

The Gross–Pitaevski (GP) equation and its multicomponent generalizations are important tools for analysing and studying the dynamics of the Bose–Einstein condensates (BEC), see the monographs [11,15,24], the review papers [3,4,14], and the numerous references therein. Among them we mention [5,6,12,19,21,22,29]; for physical relevance and applications see [17,23,31]. In the 3-dimensional case these equations can be analysed only numerically. For the quasi-one-dimensional BEC the GP equations reduces either to the scalar nonlinear Schrödinger equation (NLSE) perturbed by the external potential $V(x)$:

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + |u|^2 u(x,t) = V(x)u(x,t), \tag{1}$$

or to the Manakov model (MM) [18], perturbed not only by $V(x)$

$$i\vec{u}_t + \frac{1}{2}\vec{u}_{xx} + (\vec{u}^\dagger, \vec{u})\vec{u}(x,t) = V(x)\vec{u}(x,t) + c_1\sigma_1\vec{u}(x,t), \tag{2}$$

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but also by the interchannel interaction $c_1 \neq 0$. Here the vector function $\vec{u} = (u_1, u_2)^{\mathrm{T}}$ and $\vec{u}^{\dagger} = (u_1^*, u_2^*)$ is hermitian conjugate to $\vec{u}$. Then $(\vec{u}^{\dagger}, \vec{u})$ is the scalar product of $\vec{u}^{\dagger}$ and $\vec{u}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The analytical approach to the N-soliton interactions was proposed by Zakharov and Shabat [20,30] for the scalar NLSE. They calculated the asymptotics of the exact N-soliton solution for $t \to \pm\infty$, assuming that all solitons move with different velocities. As a result, they proved that both asymptotics are sums of N one-soliton solutions with the same sets of amplitudes and velocities. The effects of the interaction were shifts in the relative centre of masses and phases of the solitons. The same approach, however, is not applicable to the MM, because the asymptotics of the soliton solution for $t \to \pm\infty$ do not commute.

The N-soliton interactions in the adiabatic approximation for the MM ($V(x) = 0$) can be modelled by CTC [10]. When $V(x) \neq 0$, a perturbed CTC (PCTC) is derived for several types of potentials: i) harmonic, ii) periodic, and iii) shallow wide well type potentials; see [8,16,28] and literature, cited there and also for the inter-channel interactions.

Below we will consider also the effects of potentials of the form:

$$V_{\mathrm{hp}} = V_2 x^2 + V_1 x + V_0 + A\cos(\Omega x), \qquad V(x) = \sum_{s=0}^{N} c_s V_s(x), \qquad \text{with} \qquad V_s = \mathrm{sech}^2(x - x_s), \tag{3}$$

which are wells (resp. humps) for $c_s < 0$ (resp. $c_s > 0$). We will also consider wide well-like potentials

$$V_{1\mathrm{ww}}(y_{\mathrm{i}}, y_{\mathrm{f}}) = \int_{y_{\mathrm{i}}}^{y_{\mathrm{f}}} cV_s(x)dx_s = c[\tanh(x - y_{\mathrm{f}}) - \tanh(x - y_{\mathrm{i}})], \tag{4}$$

and well-in-well potentials like $V_{2\mathrm{ww}} = cV_{1\mathrm{ww}}(y_{\mathrm{i}}', y_{\mathrm{f}}') + cV_{1\mathrm{ww}}(y_{\mathrm{i}}, y_{\mathrm{f}})$ with $y_{\mathrm{i}}' \ll y_{\mathrm{i}}$ and $y_{\mathrm{f}}' \gg y_{\mathrm{f}}$ (Fig. 1).

The Manakov soliton train is a special solution of the Cauchy problem for Eq. (2) with the initial condition

$$\vec{u}(x, t = 0) = \sum_{k=1}^{N} u_k(x, t = 0)\vec{n}_k, \qquad u_k(x, t) = \frac{2\nu_k e^{i\phi_k}}{\cosh(z_k)}, \tag{5}$$

where

$$z_k = 2\nu_k(x - \xi_k(t)), \qquad \xi_k(t) = 2\mu_k t + \xi_{k,0}, \qquad \phi_k = \frac{\mu_k}{\nu_k} z_k + \delta_k(t), \qquad \delta_k(t) = 2(\mu_k^2 + \nu_k^2)t + \delta_{k,0}, \tag{6}$$

μ_k are initial velocities, ν_k are initial amplitudes, $\delta_{k,0}$ are initial phases, and $\xi_{k,0}$ are initial positions of the soliton.

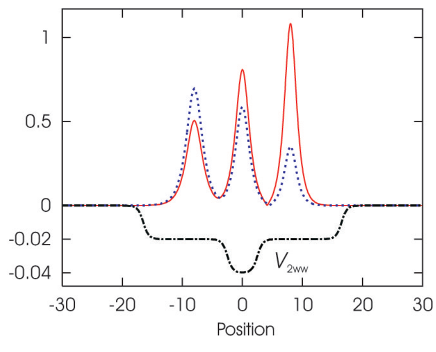

Fig. 1. Graph of the two-level external potential $V_{2ww} \equiv cV_{1\mathrm{ww}}(-16, 16) + cV_{1\mathrm{ww}}(-3, 3)$, $c = -0.01$ from Eq. (4) (cyan line) and the initial configuration of three-soliton envelopes located at $\xi_k = -8 + 8(k-1)$, $k = 1, 2, 3$ and the modules of the first (solid) and second (dashed) components of $\vec{u}$.

The polarization vectors $\vec{n}_k = \left(n_{k,1}e^{i\beta_k}, n_{k,2}e^{-i\beta_k}\right)^T$ are normalized by the conditions

$$\langle \vec{n}_k^\dagger, \vec{n}_k \rangle \equiv n_{k,1}^2 + n_{k,2}^2 = 1, \tag{7}$$

where $\vec{n}_k^\dagger$ stands for the hermitian conjugate quantity $\langle \vec{n}_k^\dagger| = (n_k^{1,*}, n_k^{2,*})$. The adiabatic approximation holds true for both equations if the soliton parameters satisfy [13]:

$$|\nu_k - \nu_0| \ll \nu_0, \quad |\mu_k - \mu_0| \ll \mu_0, \quad |\nu_k - \nu_0||\xi_{k+1,0} - \xi_{k,0}| \gg 1, \tag{8}$$

for all k, where $\nu_0 = \frac{1}{N}\sum_{k=1}^N \nu_k$, and $\mu_0 = \frac{1}{N}\sum_{k=1}^N \mu_k$ are the average amplitude and velocity, respectively. In fact we have two different scales:

$$|\nu_k - \nu_0| \simeq \varepsilon_0^{1/2}, \quad |\mu_k - \mu_0| \simeq \varepsilon_0^{1/2}, \quad |\xi_{k+1,0} - \xi_{k,0}| \simeq \varepsilon_0^{-1/2},$$

where $\varepsilon_0 \simeq 8\nu_0 r_0 e^{-2\nu_0 r_0}$ and r_0 is the distance between the neighbouring solitons.

Following Karpman and Solov′ev [13], we derive a dynamical system for the soliton parameters which describes their interaction. Using the approach by Anderson and Lisak [1,2], this idea was generalized to N-soliton interactions of scalar NLSE solitons [10] and then to the MM [5,7,9].

The paper is organized as follows. In Section 2 we formulate the PCTC model describing the N-soliton train in external potentials (harmonic, periodic, and wide wells). We also treat the inter-channel interactions. In Section 3 we compare the numerical solutions of the perturbed MM equation with the solution of the relevant PCTC and find a very good description for several important configurations of the soliton trains. We end with conclusions and briefly discuss further problems to be solved.

2. THE EFFECTS OF THE EXTERNAL POTENTIALS – THEORETICAL ASPECTS

It is well known that the scalar soliton trains in external potentials are modelled by the PCTC [5,10,28]. The MM is treated similarly to the scalar NLSE. According to [5], CTC, describing the evolution of the trains of Manakov solitons must be modified by attaching the scalar products of the relevant polarization vectors to the exponential factors. The modifications needed to account for the external potentials, due to the normalization condition (7) in fact must coincide with the ones for the scalar case. As a result we obtain the following PCTC system:

$$\begin{aligned} \frac{d\lambda_k}{dt} &= -4\nu_0 \left(e^{q_{k+1}-q_k}(\vec{n}_{k+1}^\dagger, \vec{n}_k) - e^{q_k - q_{k-1}}(\vec{n}_k^\dagger, \vec{n}_{k-1}) \right) + M_k + iN_k, \\ \frac{dq_k}{dt} &= -4\nu_0\lambda_k + 2i(\mu_0 + i\nu_0)\Xi_k - iX_k, \qquad \frac{d\vec{n}_k}{dt} = \mathscr{O}(\varepsilon), \end{aligned} \tag{9}$$

$$q_k = -2\nu_0\xi_k + k\ln(4\nu_0^2) - i(\delta_k + \delta_0 + k\pi - 2\mu_0\xi_k), \qquad \lambda_k = \mu_k + i\nu_k, \qquad \delta_0 = \frac{1}{N}\sum_{k=1}^N \delta_k. \tag{10}$$

The integrals, characterizing the effect of the perturbations, are:

$$N_k = -\frac{1}{2}\int_{-\infty}^{\infty} \frac{dz_k}{\cosh z_k} \mathrm{Im}\,\left(V(y_k)u_k \mathrm{e}^{-i\phi_k}\right), \qquad M_k = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dz_k \sinh z_k}{\cosh^2 z_k} \mathrm{Re}\,\left(V(y_k)u_k \mathrm{e}^{-i\phi_k}\right), \tag{11}$$

$$\Xi_k = -\frac{1}{4\nu_k^2}\int_{-\infty}^{\infty} \frac{dz_k\, z_k}{\cosh z_k} \mathrm{Im}\,\left(V(y_k)u_k \mathrm{e}^{-i\phi_k}\right), \qquad D_k = \frac{1}{2\nu_k}\int_{-\infty}^{\infty} \frac{dz_k\,(1 - z_k\tanh z_k)}{\cosh z_k} \mathrm{Re}\,\left(V(y_k)u_k \mathrm{e}^{-i\phi_k}\right), \tag{12}$$

where $y_k = z_k/(2\nu_0) + \xi_k$ and $X_k = 2\mu_k\Xi_k + D_k$. Obviously these integrals vanish when the external potential is not present. Along with PCTC we must take into account also the evolution of the polarization vectors.

For the potentials, described above we found that the evolution of the $|\vec{n}_k\rangle$ is of the order of ε. Also the evolution of the scalar products $(\vec{n}^{\dagger}_{k+1}, \vec{n}_k)$ will deviate from their initial values by terms of the order of ε. But $(\vec{n}^{\dagger}_{k+1}, \vec{n}_k)$ multiply exponentially small terms whose modules $|e^{q_{k+1}-q_k}| \simeq \varepsilon$. Therefore the evolution of the polarization vectors can be neglected and we replace the scalar products $(\vec{n}^{\dagger}_{k+1}, \vec{n}_k)$ by their initial values. Note that N_k, M_k, Ξ_k and P_k depend only on the parameters of the kth soliton; i.e., they are 'local' in k.

2.1. Harmonic and periodic potentials

For the class of harmonic and periodic potentials (3) we have (see, for example [5,8,9]):

$$\begin{aligned} &N_k[u] = 0, \quad M_k[u] = -\frac{1}{4\nu_k}(V_1 + 2V_2\xi_k) + \frac{\pi A\Omega^2}{8\nu_k \sinh Z_k}\sin(\Omega\xi_k + \Omega_0), \\ &\Xi_k[u] = 0, \quad D_k[u] = -\frac{1}{2}V(\xi_k) + \frac{\pi^2 V_2}{96\nu_k^2} - \frac{\pi^2 A\Omega^2}{16\nu_k^2}\frac{\cosh Z_k}{\sinh^2 Z_k}\cos(\Omega\xi_k + \Omega_0), \end{aligned} \tag{13}$$

where $Z_k = \Omega\pi/(4\nu_k)$.

2.2. Wide well-like potentials

The 'narrow' sech-like potentials V_s, Eq. (3) have been treated in [8,9] with the result

$$M_k = 2c_s\nu_k P(\Delta_{k,s}), \qquad N_k = 0, \qquad \Xi_k = 0, \qquad D_k = c_s R(\Delta_{k,s}), \tag{14}$$

where $\Delta_{k,s} = 2\nu_0\xi_k - y_s$ and the integrals describing the interaction of the solitons are equal to:

$$P(\Delta) = \frac{\Delta + 2\Delta\cosh^2(\Delta) - 3\sinh(\Delta)\cosh(\Delta)}{\sinh^4(\Delta)}, \qquad R(\Delta) = \frac{\Delta\sinh(2\Delta) - (2\Delta^2+3)\sinh^2(\Delta) - 3\Delta^2}{2\sinh^4(\Delta)}. \tag{15}$$

The shallow but wide well-like potentials (4) (Fig. 2) are obtained by integrating over Δ:

$$P_0(\Delta) = \frac{\sinh(\Delta) - \Delta\cosh(\Delta)}{\sinh^3(\Delta)}, \qquad R_0(\Delta) = \frac{e^{-\Delta}\sinh^2(\Delta) + \Delta^2\cosh(\Delta) - 2\Delta\sinh(\Delta)}{2\sinh^3(\Delta)}. \tag{16}$$

Then M_k and D_k in Eq. (14) must be replaced by

$$M_{0,k} = 2c_0\nu_k\left[P_0(z_k - y_{\mathrm{f}}) - P_0(z_k - y_{\mathrm{f}})\right], \qquad D_{0,k} = \frac{c_0}{2\nu_0}\left[R_0(z_k - y_{\mathrm{f}}) - R_0(z_k - y_{\mathrm{f}})\right]. \tag{17}$$

2.3. Interchannel interactions. Linear coupling

The interchannel interactions, called also linear coupling between the components of $\vec{u}$, are treated analogously and also lead to PCTC, in which the evolution of the polarization vectors depends on c_1 as follows:

$$\begin{aligned} &i\frac{\partial n_{k,1}}{\partial t} - c_1\nu_k n_{k,2} + \mathscr{O}(\varepsilon) = 0, \\ &i\frac{\partial n_{k,2}}{\partial t} - c_1\nu_k n_{k,1} + \mathscr{O}(\varepsilon) = 0. \end{aligned} \tag{18}$$

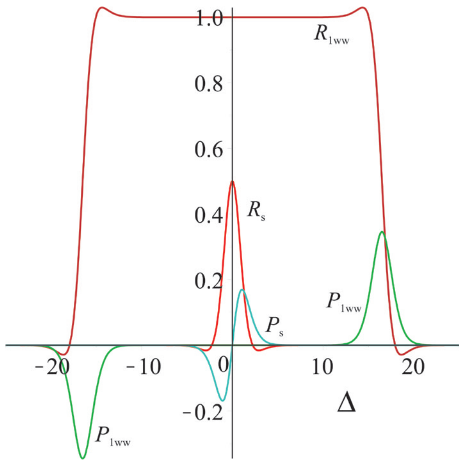

Fig. 2. The functions P and R for a single sech-potential centered at the origin, and for potential $V_{1ww}(-16,16)$.

In fact, if we keep only terms of the order of ε in PCTC, we can drop also the terms $\nu_k - \nu_0 \simeq \sqrt{\varepsilon}$ in the evolution of $|\vec{n}_k\rangle$ and replace ν_k by ν_0. Then the solution of (18) is

$$\vec{n}_k(t) = (\cos(c_1\nu_0 t)\mathbb{1} - i\sin(c_1\nu_0 t)\sigma_1)\vec{n}_k(0). \tag{19}$$

Since we assumed the constant c_1 to be real, this means that: i) $(\vec{n}_k^\dagger(t), \vec{n}_k(t)) = (\vec{n}_k^\dagger(0), \vec{n}_k(0)) = 1$, i.e., the unit norm of each of the polarization vectors is preserved, and ii) $i\frac{\partial}{\partial t}(\vec{n}_{k+1}^\dagger(t), \vec{n}_k(t)) = 0$, i.e.,

$$(\vec{n}_{k+1}^\dagger(t), \vec{n}_k(t)) = (\vec{n}_{k+1}^\dagger(0), \vec{n}_k(0)) + \mathscr{O}(\varepsilon). \tag{20}$$

As a result, we can replace the scalar products of the PCTC by their initial values in the case of inter-channel interactions as well. Note that if we treat only purely inter-channel interactions, we obtain pure CTC, since all the additional integrals N_k, M_k, Ξ_k, D_k vanish. This is compatible with the fact that a change of variables takes away the inter-channel interactions (see [26] and literature cited there).

3. CTC AND THE ASYMPTOTIC REGIMES OF N-SOLITON TRAINS

The main advantage of the CTC is that it is an integrable dynamical model and admits the Lax representation $\dot{L} = [B, L]$. This allows one to predict the asymptotic behaviour of the solitons [10]. The CTC has N complex-valued integrals of motion provided by the eigenvalues $\zeta_k = \kappa_k + i\eta_k$, $k = 1, \ldots, N$ of L. One can show that Re ζ_k determine the asymptotic velocities of the solitons. Thus three types of asymptotic regimes are possible: asymptotically free regime (AFR) when $\kappa_k \neq \kappa_j$ for $k \neq j$, i.e., all the asymptotic velocities are different [6,10]; bound state regime (BSR) when $\kappa_1 = \cdots = \kappa_N = 0$. All soliton envelopes move with the same mean asymptotic velocity; mixed asymptotic regimes (MAR) when one or more groups of soliton envelopes move with the same mean asymptotic velocity; then they would form one (or more) bound state(s) and the rest of the particles will have free asymptotic motion.

The PCTC take into account the effects of external potentials. Generically they are not integrable and do not admit Lax representation. In order to solve them we use reliable numerical methods based on a fully implicit conservative difference scheme [26] for MM and Runge–Kutta procedure for PCTC [28]. Our main aim here is to find out soliton configurations which, due to the external potential, result in transition from one asymptotic regime to another. A typical choice of initial soliton parameters used below is:

$$\mu_k(0) = 0,\ \nu_k(0) = \frac{1}{2},\ \xi_{k+1}(0) - \xi_k(0) = r_0,\ \delta_{k+1}(0) - \delta_k(0) = \pi,\ \theta_{k+1}(0) - \theta_k(0) = \frac{\pi}{8},\ k = 1, \ldots, 5, \tag{21}$$

which ensures that the solitons go into AFR. The predictions of the MM are plotted in Figs 3 to 6 in solid, while the CTC and PCTC – in dashed.

3.1. Harmonic and periodic potentials

It is natural to expect that every harmonic potential will always constrain the AFR into a bound state regime. This can be viewed in Figs 3 and 4. The initial parameters of the 5-soliton train given by Eq. (21) ensures AFR (see the left panel of Fig. 3). The motion on the right panel is a periodic one. Here the center of mass of the soliton train at time $t = 0$ is $\xi_0(0) = 0$ and coincides with the minimum of the potential $V(x) = 0.000036x^2$. Therefore the central soliton remains at rest, while the other solitons oscillate slightly around their initial positions.

The situation on the right panel of Fig. 4 is different, because the minimum of the potential $V(x) = 0.000036(x+15)^2$ is shifted with respect to $\xi_0(0)$. The motion is again a periodic one, but now it is $\xi_0(t)$ that oscillates. Indeed, we can sum up all the equations in the PCTC and derive the following approximate system for the centre of mass $\xi_0 = 1/N\sum_{k=1}^{N}\xi_k$ of the soliton train:

$$\frac{\partial \mu_0}{\partial t} = -\frac{V_1}{4\nu_0} - \frac{V_2}{2\nu_0}\xi_0(t), \qquad \frac{\partial \xi_0}{\partial t} = 2\mu_0(t), \qquad \frac{\partial \nu_0}{\partial t} = 0. \tag{22}$$

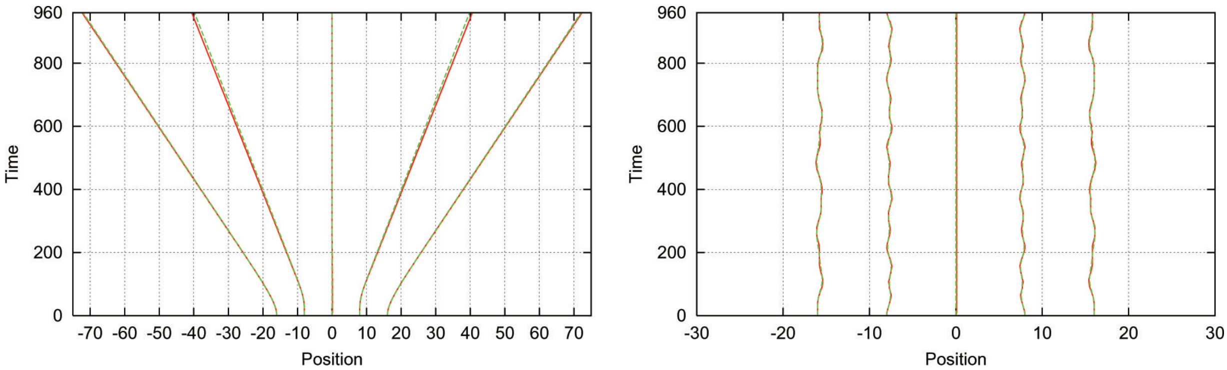

Fig. 3. Five-soliton train with initial parameters given by (21) with $r_0 = 8$, $\xi_k(0) = -24+8k$, $k = 1,\dots,5$ and $V(x) = 0$ (left panel); the same soliton train but with potential $V(x) = 0.000036x^2$ (right panel).

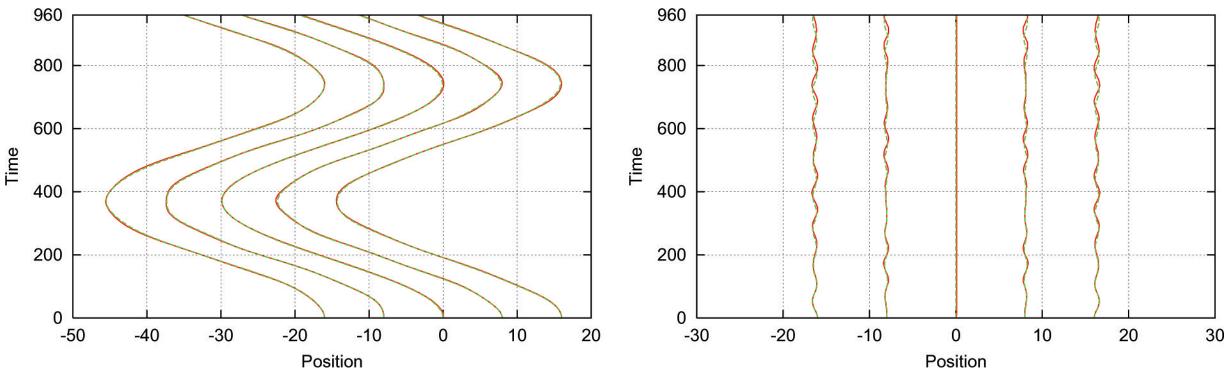

Fig. 4. Five-soliton train with initial parameters given by (21) with $r_0 = 8$, $\xi_k(0) = -24+8k$, $k = 1,\dots,5$ and $V(x) = 0.000036(x+15)^2$ (left panel); periodic potential on 5-soliton trains with $r_0 = 8$, $V(x) = -0.0060\cos(\pi x/4)$ (right panel).

The solution of this system is given by:

$$\xi_0(t) = \tilde{\xi}_{00}\cos(z_0 t) + \frac{2\mu_{00}}{z_0}\sin(z_0 t) - \frac{V_1}{2V_2}, \qquad \mu_0(t) = \mu_{00}\cos(z_0 t) - \frac{z_0}{2}\tilde{\xi}_0(0)\sin(z_0 t), \qquad \nu_0 = \nu_0(0),$$

where $z_0 = \sqrt{V_2/\nu_0}$, $\mu_{00} = \mu_0(0)$, and $\tilde{\xi}_0(0) = \xi_0(0) + V_1/(2V_2)$. The period of the motion is determined by V_2 only and equals $2\pi\sqrt{\nu_0/V_2}$. If $V(x) = 0.000036(x+15)^2$, we obtain $2\pi\sqrt{0.5/0.000036} \simeq 740.48$ which agrees very well with the left panel of Fig. 4.

The right panel of Fig. 4 demonstrates the stabilizing role of the periodic potentials provided very fine tuning is achieved. This includes: i) the period of the potential coincides with the distance between the neighbouring solitons $\Omega = 2\pi/r_0$ and ii) the initial positions of the solitons are located at the minima of the potential $V(x) = -A\cos(\Omega x)$. Then, if the potential strength is above some critical value, it will prevail the soliton repulsion and will pack them into a bound state.

3.2. Wide well-like potentials

Such potentials may be more practical because they do not require the fine tuning as, e.g., the periodic ones. Even for small intensities they can confine the solitons in their region. On the left panel of Fig. 5 we show 5-soliton train with initial conditions (21) in a weak but wide well-like potential $V(x) = -0.01V_{1\mathrm{ww}}(-24, 24)$. Even such shallow potential converts the asymptotic free regime into a bound state one and the solitons remain confined in the potential well (shaded region).

On the right panel of Fig. 5 we demonstrate the effects of a shallow well-in-well like potential. We find that the three central solitons are confined to the deeper well (doubly shadowed region) while the first and the fifth solitons remain in the shallower well (shadowed region).

3.3. Inter-channel interactions

Here we will pay more attention to the inter-channel effects. The effects will be more evident if we choose a three soliton configuration with initial parameters:

$$\begin{aligned} &\mu_k = 0, \quad \nu_{1,3} = \nu_2 \pm 0.07, \quad \nu_2 = 0.5, \\ &\delta_1 = 0, \quad \delta_{2,3} = \pm\frac{\pi}{2}, \quad \theta_k = \theta_{k-1} - \frac{\pi}{10}. \end{aligned} \tag{23}$$

The predictions of the MM and CTC are in good qualitative agreement (Fig. 6). In the next three figures we study the effects of the interchannel interactions on this three-soliton train. We plot the numerical solution

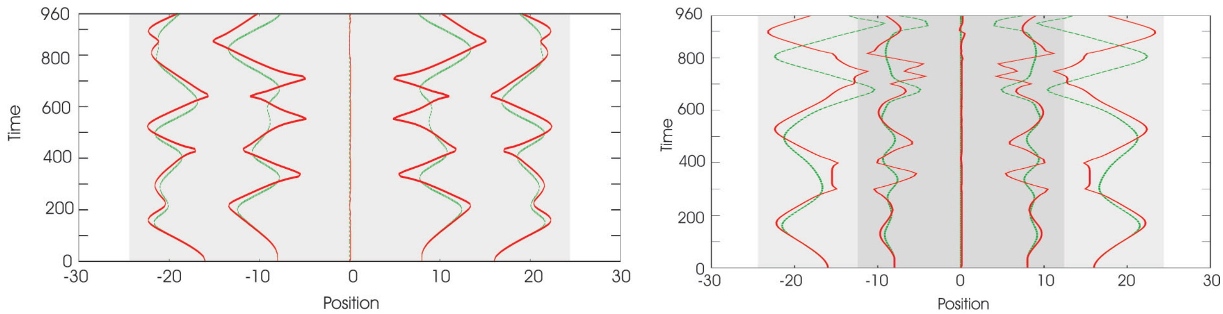

Fig. 5. Five-soliton train with initial parameters given by (21) with $r_0 = 8$, $\xi_k(0) = -24 + 8k$, $k = 1, \ldots, 5$ with $V(x) = -0.01V_{1\mathrm{ww}}(-24, 24)$ (left panel); the same but with $V(x) = -0.01(V_{1\mathrm{ww}}(-24, 24) + V_{1\mathrm{ww}}(-12, 12))$.

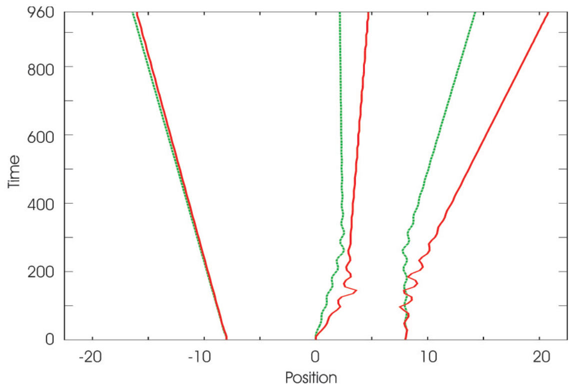

Fig. 6. Three solitons in AFR regime.

of the relevant MM with the initial conditions (23) drawing both components of each of the solitons. One can see that the amplitudes of the first (solid line) and the second components (dashed line) oscillate, but keep the normalization condition (7) irrespective of the strength of the inter-channel interaction (Figs 7, 8 and 9).

In Figs 10 and 11 we have plotted the oscillations of the polarization angles for the three solitons and the oscillations of the amplitudes of each component, respectively. The period is determined by the magnitude of the coefficient c_1 [25]. We can say, that in this case the solitons have 'breather'-like behaviour. Let us emphasize that it is possible only when c_1 is a real number [26,27]. The breathing behaviour is determined

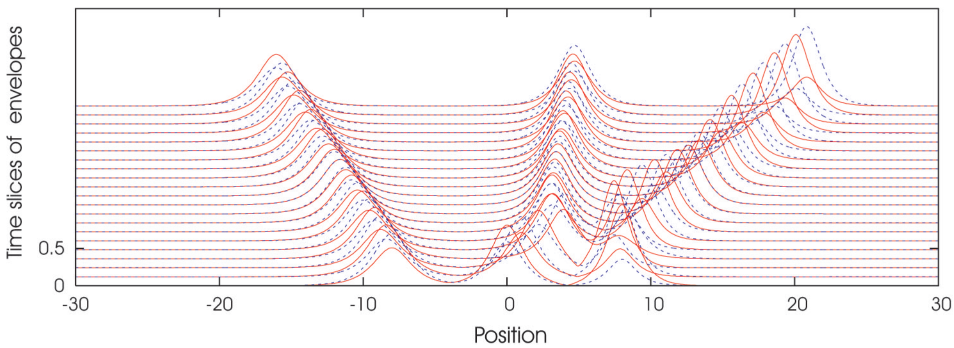

Fig. 7. Inter-channel interaction of a 3-soliton train with initial parameters as in (23) and $c_1 = 0.1$ for times up to 960.

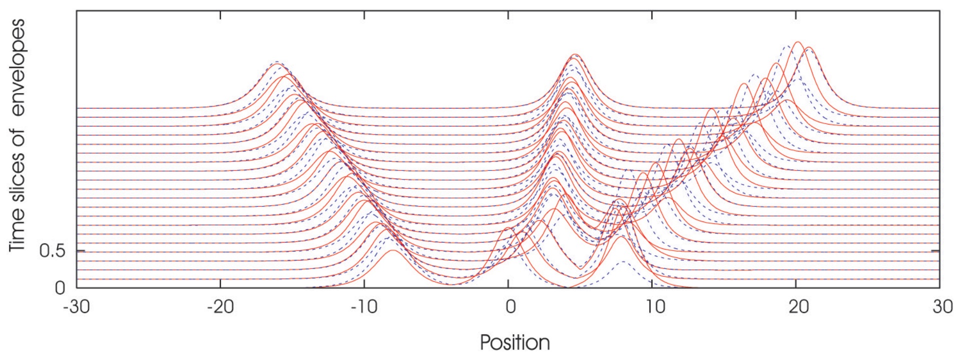

Fig. 8. Inter-channel interaction of a 3-soliton train with initial parameters as in (23) and $c_1 = 0.5$ for times up to 960.

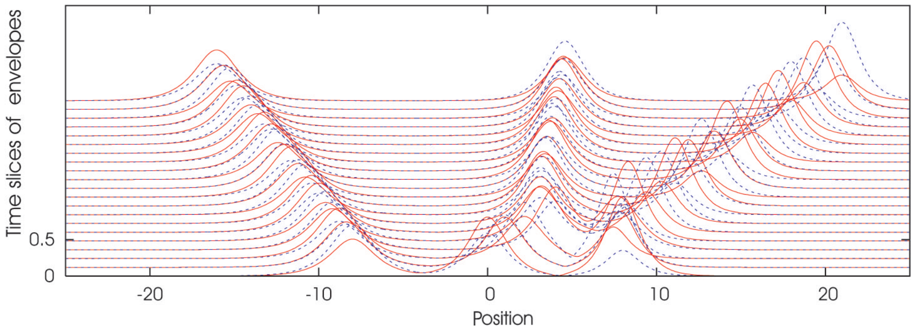

Fig. 9. Inter-channel interaction of a 3-soliton train with initial parameters as in (23) and $c_1 = 1$ for times up to 960.

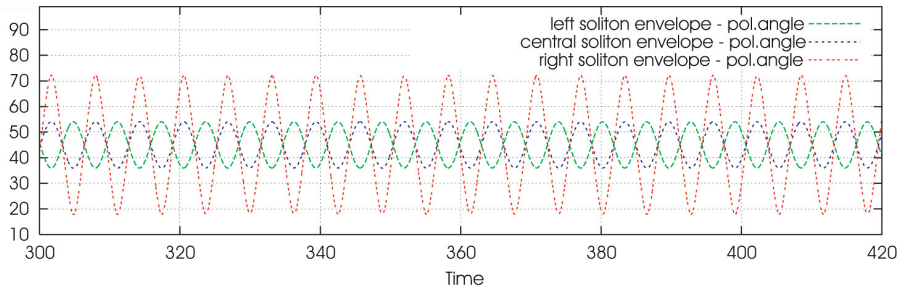

Fig. 10. Adiabatic approximation. Polarization angles (in degrees) of envelopes for $c_1 = 0.5$ as functions of the time.

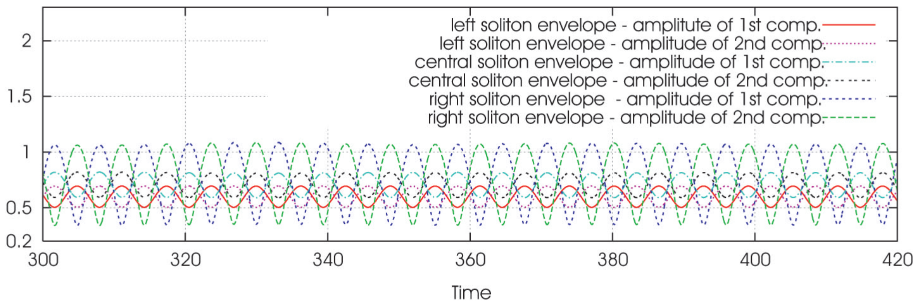

Fig. 11. Adiabatic approximation. Component amplitudes of breathing soliton envelopes for $c_1 = 0.5$ as functions of the time.

by the solution of additional equations (18) for polarization vectors in PCTC (Fig. 10). Obviously it has well presented asymptotic behaviour for big times (Fig. 11).

4. CONCLUSIONS

This paper is a natural extension of the results in [8,9,28] where various combinations of different types of external potentials were considered. In all cases we formulate the nonintegrable PCTC model [7,8], which has no Lax pair and therefore cannot be used for predicting the asymptotic behaviour of the soliton trains. One of our main results is to compare the PCTC with the perturbed MM and to demonstrate an excellent match between them for 5-soliton trains.

Another aspect, complementing our previous studies of PCTC consists in the analysis of inter-channel interactions. It undoubtedly enriches the phenomenology of the soliton interactions described by this model. The results are that the inter-channel interaction affects *only* the evolution of the the polarization vectors.

However, if the constant c_1 is real, the new evolution of $\vec{n}_k$ is an unitary one, preserving the scalar products $(\vec{n}^{\dagger}_{k+1}, \vec{n}_k)$ up to terms of the order of $\varepsilon^{1/2}$. Thus the main effect of the inter-channel interaction is the rotation of the polarization vectors, or in other words, the 'breathing' of the solitons.

A number of other configurations of the soliton trains and external potentials can be treated both analytically using PCTC and numerically by solving the corresponding perturbed MM. As such we mention the well-in-well potential in Fig. 1, quartic potentials and various combinations of them. Our experience shows that the PCTC provides adequate description for a large variety of soliton trains containing from 2 to 9 solitons and going into different asymptotic regimes. Of course, if we pick stronger potentials then the time of validity of the PCTC diminishes. In our simulations above we have demonstrated that we have very good match between the two models for times up to 960, which is about 10 times the magnitude of ε^{-1}. Of course such good matches do not hold true for all choices of the soliton parameters.

Considering the effects of the perturbations one needs a criterium which would ensure that a given perturbation (resp. given potential) can be considered adiabatically small. Obviously such criterium must depend not only on $V(x)$ but also on the initial conditions for the soliton train.

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Multisolitonide interaktsioonid välisele liitpotentsiaalile allutatud Manakovi tüüpi süsteemis

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Solitonide interaktsioonid Manakovi tüüpi solitonide jadas on välise liitpotentsiaali korral modelleeritud häiritusega kompleksse Toda ahela abil. Käesolevas artiklis vaatleme mitmeid potentsiaalide klasse ja näitame, et solitonide jada asümptootilised režiimid sõltuvad potentsiaali valikust. Meie saadud tulemusi saab rakendada näiteks Bose-Einsteini kondensaatidega tehtavatel eksperimentidel või solitonide liikumise juhtimisel. Meie numbrilised eksperimendid näitavad, et kompleksseest Toda ahelast saadavad hinnangud ühtivad pikkade ajavahemike jooksul väga hästi Manakovi mudeli numbrilise lahenduse tulemustega. Sama kehtib häiritusega kompleksse Toda ahela ja häiritusega Manakovi mudeli omavahelise sobivuse kohta. Teisisõnu, nii kompleksne Toda ahel kui ka häiritusega kompleksne Toda ahel on usaldusväärsed dünaamilised mudelid selleks, et anda hinnanguid Manakovi mudeli adiabaatilise aproksimatsiooni korral tekkiva solitonide jada dünaamika kohta. Saadud tulemused on laienduseks meie varem avaldatud tulemustele, kus on vaadeldud analoogilist probleemi skalaarse solitonide jada korral.