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MATHEMATICS

Injective hulls for posemigroups

Xia Zhang[a,b*] and Valdis Laan[c]

[a] School of Mathematical Sciences, South China Normal University, 510631 Guangzhou, China
[b] Department of Mathematics, Southern Illinois University Carbondale, 62901 Carbondale, USA
[c] Institute of Mathematics, University of Tartu, J. Liivi 2, 50409 Tartu, Estonia

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Abstract. We show that injectives with respect to a specific class of order embeddings in the category of posemigroups with submultiplicative morphisms are quantales and construct injective hulls for a certain class of posemigroups with respect to this specific class of order embeddings.

Key words: ordered semigroup, injectivity, injective hull, quantale.

1. INTRODUCTION

Bruns and Lakser in their paper [2] characterized injective hulls in the category of semilattices. In a recent article [6], Lambek et al. considered injective hulls in the category of pomonoids and submultiplicative identity and order-preserving mappings.

Inspired by these results, we construct injective hulls in the category of posemigroups and submultiplicative order-preserving mappings with respect to certain class $\mathscr{E}_{\leq}$ of morphisms in this work. In fact, Theorem 5.8 of [6] becomes a consequence of our main theorem (Theorem 7).

As usual, a *posemigroup* $(S,\cdot,\leq)$ is a semigroup $(S,\cdot)$ equipped with a partial ordering $\leq$ which is compatible with the semigroup multiplication, that is, $a_1a_2 \leq b_1b_2$ whenever $a_1 \leq b_1$ and $a_2 \leq b_2$, for any $a_1,a_2,b_1,b_2 \in S$. *Posemigroup homomorphisms* are monotone (i.e. order-preserving) semigroup homomorphisms between two posemigroups. A subsemigroup R of S equipped with the partial order $(R\times R)\cap \leq$ is called a *subposemigroup* of S. An *order embedding* from a poset $(A,\leq_A)$ to a poset $(B,\leq_B)$ is a mapping $h : A \to B$ such that $a \leq_A a'$ iff $h(a) \leq_B h(a')$, for all $a,a' \in A$. Every order embedding is necessarily an injective mapping.

Let $\mathscr{C}$ be a category and let $\mathscr{M}$ be a class of morphisms in $\mathscr{C}$. We recall that an object S from $\mathscr{C}$ is *$\mathscr{M}$-injective* in $\mathscr{C}$ provided that for any morphism $h : A \to B$ in $\mathscr{M}$ and any morphism $f : A \to S$ in $\mathscr{C}$ there exists a morphism $g : B \to S$ such that $gh = f$.

A morphism $\eta : A \to B$ in $\mathscr{M}$ is called *$\mathscr{M}$-essential* (cf. [1]) if every morphism $\psi : B \to C$ in $\mathscr{C}$, for which the composite $\psi\eta$ is in $\mathscr{M}$, is itself in $\mathscr{M}$. An object $H \in \mathscr{C}$ is called an *$\mathscr{M}$-injective hull* of an object S if H is $\mathscr{M}$-injective and there exists an $\mathscr{M}$-essential morphism $S \to H$.

It is natural to consider injectivity in the category of posemigroups with respect to posemigroup homomorphisms that are order embeddings. However, injective objects in this sense are only one-element

* Corresponding author, xiazhang@scnu.edu.cn, xiazhang_1@yahoo.com

posemigroups. Indeed, if a posemigroup $(S,\cdot,\leq)$ is injective, then the underlying semigroup $(S,\cdot)$ is injective in the category of all semigroups, because every semigroup can be considered as a discretely ordered posemigroup, and homomorphisms of discretely ordered posemigroups are just the homomorphisms of underlying semigroups. But injective semigroups are only the trivial ones (see [10]).

Allowing more morphisms between posemigroups, it is still possible to obtain nontrivial injectives. This approach is taken in [6]. Namely, one can consider order-preserving *submultiplicative mappings* $f : A \to B$ between posemigroups A and B, i.e. mappings with $f(a)f(a') \leq f(aa')$ for all $a, a' \in A$. We denote by $\mathsf{PoSgr}_{\leq}$ the category where objects are posemigroups and morphisms are submultiplicative order-preserving mappings.

A *quantale* (cf. [8]) is a posemigroup $(S,\cdot,\leq)$ such that
(1) the poset $(S,\leq)$ is a complete lattice;
(2) $s(\bigvee M) = \bigvee\{sm \mid m \in M\}$ and $(\bigvee M)s = \bigvee\{ms \mid m \in M\}$ for each subset M of S and each $s \in S$.

We note that compatibility of multiplication and order actually follows from condition (2). Indeed, if $s, a, b \in S$ and $a \leq b$, then $sb = s(\bigvee\{a,b\}) = \bigvee\{sa, sb\}$, and so $sa \leq sb$. Similarly $as \leq bs$.

2. INJECTIVE POSEMIGROUPS

Let $\mathscr{E}_{\leq}$ denote the class of all morphisms $h : A \to B$ in the category $\mathsf{PoSgr}_{\leq}$ which are order-preserving, submultiplicative, and satisfy the following condition: $h(a_1)\dots h(a_n) \leq h(a)$ implies $a_1 \dots a_n \leq a$ for all $a_1, \dots, a_n, a \in A$. Each such morphism is necessarily an order-embedding. In this section we show that $\mathscr{E}_{\leq}$-injective objects in the category $\mathsf{PoSgr}_{\leq}$ are precisely the quantales. This is largely a restatement of arguments from [6] for posemigroups.

Proposition 1. *Quantales are $\mathscr{E}_{\leq}$-injective objects in the category* $\mathsf{PoSgr}_{\leq}$.

Proof. Let S be a quantale, $h : A \to B$ be a morphism in $\mathscr{E}_{\leq}$, and let $f : A \to S$ be a morphism in $\mathsf{PoSgr}_{\leq}$. Define a mapping $g : B \to S$ by

$$g(b) = \bigvee\{f(a_1)\dots f(a_n) \mid h(a_1)\dots h(a_n) \leq b,\ a_1, \dots, a_n \in A\},$$

for any $b \in B$. Then g is clearly an order-preserving mapping. The fact that g is submultiplicative and satisfies $gh = f$ follows from the proof of [6] Theorem 4.1. ☐

Proposition 2. *In the category* $\mathsf{PoSgr}_{\leq}$, *every retract of a quantale is a quantale.*

Proof. Let $(E, \circ, \leq_E)$ be a quantale and let $(S, \cdot, \leq_S)$ be a retract of E. Then there exist submultiplicative order-preserving mappings $i : S \to E$ and $g : E \to S$ such that $gi = id_S$, where id_S is the identity mapping on S. It is obvious that $(S, \leq_S)$ is complete.

Let $s \in S$ and $M \subseteq S$. Obviously, $s(\bigvee M)$ is an upper bound of the set $\{sm \mid m \in M\}$. Suppose that u is an upper bound of $\{sm \mid m \in M\}$ in S. Then

$$\begin{aligned} u &= g(i(u)) \\ &\geq g\left(\bigvee_E \{i(sm) \mid m \in M\}\right) \\ &\geq g\left(\bigvee_E \{i(s) \circ i(m) \mid m \in M\}\right) \\ &= g\left(i(s) \circ \bigvee_E \{i(m) \mid m \in M\}\right) \end{aligned}$$

$$\begin{aligned} &\geq \; g(i(s))g\left(\bigvee_E\{i(m) \mid m \in M\}\right) \\ &= \; sg\left(\bigvee_E\{i(m) \mid m \in M\}\right) \\ &\geq \; s\left(\bigvee_S M\right). \end{aligned}$$

So $s(\bigvee M)$ is the least upper bound of $\{sm \mid m \in M\}$, that is,

$$s\left(\bigvee M\right) = \bigvee\{sm \mid m \in M\}.$$

Similarly one can prove the equality

$$\left(\bigvee M\right)s = \bigvee\{ms \mid m \in M\}.$$ □

A subset A of a poset $(S, \leq)$ is said to be a *down-set* if $s \in A$ whenever $s \leq a$ for $s \in S$, $a \in A$. For any $I \subseteq S$, we denote by $I\!\downarrow$ the down-set $\{x \in S \mid x \leq i \text{ for some } i \in I\}$ and by $a\!\downarrow$ the down-set $\{s \in S \mid s \leq a\}$ for $a \in S$.

Now one can construct an $\mathscr{E}_{\leq}$-injective posemigroup starting from any posemigroup.

Let $(S, \cdot, \leq)$ be a posemigroup, and let $\mathscr{P}(S)$ be the set of all down-sets of S. Define a multiplication $\cdot$ on $\mathscr{P}(S)$ by

$$I \cdot J = (IJ)\!\downarrow = \{x \in S \mid x \leq ij \text{ for some } i \in I,\ j \in J\}. \tag{1}$$

As in [6], $(\mathscr{P}(S), \cdot, \subseteq)$ is a quantale. Hence, by Proposition 1 we have the following result.

Proposition 3. *Let $(S, \cdot, \leq)$ be a posemigroup. Then $(\mathscr{P}(S), \cdot, \subseteq)$ is $\mathscr{E}_{\leq}$-injective in the category* $\mathsf{PoSgr}_{\leq}$.

Theorem 4. *Let $(S, \cdot, \leq)$ be a posemigroup. Then S is $\mathscr{E}_{\leq}$-injective in* $\mathsf{PoSgr}_{\leq}$ *if and only if S is a quantale.*

Proof. Sufficiency follows by Proposition 1.

Necessity. The mapping $\eta : (S, \cdot, \leq) \to (\mathscr{P}(S), \cdot, \subseteq)$, given by $\eta(a) = a\!\downarrow$ for each $a \in S$, is clearly an order-embedding of the poset $(S, \leq)$ into the poset $(\mathscr{P}(S), \subseteq)$. It is routine to check that η preserves multiplication and hence η is also submultiplicative. Being a multiplicative order-embedding, η belongs to $\mathscr{E}_{\leq}$.

Since S is $\mathscr{E}_{\leq}$-injective by assumption, there exists $g : \mathscr{P}(S) \to S$ such that $g\eta = id_S$, so S is a retract of $\mathscr{P}(S)$. Consequently, $(S, \cdot, \leq)$ is a quantale by Proposition 2. □

3. ON INJECTIVE HULLS OF POSEMIGROUPS

In this section we show that, for a certain class of posemigroups, $\mathscr{E}_{\leq}$-injective hulls exist. This class will include all pomonoids, but not only those. Similarly to Proposition 2.1 in [6] it can be shown that $\mathscr{E}_{\leq}$-injective hulls are unique up to isomorphism.

For any down-set I of a posemigroup S we define its closure by

$$\mathsf{cl}(I) := \{x \in S \mid aIc \subseteq b\!\downarrow \text{ implies } axc \leq b \text{ for all } a, b, c \in S\}.$$

Let I be a down-set and $s \leq x \in \mathsf{cl}(I)$. Suppose that $aIc \subseteq b\!\downarrow$. Since $x \in \mathsf{cl}(I)$, $axc \leq b$. But then also $asc \leq axc \leq b$, which means that $s \in \mathsf{cl}(I)$. Thus $\mathsf{cl}(I)$ is a down-set and we may consider the mapping $\mathsf{cl} : \mathscr{P}(S) \to \mathscr{P}(S)$.

Recall (see [8], Definition 3.1.1) that a *quantic nucleus* on a quantale Q is a submultiplicative closure operator on Q.

Lemma 5. *The mapping* cl *is a quantic nucleus on the quantale* $\mathscr{P}(S)$.

Proof. First, let us show that cl is a closure operator.

If $aIc \subseteq b\downarrow$, then clearly $axc \le b$ for every $x \in I$. Hence $I \subseteq \mathsf{cl}(I)$ and cl is extensive.

Let $I \subseteq J$, $x \in \mathsf{cl}(I)$, and $aJc \subseteq b\downarrow$. Then we have $aIc \subseteq aJc \subseteq b\downarrow$ and hence $axc \le b$. So $x \in \mathsf{cl}(J)$ and we have proven that cl is order-preserving.

The inclusion $\mathsf{cl}(I) \subseteq \mathsf{cl}(\mathsf{cl}(I))$ holds because cl is extensive and order-preserving. Conversely, suppose that $aIc \subseteq b\downarrow$ and $y \in \mathsf{cl}(\mathsf{cl}(I))$. Then $axc \le b$ for any $x \in \mathsf{cl}(I)$. This means $a\mathsf{cl}(I)c \subseteq b\downarrow$. So $ayc \le b$ by the definition of cl, and $y \in \mathsf{cl}(I)$. Thus cl is also idempotent, and therefore a closure operator.

It remains to prove that cl is a submultiplicative mapping. To this end, let us first prove that $\mathsf{cl}(I) \cdot J \subseteq \mathsf{cl}(I \cdot J)$ for all $I, J \in \mathscr{P}(S)$. Take $z \in \mathsf{cl}(I) \cdot J$ and suppose that $a(I \cdot J)c \subseteq b\downarrow$. Then for any $j \in J$ the inclusion $Ij \subseteq I \cdot J$ implies that $aI(jc) \subseteq b\downarrow$. We have $z \le mj$ for some $m \in \mathsf{cl}(I)$ and $j \in J$. So $am(jc) \le b$. It follows that $azc \le amjc \le b$, which results in $z \in \mathsf{cl}(I \cdot J)$, as needed.

Similarly, $I \cdot \mathsf{cl}(J) \subseteq \mathsf{cl}(I \cdot J)$ holds. Consequently, we obtain that

$$\mathsf{cl}(I) \cdot \mathsf{cl}(J) \subseteq \mathsf{cl}(I \cdot \mathsf{cl}(J)) \subseteq \mathsf{cl}(\mathsf{cl}(I \cdot J)) = \mathsf{cl}(I \cdot J). \qquad \square$$

One can immediately get the following corollary.

Corollary 6. *For a posemigroup* S *and* $I, J \in \mathscr{P}(S)$, *we have*

$$\mathsf{cl}(\mathsf{cl}(I) \cdot \mathsf{cl}(J)) = \mathsf{cl}(I \cdot J).$$

We put

$$\mathscr{Q}(S) := \{I \in \mathscr{P}(S) \mid I = \mathsf{cl}(I)\}$$

and define a multiplication $\circ$ on $\mathscr{Q}(S)$ by

$$I \circ J := \mathsf{cl}(I \cdot J). \tag{2}$$

By Theorem 3.1.1 of [8] we immediately have that, for every posemigroup S, $(\mathscr{Q}(S), \circ, \subseteq)$ is a quantale which is the image of the quantic nucleus cl. From Theorem 4 we conclude that $\mathscr{Q}(S)$ is $\mathscr{E}_{\le}$-injective in the category $\mathsf{PoSgr}_{\le}$.

Now we can prove our main result.

Theorem 7. (cf. Theorem 5.8 in [6]). *Let* S *be a posemigroup such that* $\mathsf{cl}(s\downarrow) = s\downarrow$ *for every* $s \in S$. *Then* $\mathscr{Q}(S)$ *is an* $\mathscr{E}_{\le}$*-injective hull of* S *in* $\mathsf{PoSgr}_{\le}$.

Proof. Since $\mathsf{cl}(s\downarrow) = s\downarrow$, we can consider the mapping $\eta : S \to \mathscr{Q}(S), a \mapsto a\downarrow$. We shall prove that η is an $\mathscr{E}_{\le}$-essential morphism in $\mathsf{PoSgr}_{\le}$.

Let us show that η is a posemigroup homomorphism. Take $a, b \in S$. It is easy to see (see also the proof of Proposition 3.3 in [6]) that $(ab)\downarrow = ((a\downarrow)(b\downarrow))\downarrow$. Hence, using (2) and (1), we have

$$\eta(a) \circ \eta(b) = \mathsf{cl}(a\downarrow \cdot b\downarrow) = \mathsf{cl}(((a\downarrow)(b\downarrow))\downarrow) = \mathsf{cl}((ab)\downarrow) = (ab)\downarrow = \eta(ab),$$

i.e. η is a semigroup homomorphism. For every $a, b \in S$, $a \le b$ if and only if $a\downarrow \subseteq b\downarrow$. This means that η is both monotone and an order-embedding. If now $\eta(a_1) \circ \ldots \circ \eta(a_n) \subseteq \eta(a)$, then $\eta(a_1 \ldots a_n) \subseteq \eta(a)$, which implies $a_1 \ldots a_n \le a$. Thus η belongs to $\mathscr{E}_{\le}$.

Finally, let $\psi : \mathscr{Q}(S) \to B$ be a morphism in $\mathsf{PoSgr}_{\le}$ such that $\psi\eta \in \mathscr{E}_{\le}$. We have to show that $\psi \in \mathscr{E}_{\le}$. Suppose that $\psi(I_1) \ldots \psi(I_n) \le \psi(J)$ in B, where $I_1, \ldots, I_n, J \in \mathscr{Q}(S)$. First we prove that

$$(\forall a, b, c \in S)(aJc \subseteq b\downarrow \Longrightarrow a(I_1 \circ \ldots \circ I_n)c \subseteq b\downarrow). \tag{3}$$

Suppose that $aJc \subseteq b\!\downarrow$, $a,b,c \in S$. Then also $a\!\downarrow \cdot J \cdot c\!\downarrow \subseteq b\!\downarrow$. Let us show that

$$a(I_1 \dots I_n)c \subseteq b\!\downarrow. \tag{4}$$

Take $i_1 \in I_1, \dots, i_n \in I_n$. Then

$$\begin{aligned}(\psi\eta)(a)(\psi\eta)(i_1)\dots(\psi\eta)(i_n)(\psi\eta)(c) &= \psi(a\!\downarrow)\psi(i_1\!\downarrow)\dots\psi(i_n\!\downarrow)\psi(c\!\downarrow)\\ &\leq \psi(a\!\downarrow)\psi(I_1)\dots\psi(I_n)\psi(c\!\downarrow)\\ &\leq \psi(a\!\downarrow)\psi(J)\psi(c\!\downarrow)\\ &\leq \psi(a\!\downarrow \circ J \circ c\!\downarrow)\\ &= \psi(\mathsf{cl}(a\!\downarrow \cdot J \cdot c\!\downarrow))\\ &\leq \psi(\mathsf{cl}(b\!\downarrow))\\ &= \psi(b\!\downarrow)\\ &= (\psi\eta)(b).\end{aligned}$$

Since $\psi\eta \in \mathscr{E}_{\leq}$, we conclude that $ai_1 \dots i_n c \leq b$. Consequently, $a(I_1 \dots I_n)c \subseteq b\!\downarrow$.

Now (4) implies $a((I_1 \dots I_n)\!\downarrow)c \subseteq b\!\downarrow$. If $x \in \mathsf{cl}((I_1 \dots I_n)\!\downarrow) = \mathsf{cl}(I_1 \dots I_n) = I_1 \circ \dots \circ I_n$, then $a((I_1 \dots I_n)\!\downarrow)c \subseteq b\!\downarrow$ implies $axc \leq b$ by the definition of closure. Thus we have proven (3).

To prove that $I_1 \circ \dots \circ I_n \subseteq J$, let $x \in I_1 \circ \dots \circ I_n$. Suppose that $a,b,c \in S$ and $aJc \subseteq b\!\downarrow$. By (3), $a(I_1 \circ \dots \circ I_n)c \subseteq b\!\downarrow$. Since $x \in I_1 \circ \dots \circ I_n = \mathsf{cl}(I_1 \circ \dots \circ I_n)$, we have $axc \leq b$. Hence $x \in \mathsf{cl}(J) = J$. □

It turns out that the assumptions of Theorem 7 are satisfied for several natural classes of posemigroups.

A posemigroup S is *negatively ordered* (cf. [9]) if $st \leq s$ and $st \leq t$ for all $s,t \in S$. Negatively ordered semigroups and monoids arise naturally in various semigroup theoretic contexts; see, for example, [3,4,11].

Example 8.

(1) Every lower semilattice with respect to its natural order is negatively ordered.

(2) If S is any semigroup, then the set $Id(S)$ of all its ideals is a negatively ordered posemigroup with respect to inclusion and the product $IJ = \{ij \mid i \in I, j \in J\}$.

(3) The real interval $[0,1]$ is negatively ordered with respect to the usual multiplication and order of real numbers.

(4) In [7], negatively ordered semigroups with respect to natural partial order in many classes of semigroups are determined.

A semigroup S has *weak local units* (see, e.g., [5]) if for every $s \in S$ there exist $u,v \in S$ such that $s = su = vs$.

Corollary 9. *The posemigroup $\mathscr{Q}(S)$ is an $\mathscr{E}_{\leq}$-injective hull of S in $\mathsf{PoSgr}_{\leq}$ in any of the following four cases*:

(1) *S is a pomonoid*;

(2) *S is a negatively ordered posemigroup with weak local units*;

(3) *S is a linearly ordered cancellative posemigroup*;

(4) *S is an upper semilattice with natural order.*

Proof. We shall show that the assumption of Theorem 7 is fulfilled in all these cases. Since $s\!\downarrow \subseteq \mathsf{cl}(s\!\downarrow)$ holds always, we have to prove that $\mathsf{cl}(s\!\downarrow) \subseteq s\!\downarrow$ for every $s \in S$.

(1) Suppose that $x \in \mathsf{cl}(s\!\downarrow)$. Since $1(s\!\downarrow)1 \subseteq s\!\downarrow$, we have that $x = 1x1 \leq s$, that is, $x \in s\!\downarrow$.

(2) Take $x \in \mathsf{cl}(s\!\downarrow)$. By assumption there exist $u,v \in S$ such that $x = ux = xv$. Since S is negatively ordered, we have $usv \leq s$. This implies $u(s\!\downarrow)v \subseteq s\!\downarrow$, and hence, by the definition of $\mathsf{cl}(s\!\downarrow)$, $x = uxv \leq s$. Thus, $\mathsf{cl}(s\!\downarrow) \subseteq s\!\downarrow$, as needed.

(3) To prove that $\mathsf{cl}(s{\downarrow}) \subseteq s{\downarrow}$ for every $s \in S$, we show that $x \notin s{\downarrow}$ implies $x \notin \mathsf{cl}(s{\downarrow})$ for every $x \in S$. So let $x \notin s{\downarrow}$, i.e. $s < x$. Suppose that $x \in \mathsf{cl}(s{\downarrow})$. Choose arbitrary $a, c \in S$, and put $b := asc$. Then $a(s{\downarrow})c \subseteq b{\downarrow}$, and hence $axc \leq b$, because $x \in \mathsf{cl}(s{\downarrow})$. Consequently, $b = asc \leq axc \leq b$, which gives $asc = axc$. Cancelling a and c, we obtain $s = x$, contradicting inequality $s < x$. Thus $x \notin \mathsf{cl}(s{\downarrow})$.

(4) Let $(S, \vee, \leq)$ be an upper semilattice with its natural order. Assume $x \in \mathsf{cl}(s{\downarrow})$. Since $s \vee (s{\downarrow}) \vee s \subseteq s{\downarrow}$, it follows that $s \vee x \vee s \leq s$. Hence $x \in s{\downarrow}$. □

Example 10. Both additive and multiplicative posemigroups of natural numbers are linearly ordered and cancellative. Note that neither of them is a pomonoid or negatively ordered.

There exist semigroups S for which $\mathscr{Q}(S)$ is not an $\mathscr{E}_{\leq}$-injective hull of S in $\mathsf{PoSgr}_{\leq}$.

Example 11. Let $S = \{a, b, c\}$ be a left zero semigroup with the ordering $a \leq c,\ b \leq c$. Then

$$\mathscr{P}(S) = \{a{\downarrow}, b{\downarrow}, c{\downarrow}, \emptyset, \{a, b\}\},$$

where $a{\downarrow} = \{a\}$ and $\mathsf{cl}(a{\downarrow}) = S \neq a{\downarrow}$. In fact, $\mathsf{cl}(a{\downarrow}) = \mathsf{cl}(b{\downarrow}) = \mathsf{cl}(c{\downarrow}) = \mathsf{cl}(\{a, b\}) = S$. The reason is that for any $u, v, w \in S$ and nonempty $I \in \mathscr{P}(S)$, if $uIv = \{u\} \subseteq w{\downarrow}$, then $uxv = u \leq w$ for any $x \in S$. Therefore, $\mathscr{Q}(S) = \{S, \emptyset\}$ and there is no $\mathscr{E}_{\leq}$-essential morphism from S to $\mathscr{Q}(S)$, because such a morphism would have to be an order-embedding and hence injective. Consequently, $\mathscr{Q}(S)$ is not an $\mathscr{E}_{\leq}$-injective hull of S in $\mathsf{PoSgr}_{\leq}$.

As the last thing we show that Theorem 5.8 of [6] follows from Theorem 7.

Let $\mathsf{PoMon}^1_{\leq}$ be the category where objects are pomonoids and morphisms are submultiplicative order-preserving mappings which preserve identity (this is the category considered in [6]). Thus $\mathsf{PoMon}^1_{\leq}$ is a subcategory of $\mathsf{PoSgr}_{\leq}$. By $\mathscr{E}^1_{\leq}$ we denote the class of those morphisms which belong to $\mathsf{PoMon}^1_{\leq}$ and $\mathscr{E}_{\leq}$.

Corollary 12. *Let S be a pomonoid. Then $\mathscr{Q}(S)$ is an $\mathscr{E}^1_{\leq}$-injective hull of S in the category* $\mathsf{PoMon}^1_{\leq}$.

Proof. From the proof of Corollary 9 we know that $\mathsf{cl}(s{\downarrow}) = s{\downarrow}$ for every $s \in S$. Observe that $\mathscr{Q}(S)$ is a pomonoid with the identity element $1{\downarrow}$. Let us show that $\mathscr{Q}(S)$ is $\mathscr{E}^1_{\leq}$-injective in the category $\mathsf{PoMon}^1_{\leq}$. Consider a morphism $h : A \to B$ in $\mathscr{E}^1_{\leq}$ and any morphism $f : A \to \mathscr{Q}(S)$ in $\mathsf{PoMon}^1_{\leq}$. Since $\mathscr{Q}(S)$ is $\mathscr{E}_{\leq}$-injective in $\mathsf{PoSgr}_{\leq}$, there exists $g : B \to \mathscr{Q}(S)$ in $\mathsf{PoSgr}_{\leq}$ such that $gh = f$. Then $1{\downarrow} = f(1) = (gh)(1) = g(1)$ and g is a morphism in $\mathsf{PoMon}^1_{\leq}$. A similar argument shows that $\eta : S \to \mathscr{Q}(S), s \mapsto s{\downarrow}$ is an $\mathscr{E}^1_{\leq}$-essential morphism in $\mathsf{PoMon}^1_{\leq}$. □

As the authors of [6] mention, in the category of pomonoids it would be natural to require $1 \leq f(1)$ instead of $1 = f(1)$ from a morphism f. It is an open problem if in such a category injective hulls can be constructed in a similar way.

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Järjestatud poolrühmade injektiivsed katted

Xia Zhang ja Valdis Laan

On tõestatud, et injektiivsed objektid teatud sisestuste klassi suhtes kategoorias, mille objektideks on järjestatud poolrühmad ja morfismideks submultiplikatiivsed kujutused, on kvantaalid. Samuti on näidatud, kuidas teatud poolrühmade klassi jaoks saab konstrueerida injektiivseid katteid vaadeldava sisestuste klassi suhtes.