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MATHEMATICS

Factorable matrices and their associated Riesz matrices

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Abstract. A factorable matrix is a natural generalization of a Riesz matrix. When considering the properties of factorable matrices, many authors have used methods similar to the methods for Riesz matrices. So, a property having a long proof for Riesz matrices generated a long proof for a factorable matrix. In this paper for any factorable matrix we introduced its associated Riesz matrix. With its help many properties of a factorable matrix can be easily and briefly deduced from the corresponding properties of the associated Riesz matrix.

Key words: factorable matrices, Riesz matrices, Hahn properties, 0–1 sequences, summability domain, Tauberian theorems.

1. INTRODUCTION

One of the most thoroughly studied matrix methods in summability is a Riesz method. This is a summability method associated with a Riesz matrix which is defined as follows. Let $p = (p_n)$ be a real sequence with

$$p_1 > 0, \quad p_k \geq 0 \quad (k \in \mathbb{N}) \quad \text{and} \quad P_n := \sum_{k=1}^{n} p_k \quad (n \in \mathbb{N}).$$

The Riesz matrix $R_p = (r_{nk})$ (associated with p) is a lower triangular matrix with $r_{nk} := \frac{p_k}{P_n}$ if $k \leq n$. The Riesz matrix R_p is conservative and either regular (equivalent to $P_n \to \infty$) or coercive. In view of their simple structure, handling the properties of Riesz methods is relatively easy compared to other classes of matrix methods.

A natural generalization of a Riesz matrix is a factorable matrix. A factorable matrix is a lower triangular matrix with nonnegative entries $b_k c_n$, $0 \leq k \leq n$. In this paper we concentrate on conservative factorable matrices with $b_1 > 0$ and $c_n > 0$ $(n \in \mathbb{N})$.

When considering the properties of factorable matrices, many authors have used methods similar to the methods for Riesz matrices (see for example [1,7,9,10]), which often generated quite lengthy proofs. In this paper for each factorable matrix $A = (b_k c_n)$ we use the associated Riesz matrix R_b. We will see that many properties of a factorable matrix can be easily and briefly deduced from the properties of the associated Riesz matrix.

In Section 2 we consider the relation of summability domains of a factorable matrix and its associated Riesz matrix. We observe that coregularity of a factorable matrix and regularity of its associated Riesz matrix are connected to each other. We also correlate coercivity of these matrices. Using these relations, we easily find a characterization of those conservative matrices which are stronger than a given factorable matrix.

As further applications of our methods in Section 3 we prove a limitation theorem and two Tauberian theorems for factorable matrices.

In Section 4 we apply our methods to describe summation of 0–1 sequences by factorable matrices. More precisely, we give a characterization of those conservative factorable matrices for which the bounded summability domain has the Hahn property.

2. SUMMABILITY DOMAINS OF FACTORABLE MATRICES

From the Silverman–Toeplitz conditions we can easily deduce that a factorable matrix $A = (b_k c_n)$ is conservative if and only if there exist the finite limits

$$\gamma := \lim_n c_n \quad \text{and} \quad t := \lim_n c_n B_n, \tag{2.1}$$

where $B_n = \sum_{k=1}^n b_k$ ($n \in \mathbb{N}$), and a factorable matrix is regular if and only if $\gamma = 0$ and $t = 1$. A conservative factorable matrix $A = (b_k c_n)$ is regular for null sequences if and only if $\gamma = 0$.

A conservative matrix $D = (d_{nk})$ is called coregular if

$$\chi(D) := \lim_n \sum_k d_{nk} - \sum_k \lim_n d_{nk} \neq 0$$

and conull if $\chi(D) = 0$. A conservative factorable matrix $A = (b_k c_n)$ is coregular if and only if (cf. [9], p. 95)

$$\gamma = \lim_n c_n = 0 \quad \text{and} \quad t = \lim_n c_n B_n \neq 0. \tag{2.2}$$

A matrix $D = (d_{nk})$ is called coercive if $\ell^\infty \subset c_D$. From the Schur Theorem (cf. Theorem 2.4.1 in [3]) we get that a conservative factorable matrix $A = (b_k c_n)$ is coercive if and only if $\lim_n (c_n - \gamma) B_n = 0$.

Lemma 2.1. *Let $A = (b_k c_n)$ be a conservative factorable matrix. Then A is either coregular or coercive.*

Proof. Suppose $\lim_n c_n B_n = 0$. Then

$$c_n = c_n B_n \cdot \frac{1}{B_n} \leq c_n B_n \cdot \frac{1}{B_1} \to 0.$$

Hence A is coercive.

If $\lim_n c_n B_n \neq 0$ and $\lim_n c_n = 0$, then A is coregular. If $\lim_n c_n B_n \neq 0$ and $\lim_n c_n \neq 0$, then $\lim_n B_n =: B < \infty$. So $\lim_n (c_n - \gamma) B_n = 0 \cdot B = 0$. Hence A is coercive. □

It turns out that summability domains of a factorable matrix A and the associated Riesz matrix R_b coincide in many cases.

Proposition 2.2. *Let $A = (b_k c_n)$ be a conservative factorable matrix.*
(i) *Then $c_{R_b} \subset c_A$ and $\lim_A x = t \lim_{R_b} x$ ($x \in c_{R_b}$).*
(ii) *If $b_k = 0$ for $k > k_0$ and some $k_0 \in \mathbb{N}$, then $c_A = c_{R_b} = \omega$, the space of all sequences.*
(iii) *If $\{k \in \mathbb{N} : \ b_k \neq 0\}$ is infinite, then the inclusion $c_A \subset c_{R_b}$ holds if and only if $\lim_n c_n B_n \neq 0$.*

Proof.
(i) Let $x \in c_{R_b}$, then in view of

$$[Ax]_n = c_n \sum_{k=1}^n b_k x_k = c_n B_n \cdot \frac{1}{B_n} \sum_{k=1}^n b_k x_k = c_n B_n \cdot [R_b x]_n \tag{2.3}$$

and the conservativity of A we get $x \in c_A$. Moreover, by (2.3) we get $\lim_A x = t \lim_{R_b} x$.

(ii) The statement $c_{R_b} = \omega = c_A$ is evident because both matrices A and R_b are conservative matrices with entries unequal to zero in finitely many columns at the most.

(iii) If $\lim_n c_n B_n \neq 0$, then the inclusion $c_A \subset c_{R_b}$ follows from (2.3). Now suppose that $\lim_n c_n B_n = 0$ and let (n_i) be the index sequence such that $b_{n_i} \neq 0$ $(i \in \mathbb{N})$ and $b_n = 0$ otherwise. For $i \in \mathbb{N}$ let

$$k_i := \min\{n_i \leq \nu < n_{i+1} : \; c_\nu = \max\{c_k : \; n_i \leq k < n_{i+1}\}\}.$$

We define inductively $(x_n) \in c_A \backslash c_{R_p}$ by the setting $x_1 := 1$ and

$$x_n := \begin{cases} \frac{1}{b_n}\left(\sqrt{\frac{B_{k_i}}{c_{k_i}}} - \sum_{k=1}^{n-1} b_k x_k\right) & \text{if } n = n_i \text{ for some } i \in \mathbb{N}, \\ 0 & \text{if } n \notin \{n_i | \; i \in \mathbb{N}\}. \end{cases}$$

For any $i \in \mathbb{N}$ we get

$$[Ax]_{n_i} = c_{n_i}\left(\sqrt{\frac{B_{k_i}}{c_{k_i}}} - \sum_{k=1}^{n_i - 1} b_k x_k + \sum_{k=1}^{n_i - 1} b_k x_k\right) = \frac{c_{n_i}}{c_{k_i}}\sqrt{B_{k_i} c_{k_i}} \leq \sqrt{B_{k_i} c_{k_i}} \to 0.$$

Now let n be an integer with $n_i < n < n_{i+1}$ for some $i \in \mathbb{N}$. Then

$$[Ax]_n = c_n \sum_{k=1}^{n_i} b_k x_k = \frac{c_n}{c_{n_i}}[Ax]_{n_i} = \frac{c_n}{c_{n_i}}\frac{c_{n_i}}{c_{k_i}}\sqrt{B_{k_i} c_{k_i}} = \frac{c_n}{c_{k_i}}\sqrt{B_{k_i} c_{k_i}} \leq \sqrt{B_{k_i} c_{k_i}}.$$

So $[Ax]_k \to 0$ as $k \to \infty$. On the other hand, by (2.3) we have

$$[R_p x]_{k_i} = \frac{1}{B_{k_i} c_{k_i}} \cdot \frac{c_{k_i}}{c_{k_i}} \sqrt{B_{k_i} c_{k_i}} = \frac{1}{\sqrt{B_{k_i} c_{k_i}}} \to \infty.$$

Hence $x \notin c_{R_p}$. □

The following result demonstrates that coregularity of A and regularity of R_b are also related to each other.

Proposition 2.3. *Let $A = (b_k c_n)$ be a conservative factorable matrix.*
(i) *If A is coregular, then R_b is regular.*
(ii) *If R_b is regular, then A is regular for null sequences.*

Proof.
(i) This statement follows from (2.2) and the relation

$$B_n = c_n B_n \cdot \frac{1}{c_n} \quad (n \in \mathbb{N}).$$

(ii) Since $B_n \to \infty$ and A is conservative, it follows

$$\lim_n c_n = \lim_n \frac{1}{B_n} \cdot \lim_n c_n B_n = 0.$$ □

Remark 2.4.

(i) Conservativity (even regularity) of R_b does not imply that a factorable matrix A is conservative: take any nonnegative sequences (b_n) and (c_n) with $\lim_n B_n = \infty$ and $(c_n) \notin c$.

(ii) If R_b is regular, then a conservative factorable matrix A can be both either conull or coregular. Take $b_k = 1$ $(k \in \mathbb{N})$, then R_b is the Cesàro method C_1 which is regular. Moreover, if we set $c_n := \frac{1}{n}$ $(n \in \mathbb{N})$, then A is coregular, and it is conull if we put $c_n := \frac{1}{n^2}$ $(n \in \mathbb{N})$.

Let us also consider the relation of coercivity of a factorable matrix A and its associated Riesz matrix R_b.

Proposition 2.5. *Let $A = (b_k c_n)$ be a conservative factorable matrix.*
(i) *If R_b is coercive, then A is coercive.*
(ii) *If A is coercive and $\gamma = \lim_n c_n \neq 0$, then R_b is coercive.*

Proof.
(i) Since R_b is coercive, there exists $B = \lim_n B_n < \infty$. Hence

$$\lim_n (c_n - \gamma) B_n = 0 \cdot B = 0.$$

(ii) If A is coercive and $\lim_n c_n \neq 0$, then $\lim_n c_n B_n \neq 0$, so

$$\lim_n B_n = \lim_n c_n B_n \cdot \lim_n \frac{1}{c_n} < \infty.$$

Therefore R_b is coercive. □

Remark 2.6. If A is a coercive factorable matrix with $\gamma = 0$, then R_b can be both either coercive or regular. Take $c_n := \frac{1}{n^2}$ $(n \in \mathbb{N})$. If $b = (b_k) \in \ell^1$, then both A and R_b are coercive; if $b_k = 1$ $(k \in \mathbb{N})$, then A is coercive while R_b is regular.

In the next result we characterize those conservative matrices which are stronger than a given factorable matrix.

Theorem 2.7. *Let $A = (b_k c_n)$ be a positive coregular factorable matrix and $D = (d_{nk})$ be a conservative matrix method. Then D is stronger than A if and only if the following conditions hold:*
(i) $\left(\frac{d_{nk}}{b_k}\right)_k \in c_0$ $(n \in \mathbb{N})$;
(ii) $\sup_n \sum_k B_k \left| \frac{d_{nk}}{b_k} - \frac{d_{n,k+1}}{b_{k+1}} \right| < \infty$.

Proof. Since A is coregular, Proposition 2.3 (i) implies that R_b is regular, moreover, in view Proposition 2.2 we have $c_A = c_{R_b}$. Now the statement of the theorem follows from the corresponding result for Riesz matrices (cf. Theorem 3.2.8 in [3]). □

Corollary 2.8. *Let $A = (b_k c_n)$ be a positive coregular factorable matrix. Then the method A is equiconvergent if and only if $\left(\frac{B_n}{b_n}\right) \in \ell^\infty$.*

Proof. This is an immediate consequence of the arguments in the proof of Theorem 2.7 and Corollary 3.2.10 in [3]. □

Remark 2.9.
(i) Since $\left(\frac{B_n}{b_n}\right) \in \ell^\infty$ is equivalent to the condition $\liminf_n \frac{b_{n+1}}{B_n} > 0$, Corollary 2.8 is just Theorem 1 of Rhoades [9] with a much shorter proof.
(ii) In contrast to Riesz matrices (cf. Corollary 3.2.10 in [3]), we cannot omit the coregularity assumption in Corollary 2.8: taking (b_n) and (c_n) such that $\left(\frac{B_n}{b_n}\right) \in \ell^\infty$ and $c_n B_n \to 0$, we have $\ell^\infty \subset c_A$.

3. LIMITATION THEOREM AND TAUBERIAN THEOREMS FOR FACTORABLE MATRICES

The following proposition is a limitation theorem for factorable matrices generalizing the corresponding result of Hardy ([6], Theorem 13) for Riesz methods.

Proposition 3.1. *Let $A = (b_k c_n)$ be a positive coregular factorable matrix. If $x \in c_A$ and $\lim_A x = s$, then*

$$x_n - \frac{s}{t} = o\left(\frac{1}{b_n c_n}\right).$$

Proof. Let $x \in c_A$, then by Proposition 2.2 (iii) $x \in c_{R_b}$. Moreover, by Proposition 2.3 (i) R_b is regular and $\lim_{R_b} x = t^{-1} \lim_A x = t^{-1}s$. Hence, by Theorem 13 of [6] it follows that $(x_n - t^{-1}s)b_n/B_n \to 0$. Therefore

$$\left(x_n - \frac{s}{t}\right) b_n c_n = \frac{\left(x_n - \frac{s}{t}\right) b_n}{B_n} c_n B_n \to 0. \qquad \square$$

Corollary 3.2. *Let $A = (b_k c_n)$ be a positive regular factorable matrix. If $x \in c_A$ and $\lim_A x = s$, then*

$$x_n - s = o\left(\frac{1}{b_n c_n}\right).$$

In the following result, again based on properties of Riesz matrices, we generalize Theorem 2 of Rhoades [9] from regular to coregular factorable matrices.

Corollary 3.3. *Let $A = (b_k c_n)$ be a positive coregular factorable matrix. If*

$$\lim_n \frac{c_n}{c_{n-1}} = 1, \tag{3.1}$$

then $x \in c_A$ implies that $x_n = o(\frac{1}{b_n c_n})$.

Proof. Since

$$\lim_n \frac{B_{n-1}}{B_n} = \lim_n \frac{B_{n-1}c_{n-1}}{B_n c_n} \cdot \frac{c_n}{c_{n-1}} = \frac{t}{t} \cdot 1 = 1, \tag{3.2}$$

we obtain

$$\lim_n b_n c_n = \lim_n c_n B_n \frac{B_n - B_{n-1}}{B_n} = t(1-1) = 0.$$

So, by Proposition 3.1

$$b_n c_n x_n = \left(x_n - \frac{s}{t}\right) b_n c_n + \frac{s}{t} b_n c_n \to 0. \qquad \square$$

As a further application of our methods we prove two Tauberian theorems for factorable matrices.

Theorem 3.4. (*O-Tauberian theorem for a factorable matrix*). *Let $A = (b_k c_n)$ be a coregular factorable matrix. Then each A-summable sequence (x_n) which satisfies the Tauberian condition $x_{n+1} - x_n = O(b_n c_n)$ is convergent.*

Proof. Let (x_n) be an A-summable sequence which satisfies the Tauberian condition $x_{n+1} - x_n = O(b_n c_n)$ and let $M > 0$ be such that $|x_{n+1} - x_n| \le M b_n c_n$ $(n \in \mathbb{N})$. By Proposition 2.2 (iii) $x \in c_{R_b}$ and by Proposition 2.3 (i) R_b is regular. Since A is conservative, $B_n c_n \le C$ $(n \in \mathbb{N})$ for some $C > 0$. Hence

$$|x_{n+1} - x_n| \le M b_n c_n = M c_n B_n \frac{b_n}{B_n} \le MC \frac{b_n}{B_n}.$$

So $x_{n+1} - x_n = O(\frac{b_n}{B_n})$. Now, by the O-Tauberian theorem for Riesz matrices (cf. Theorem 4.2.5 in [3]), (x_n) is convergent. $\square$

Theorem 3.5. (*one-sided oscillation Tauberian theorem for a factorable matrix*). *Let* $A = (b_k c_n)$ *be a positive coregular factorable matrix such that*

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} > 0.$$

Then each A-summable sequence (x_n) *which satisfies the Tauberian condition*

$$\liminf_{\frac{c_r}{c_n}\to 1, r\geq n\to\infty} (x_r - x_n) \geq 0 \tag{3.3}$$

is convergent.

Proof. First note that

$$\liminf_{n\to\infty} \frac{B_n}{B_{n+1}} = \liminf_{n\to\infty} \frac{c_{n+1}}{c_n} > 0$$

(cf. (3.2)). Moreover, if $r \geq n \to \infty$ such that $\frac{B_r}{B_n} \to 1$, then

$$\frac{c_r}{c_n} = \frac{B_r c_r}{B_n c_n} \cdot \frac{B_n}{B_r} \to 1.$$

Therefore

$$\liminf_{\frac{B_r}{B_n}\to 1, r\geq n\to\infty} (x_r - x_n) \geq 0$$

for a given A-summable sequence (x_n) which satisfies the Tauberian condition (3.3). Hence, by the one-sided oscillation Tauberian theorem for Riesz matrices (cf. Theorem 4.2.11 in [3]), (x_n) is convergent. □

4. SUMMABILITY OF SEQUENCES OF 0'S AND 1'S BY FACTORABLE MATRICES

Using the methods developed in the previous section, we will also study summation of 0–1 sequences by factorable matrices.

We start with some preliminaries. For other notations and preliminary results we refer the reader to [2–4].

Let χ denote the set of all sequences of 0's and 1's and let $\chi(E)$ denote the linear hull of $\chi \cap E$.

An FK-space is a sequence space endowed with a complete, metrizable, locally convex topology under which all coordinate mappings $x = (x_j) \to x_k$ $(k \in \mathbb{N})$ are continuous.

A sequence space E is said to have the *Hahn property*, the *separable Hahn property*, and *the matrix Hahn property*, if $\chi(E) \subset F$ implies $E \subset F$ whenever F is any FK-space, a separable FK-space, and a matrix domain c_B, respectively. Obviously, the Hahn property implies the separable Hahn property, and the latter implies the matrix Hahn property.

In case of Riesz matrices R_p, Hahn properties of their bounded summability domain $\ell^\infty \cap c_{R_p}$ are very well studied: if $p \in \ell^1$, then $\ell^\infty \cap c_{R_p}$ has the Hahn property since R_p is coersive; if $p \notin \ell^1$, then $\ell^\infty \cap c_{R_p}$ has the Hahn property if and only if $(\frac{p_n}{P_n}) \in c_0$ (cf. Corollary 3.9 in [5]). Keeping this characterization in mind, we describe Hahn properties of bounded summability domain for a factorable matrix.

Theorem 4.1. *Let* $A = (b_k c_n)$ *be a coregular factorable matrix. Then the following conditions are equivalent*:
a) $b_n c_n \to 0$;
b) *A has spreading rows*, *that is*, $\lim_n c_n \sup_{1\leq k\leq n} b_k = 0$;
c) $A \in KG$ (*that is*, *each matrix D with* $\chi(\ell^\infty \cap c_A) \subset c_D$ *is conservative*);

d) $\ell^\infty \cap c_A$ *has the matrix Hahn property*;
e) $\ell^\infty \cap c_A$ *has the separable Hahn property*;
f) $\ell^\infty \cap c_A$ *has the Hahn property*;
g) $\chi(c_A \cap \ell^\infty)^\beta = \ell^1$.

Proof. Since A is coregular, $\ell^\infty \cap c_A = \ell^\infty \cap c_{R_b}$ and R_b is regular. Hence the equivalences

$$\text{a')} \Leftrightarrow \text{b')} \Leftrightarrow \text{c)} \Leftrightarrow \text{d)} \Leftrightarrow \text{e)} \Leftrightarrow \text{f)},$$

where a') $b_n/B_n \to 0$, b') R_b has spreading rows, follow from Corollary 3.9 in [5]. The implication c) $\Leftrightarrow$ g) follows from Corollary 2.4 in [11]. Since

$$b_n c_n = c_n B_n \frac{b_n}{B_n}, \qquad c_n \sup_{1\le k\le n} b_k = c_n B_n \sup_{1\le k\le n} \frac{b_k}{B_n},$$

and $\lim_n c_n B_n \neq 0$, the equivalence of all conditions in the theorem follow. □

Note that the application of the correspondence of factorable matrices and their associated Riesz matrix allowed us to avoid long proofs related to Hahn properties theorems.

Remark 4.2. The equivalence of conditions a) $\Leftrightarrow$ d) was first shown in [7], however, the methods analogical to the methods used by Kuttner and Parameswaran for Riesz matrices [8] were applied. So the proof appeared to be quite long.

5. CONCLUSION

We introduced for a factorable matrix a notion of its associated Riesz matrix. Properties of these two matrices are closely related: using well-known properties of Riesz matrices, we can easily and briefly prove the corresponding properties of related factorable matrices. As a demonstration of our methods we found a characterization of those conservative matrices which are stronger than a given factorable matrix; we proved a limitation theorem and two Tauberian theorems for factorable matrices. We also applied our methods to characterize conservative factorable matrices for which their bounded summability domains have the Hahn property.

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Faktoriseeruvad maatriksid ja nendega seotud Rieszi maatriksid

Maria Zeltser

Käesolevas töös on antud faktoriseeruva maatriksi jaoks kasutusele võetud duaalse Rieszi maatriksi mõiste. Nende kahe maatriksi omadused on tihedalt seotud: kasutades Rieszi maatriksite hästi tuntud omadusi, saame lihtsalt ja lühidalt tõestada seotud faktoriseeruvate maatriksite vastavad omadused. Meie meetodite näitamiseks on iseloomustatud konservatiivseid maatrikseid, mis on antud faktoriseeruvast maatriksist tugevamad, ja tõestatud limiteeriv teoreem ning kaks Tauberi teoreemi faktoriseeruvate maatriksite jaoks. Samuti on rakendatud meie meetodit, et iseloomustada konservatiivseid faktoriseeruvaid maatrikseid, mille korral on nende tõkestatud summeeruvusväljad Hahni omadusega.